Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: \( \mathbb{C} \) denotes the set of complex numbers.
\( \mathbb{R} \) denotes the set of real numbers.
\( \text{Re}(z) \) denotes the real part of the complex number \( z \).
\( \text{Im}(z) \) denotes the imaginary part of the complex number \( z \).
\( \overline{z} \) denotes the complex conjugate of the complex number \( z \).
\( |z| \) denotes the absolute value of the complex number \( z \).
\( \log z \) denotes the principal branch of log \( z \).
\( \arg z \) denotes the principal branch of \( \arg z \).
\( D(z; r) \) is the open disk with center \( z \) and radius \( r \).
A domain is an open connected subset of \( \mathbb{C} \).

**MISCELLANEOUS FACTS**

\[
\begin{align*}
2 \sin a \sin b &= \cos(a - b) - \cos(a + b) \\
2 \cos a \cos b &= \cos(a - b) + \cos(a + b) \\
2 \sin a \cos b &= \sin(a + b) + \sin(a - b) \\
2 \cos a \sin b &= \sin(a + b) - \sin(a - b) \\
\sin(a + b) &= \sin a \cos b + \cos a \sin b \\
\cos(a + b) &= \cos a \cos b - \sin a \sin b \\
\tan(a + b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\
\sin^2 a &= \frac{1}{2} - \frac{1}{2} \cos(2a) \\
\cos^2 a &= \frac{1}{2} + \frac{1}{2} \cos(2a)
\end{align*}
\]
Fall 2003 # 1. a. (12 points) Describe and sketch each of the following regions. (Giving reasons for your answers.)

(i) \( A = \{ z \in \mathbb{C} : \text{Im} \left( \frac{z+1}{z-1} \right) < 0 \} \)

(ii) \( B = \{ z \in \mathbb{C} : \text{Re} \left( \frac{z+1}{z-1} \right) < 0 \} \)

b. (8 points) Find a fractional linear (Möbius) transformation \( f \) such that

\[ f(i) = -i, \quad f(0) = -1, \quad \text{and} \quad f(-1) = 0. \]

(You may do parts a and b in either order, and they may or may not be related.)

Fall 2003 # 2. Suppose \( f : \Omega \to \mathbb{C} \) is analytic on an open subset \( \Omega \) of \( \mathbb{C} \). For \( z = x + iy \) in \( \Omega \) with \( x \) and \( y \) real, let \( u(x, y) = \text{Re}(f(x + iy)) \) and \( v(x, y) = \text{Im}(f(x + iy)) \)

a. State the Cauchy-Riemann equations for \( u \) and \( v \) and show how they follow from the existence of \( f'(z) \).

b. Show that \( u \) and \( v \) are harmonic on \( \Omega \).

c. Find a harmonic conjugate \( v(x, y) \) for the function \( u(x, y) = 1 + 2x + y^3 - 3x^2y \).

Fall 2003 # 3. Find the Laurent series for \( f(z) = \frac{1}{(z-1)(z-2)} \) valid in each of the following regions.

a. \( A = \{ z \in \mathbb{C} : 0 < |z-1| < 1 \} \)

b. \( B = \{ z \in \mathbb{C} : 1 < |z-1| \} \)

Fall 2003 # 4. Let \( f(z) = \frac{z^2}{e^z - 1} \).

a. Find all the singularities of \( f \) in \( \mathbb{C} \) and classify each as a removable singularity, a pole, or an essential singularity. For poles, specify the order of the pole.

b. Evaluate \( \int_{\gamma} f(z) \, dz \) for each of the following paths \( \gamma \).

(i) the circle of radius 1 centered at 0 traveled once counterclockwise

(ii) the circle of radius 8 centered at 0 traveled once counterclockwise

Fall 2003 # 5. Let \( f : \mathbb{C} \to \mathbb{C} \) and \( g : \mathbb{C} \to \mathbb{C} \) be analytic on all of \( \mathbb{C} \).

a. Show that if \( \lim_{z \to \infty} |g(z)| = 0 \), then \( g(z) = 0 \) for all \( z \) in \( \mathbb{C} \).

b. Show that if \( \lim_{z \to \infty} |f'''(z)| = 0 \), then \( f \) must be a polynomial.
Fall 2003 # 6. Evaluate each of the following integrals. Show any curves and explain estimates needed to justify your method.

\begin{align*}
\text{a. } & \int_0^{2\pi} \frac{dt}{4 + \sin t} & \text{b. } & \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} & \text{c. } & \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} \, dx
\end{align*}

Fall 2003 # 7. Consider the problem: Find a function \( f \) with

\[ f'(z) - f(z) = z \quad \text{and} \quad f(0) = 1. \]

Suppose \( f \) has a series solution \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) valid in some neighborhood of 0.

\begin{enumerate}
\item Compute what \( a_1 \), \( a_2 \), and \( a_3 \) would have to be.
\item Find what the series would have to be.
\item Show that the series converges to a solution which is an entire function.
\end{enumerate}

(You may leave the solution as an infinite series if you need to.)

End of Exam