Answer 5 questions only. You must answer at least one from each of Groups, Rings, and Fields. Please show work to support your answers.

GROUPS:

1. Let $P$ be a Sylow $p$-subgroup of $G$. Let $N \triangleleft G$. Show:
   (a) $P \cap N$ is a Sylow $p$-subgroup of $N$.
   (b) $PN/N$ is a Sylow $p$-subgroup of $G/N$.

2. Let $H$ be a subgroup of $G$ and let $Z = Z(G)$, the center of $G$, and suppose $G = HZ$. Prove:
   (a) $H \cap Z = Z(H)$.
   (b) $G/Z = H/Z(H)$.

3. Let $G$ be a group of order 175 ($5^2 \cdot 7$). Prove that $G$ is abelian.

RINGS:

4. (a) Let $F$ be a field and let $f(x) \in F[x]$ with $\deg(f(x)) = n > 0$. Prove that $f(x)$ has at most $n$ roots in $F$.
   (b) Let $F$ be a field and let $f(x)$ and $g(x)$ be elements of $F[x]$ with $\deg(f(x))$ and $\deg(g(x))$ each at most $n$. Suppose there exist $a_1, a_2, a_3, \ldots, a_{n+1} \in F$ such that $f(a_i) = g(a_i)$ for $1 \leq i \leq n+1$. Prove that $f(x) = g(x)$.

5. Prove that the ring $F^{2 \times 2}$ of $2 \times 2$ matrices over the field $F$ has no ideals except for $\{0\}$ and $F^{2 \times 2}$.

6. Let $M$ be a proper ideal of the commutative ring $R$. Prove that $M$ is a maximal ideal if and only if $R = M + (a)$, for all $a \notin M$ (here $(a)$ = the principal ideal generated by $a$).

FIELDS:

7. Let $E$ be an algebraic extension of a field $F$. Let $\alpha \in E$ and let $p(x) \in \text{Irr}(\alpha, x, F)$, the minimal polynomial of $\alpha$ over $F$. Prove:
   (a) If the degree of $p(x)$ is 3, then $F(\alpha^2) = F(\alpha)$.
   (b) If $\beta \in E$ and $[F(\beta) : F] = 7$, then $p(x) = \text{Irr}(\alpha, x, F(\beta))$.

8. Let $E$ be the splitting field of $x^5 - 3$ over the rational numbers $Q$.
   (a) Find $[E : Q]$ and explain your answer.
   (b) Show that the Galois group $G(E/Q)$ is not abelian.

9. (a) Show that $f(x) = x^3 + 2x + 1$ is irreducible over the rational numbers $Q$.
   (b) Show that $f(x)$ has at least one real root.
   (c) Let $\alpha$ be a root of $f(x)$ in the reals and find rational numbers $b_0, b_1, b_2$ such that $(\alpha+1)^{-1} = b_0 + b_1\alpha + b_2\alpha^2$. 