Answer five (5) questions only! You must answer at least one from each of Groups, Rings, and Fields. Show enough work to adequately support your answers.

Groups

1. Let $G$ be a group of order 147. Prove that $G$ contains a nontrivial normal abelian subgroup.

2. Prove that \( \text{Aut}(\mathbb{Z}_n) \cong \mathbb{U}_n \). \([ \text{Aut}(G) = \{ \phi: G \to G \mid \phi \text{ is an isomorphism} \} ]\)
   \([ \mathbb{U}_n = \{ k \in \mathbb{Z}_n \mid \text{GCD}(k, n) = 1 \} \] is the group of units in the ring \( \mathbb{Z}_n \). Also known as \( \mathbb{Z}_n^\times \).\]

3. Let $p$ be a prime and assume $G$ is a finite $p$-group.
   a) Show that the center of $G$ is non-trivial (i.e. $Z(G) \neq \{e\}$).
   b) Let $K$ be a normal subgroup of $G$ of order $p$. Show that $K \subseteq Z(G)$.

Rings

1. Let $R$ be a finite commutative ring with more than one element and with no zero divisors. Prove that $R$ is a field.

2. Let $R$ be a commutative ring with identity. Define $\rho(I) = \{ x \in R \mid x^n \in I, \text{ for some } n \geq 1 \}$. Prove the following:
   a) If $I$ is an ideal of $R$, then $\rho(I)$ is an ideal of $R$.
   b) If $I \subseteq J$ are ideals, then $\rho(I) \subseteq \rho(J)$.
   c) $\rho(\rho(I)) = \rho(I)$.
   d) If $I$ and $J$ are ideals of $R$, then $\rho(I \cap J) = \rho(I) \cap \rho(J)$.

3. Let $F$ be a field. Prove that every ideal of $F[X]$ is principal.

Fields

1. Let $\mathbb{Q}$ be the rational numbers and let $E$ be the splitting field of $p(X) = X^4 - 2$ over $\mathbb{Q}$.
   a) Find $[E : \mathbb{Q}]$ and explain your answer.
   b) Show that $\text{Gal}(E/\mathbb{Q})$ is not abelian.

   a) Show that $F(\alpha) = F(\alpha^2)$, for all $\alpha \in E$.
   b) Show that $F(\alpha) = F(\alpha^9)$, for all $\alpha \in E$.

3. Let $G$ be a finite group. Prove that there exists a Galois field extension $K/L$ whose Galois group is isomorphic to $G$. 