Do five of the following seven problems.
Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in
ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and
follow any instructions concerning identification code numbers.

Notation: \( \mathbb{C} \) denotes the set of complex numbers.
\( \mathbb{R} \) denotes the set of real numbers.
\( \text{Re}(z) \) denotes the real part of the complex number \( z \).
\( \text{Im}(z) \) denotes the imaginary part of the complex number \( z \).
\( \overline{z} \) denotes the complex conjugate of the complex number \( z \).
\( |z| \) denotes the absolute value of the complex number \( z \).
\( \log z \) denotes the principal branch of \( \log z \).
\( \text{Arg} z \) denotes the principal branch of \( \text{arg} z \).
\( D(z; r) \) is the open disk with center \( z \) and radius \( r \).
A domain is an open connected subset of \( \mathbb{C} \).

MISCELLANEOUS FACTS

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\begin{align*}
2 \sin a \sin b &= \cos(a - b) - \cos(a + b) \\
2 \sin a \cos b &= \sin(a + b) + \sin(a - b) \\
\sin(a + b) &= \sin a \cos b + \cos a \sin b \\
\tan(a + b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\
\sin^2 a &= \frac{1}{2} - \frac{1}{2} \cos(2a) \\
\cos^2 a &= \frac{1}{2} + \frac{1}{2} \cos(2a)
\end{align*}
\]
Fall 2004 # 1. How many values are there for \((1 + i)^{2/3}\)? Write each in polar form \((re^{i\theta})\) and in rectangular form \((a + bi)\). Sketch their location(s) in the plane.

Fall 2004 # 2. For each of the following real valued functions \(u(x, y)\) determine whether it can be the real part of an analytic function \(f(z) = f(x + iy) = u(x, y) + iv(x, y)\) with \(\text{Im}(f(z)) = v(0, 0) = 0\). If it can be, find \(v(x, y)\). If it cannot, explain how you know that.

- a. \(u(x, y) = x^3 + 3x^2y - 3xy^2 - y^3\)
- b. \(u(x, y) = 4x^3y + 2xy - 1\)

Fall 2004 # 3. Evaluate each of the following integrals.

- a. \(\int_{\gamma} \frac{\sin(z^2)}{z^7} \, dz\) where \(\gamma\) is the circle of radius 1 centered at the origin and traveled once counterclockwise.
- b. \(\int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} \, dx\) Show any contours and explain estimates needed to justify your method.

Fall 2004 # 4. Let \(A\) be the disk \(A = \{z \in \mathbb{C} : |z - 1| < 1\}\).
Let \(B\) be the half-disk \(B = \{z \in \mathbb{C} : |z - 1| < 1\text{ and } \text{Im}(z) > 0\}\).
Let \(C\) be the half-plane \(C = \{z \in \mathbb{C} : \text{Re}(z) > 0\}\).
Let \(D\) be the quarter-plane \(D = \{z \in \mathbb{C} : \text{Re}(z) > 0\text{ and } \text{Im}(z) > 0\}\).
Let \(E\) be the half-plane \(E = \{z \in \mathbb{C} : \text{Im}(z) > 0\}\).

- a. Find \(f : A \to C\) mapping \(A\) one-to-one analytically onto \(C\).
- b. Find \(g : B \to E\) mapping \(B\) one-to-one analytically onto \(E\).
(Hint: How might \(D\) figure into this problem?)

Fall 2004 # 5. Let \(p(z) = z^{10} - 3z^3 + 1\). Let \(A\) be the annulus \(A = \{z \in \mathbb{C} : 1 < |z| < 2\}\).

- a. Counting possible multiplicity, how many zeros does the polynomial \(p\) have in \(A\)?
- b. Show that none of the zeros of \(p\) in \(A\) can have multiplicity larger than 1.

Fall 2004 # 6. Suppose \(f : \mathbb{C} \to \mathbb{C}\) is analytic on \(\mathbb{C}\) and that it is an isometry in the sense that \(|f(z) - f(w)| = |z - w|\) for all \(z\) and \(w\) in \(\mathbb{C}\). Show that there are constants \(a\) and \(b\) such that \(f(z) = az + b\) for all \(z\) in \(\mathbb{C}\).

Fall 2004 # 7. Let \(A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}\).
Let \(C_1\) and \(C_2\) be the boundary circles, \(C_1 = \{z \in \mathbb{C} : |z| = 1\}\) and \(C_2 = \{z \in \mathbb{C} : |z| = 2\}\).
Suppose \(f\) is a complex valued function analytic on an open set containing \(A\) such that \(|f(z)| \leq 3\) for all \(z\) on \(C_1\) and \(|f(z)| \leq 12\) for all \(z\) on \(C_2\).

Show that \(|f(z)| \leq 3|z|^2\) for all \(z\) in \(A\).

End of Exam