Do five of the following seven problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: \( \mathbb{C} \) denotes the set of complex numbers.
\( \mathbb{R} \) denotes the set of real numbers.
\( \text{Re}(z) \) denotes the real part of the complex number \( z \).
\( \text{Im}(z) \) denotes the imaginary part of the complex number \( z \).
\( \overline{z} \) denotes the complex conjugate of the complex number \( z \).
\( |z| \) denotes the absolute value of the complex number \( z \).
\( \text{Log} z \) denotes the principal branch of \( \log z \).
\( \text{Arg} z \) denotes the principal branch of \( \arg z \).
\( D(z; r) \) is the open disk with center \( z \) and radius \( r \).
A domain is an open connected subset of \( \mathbb{C} \).

Fall 2002 # 1. For \( z \) in \( \mathbb{C} \), let \( x = \text{Re}(z) \) and \( y = \text{Im}(z) \).
For each of the following functions \( u(x, y) \), determine whether there is a real valued function \( v(x, y) \) such that \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic with \( f(0) = 3i \). If there is such a function \( v \), find it. If there is not, explain how you know there is not.

a. \( u(x, y) = (1 - x)^2 y \)
b. \( u(x, y) = (1 - x)y \)

Fall 2002 # 2. Suppose the power series \( \sum_{k=0}^{\infty} a_k z^k \) has radius of convergence 1. Call the value of the sum \( f(z) \) and let \( g(z) = f(z)/(1 - z) \).

a. Explain how you know \( g(z) \) has a series representation of the form \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) valid for \( |z| < 1 \).
b. For each \( n \) find \( b_n \) in terms of \( a_0, a_1, a_2, a_3, \ldots \).
Fall 2002 # 3. Suppose $z$ and $w$ are complex numbers with $|z| \leq 1$ and $|w| < 1$.  
Show that $|z - w| \leq |1 - z\overline{w}| = |1 - \overline{zw}|$.  
(Suggestion: Do it first with $|z| = 1$.)

Fall 2002 # 4.  

a. Suppose $p$ and $q$ are polynomials with degree$(q) \geq$ degree$(p) + 2$, and let $f(z) = p(z)/q(z)$.
Show that the sum of the residues of $f$ at all of its singularities in $\mathbb{C}$ is equal to 0.

b. Let $\gamma$ be the circle of radius 2 centered at 0 and travelled once counterclockwise.
Evaluate $\int_{\gamma} \frac{1}{(z-3)(z^5-1)} \, dz$

Fall 2002 # 5. Evaluate $\int_{-\infty}^{\infty} \frac{\cos^2 x}{1 + x^2} \, dx$.
Indicate curves and estimates used to justify your method.
(Suggestion: Write $\cos x$ in terms of exponentials.)

Fall 2002 # 6.  
a. (5 points) State a version of the Schwarz lemma.
b. (15 points) Let $U$ be the open disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. Suppose $f : U \rightarrow U$ is analytic on $U$ with $f(1/2) = 0$. Show that

$$|f(z)| \leq \left| \frac{z - (1/2)}{1 - (1/2)z} \right| = \left| \frac{2z - 1}{2 - z} \right|$$

for all $z$ in $U$.

Fall 2002 # 7. Let $f(z) = z^4 + 3z + 1$
a. How many zeros, counting multiplicity, does $f$ have in the annulus $A = \{ z \in \mathbb{C} : 1 \leq |z| \leq 2 \}$?
b. Can any of the zeros found in part (a) have multiplicity larger than 1?

End of Exam