Answer 5 questions only. You must answer at least one from each of Groups, Rings, and Fields. Please show work to support your answers.

GROUPS

1. Let $p$ be a prime and $G$ be a finite $p$-group with center $Z(G)$.
   (a) Show that $Z(G) \neq \{e\}$
   (b) If $N$ is a normal subgroup with $|N| = p$, prove that $N \subseteq Z(G)$.

2. Prove that any group of order 255 is cyclic.

3. Let $G$ be an group of order 405 ($= 3^4 \cdot 5$). Prove that $G$ is solvable.

RINGS

1. Let $R$ be a commutative ring with identity and let $I$ be an ideal of $R$. Define
   $\alpha(I) = \{x \in R \mid \exists n \geq 1, \text{ with } x^n \in I\}$ and prove that:
   (a) $\alpha(I) \supseteq I$,
   (b) $\alpha(I)$ is an ideal of $R$, and
   (c) $\alpha(\alpha(I)) = \alpha(I)$.

2. Let $R$ be a ring with identity and assume that $x \in R$ has a right inverse. Prove that the following are equivalent:
   (a) $x$ has more than one right inverse,
   (b) $x$ is not a unit, and
   (c) $x$ is a left zero-divisor.

3. Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$
   Where $R$ is the set of real numbers with the usual matrix operations.
   (a) Prove that $M$ is not a field.
   (b) Prove that an element $A$ of $M$ is a zero-divisor $\iff \det A \neq 0$

FIELDS

1. Let $E$ be the splitting field of $x^6 - 3$ over the rationals $\mathbb{Q}$.
   (a) Find $[E : \mathbb{Q}]$, and explain.
   (b) Show that the Galois group $\text{Gal}(E/\mathbb{Q})$ is not abelian.

2. Prove that “algebraicness” is transitive; i.e., if $E$, $F$, and $K$ is a tower of fields with $F$ algebraic over $E$ and $K$ algebraic over $F$, then $K$ is algebraic over $E$.

3. Let $E = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ and $\alpha = \sqrt{3} + \sqrt{5}$
   Prove: (a) $[E : \mathbb{Q}] = 4$.
   (b) $E = \mathbb{Q}(\alpha)$
   (c) Describe the Galois group $\text{G}(E/\mathbb{Q})$. 