Answer FIVE questions only. You must answer AT LEAST ONE from each of GROUPS, RINGS, and FIELDS. Be sure to show enough work that your answers are adequately supported.

GROUPS

1. Let $p$ be a prime and $G$ be a $p$–group. Let $N$ be a normal subgroup of $G$ and assume $N$ has order $p$. Prove that $N$ is in the center of $G$.

2. Let $H$ be a proper subgroup of a finite group $G$ and for each $g$ in $G$ let $H^g = \{ g^{-1}hg \mid h \in H \}$. Let $K = \bigcup_{g \in G} H^g$. Prove that $K \neq G$.

3. Let $G$ be a group of order $992 = 32 \times 31$. Prove that $G$ is solvable.

RINGS

1. Let $R$ be a finite commutative ring with more than one element and with no zero divisors. Prove that $R$ is a field.

2. Let $R$ be a commutative ring with 1 such that all its ideals are finitely generated. Prove that any ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$$

must terminate in finitely many steps.

3. Prove that every ideal of a Euclidean domain is principal.

FIELDS

1. Let $E$ be the splitting field of $x^8 - 2$ over the field of the rational numbers $\mathbb{Q}$.
   i) Prove that $[E: \mathbb{Q}] = 16$.
   ii) Show that the Galois group $G(E/\mathbb{Q})$ is not abelian.

2. Let $K$ be a field extension of a field $F$ and $\alpha \in K$. Let $F[\alpha]$ be the smallest subring of $K$ that contains both $F$ and $\alpha$ and let $F(\alpha)$ be the smallest subfield of $K$ that contains both $F$ and $\alpha$. Prove that $\alpha$ is algebraic over $F$ if and only if $F[\alpha] = F(\alpha)$.

3. Let $\mathbb{Q}$ be the field of the rational numbers and let $F = \mathbb{Q}(\sqrt{3}, \sqrt[5]{5}, \sqrt[11]{11}, \sqrt[13]{13})$. Find $\alpha$ in $F$ such that $F = \mathbb{Q}(\alpha)$ and prove your result.