A THEORY OF MATHEMATICAL CORRECTNESS AND MATHEMATICAL TRUTH[1]

BY

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Abstract: A theory of objective mathematical correctness is developed. The theory is consistent with both mathematical realism and mathematical anti-realism, and versions of realism and anti-realism are developed that dovetail with the theory of correctness. It is argued that these are the best versions of realism and anti-realism and that the theory of correctness behind them is true. Along the way, it is shown that, contrary to the traditional wisdom, the question of whether undecidable sentences like the continuum hypothesis have objectively determinate truth values is independent of the question of whether mathematical realism is true.

1. Introduction

It is widely known that there are certain mathematical sentences that are undecidable in all of our currently accepted axiomatic mathematical theories. A famous example is the continuum hypothesis (CH)[2]: it can neither be proven nor disproven in any of our standard axiomatic theories, e.g., Zermelo-Fraenkel set theory (ZF). Sentences that are undecidable in this way give rise to open mathematical questions that cannot be answered by our axiomatic theories – questions like 'Is CH true or false?' – and many people have wondered whether these questions have objectively correct answers. For the time being, let us call the view that such questions do have correct answers *objectivism* and the view that they don't have correct answers *anti-objectivism*.

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Which of these views is right? Well, the traditional wisdom is that the answer to this question depends upon the answer to another question, namely, the question of whether *mathematical realism* or *anti-realism* is true, where realism is the view that (a) there exist mathematical objects – in particular, abstract (i.e., non-spatio-temporal) mathematical objects – and (b) our mathematical theories are descriptions of such objects; and anti-realism is the view that there are no such things as mathematical objects and, hence, that our mathematical theories are, in some sense or other, factually empty.[3] One might try to motivate this "traditional wisdom" in something like the following way:

If realism is true – if there is a universe of sets "out there in the world", or in "Platonic Heaven" – then CH is either true of that universe or false of it; we might not know what the right answer is, but that is irrelevant to the fact that there *is* a right answer. But if anti-realism is true – if there are no such things as mathematical objects – then there is nothing for CH to be right or wrong *about*, and so there is no objectively correct answer to the CH question; there will be various (internally consistent) set theories that answer the CH question differently – e.g., ZF+CH and ZF+~CH, to name just two – but it would be wrong to say that some of these theories are correct and others are incorrect. Thus, overall, it seems that the objectivism–anti-objectivism debate is decided by the realism–anti-realism debate. For (a) mathematical realism entails objectivism about undecidable sentences like CH, and (b) mathematical anti-realism entails anti-objectivism about such sentences.

I will argue that this traditional view is confused. I am going to develop and motivate a novel theory of objective mathematical correctness, i.e., a theory of what such correctness ultimately consists in. Now, one thing we'll see here is that the simple objectivism–anti-objectivism taxonomy is over-simplified. More specifically, we'll see that we shouldn't endorse objectivism or anti-objectivism "across the board"; rather, we should take each open question as a separate case and decide whether *it* has an objectively correct answer. Moreover, we'll see that whether a given mathematical question has an objectively correct answer is determined by something very specific that has nothing whatsoever to do with the question of whether there exist any mathematical objects. Thus, in accordance with this, my theory of mathematical correctness is going to be perfectly consistent with both realism and anti-realism. And to back this up, I am going to formulate versions of realism and anti-realism that fit perfectly with my theory of correctness (and, hence, do not entail objectivism or anti-objectivism) and in arguing for my theory of correctness, I will also be arguing that realists and anti-realists ought to formulate their views in the ways that I suggest. But my central thesis is that regardless of whether we endorse realism or anti-realism (or neither), we ought to endorse my theory of correctness.

(I've called the view that the objectivism debate is decided by the realism debate the "traditional wisdom", but it is worth noting that *some* people have questioned it. Putnam very famously rejected half of it, arguing that realism doesn't deliver objectivism. Field has suggested that anti-realists can salvage some measure of objectivity in mathematics – although not enough to make the CH question objective. And Maddy takes this all the way: she thinks we can have full-blown objectivity without realism.[4] But in any event, the analysis that I provide here of *why* the traditional wisdom is wrong, as well as the positive view that I erect in its place, are both original.)

2. *A Quick Argument Against Traditional Anti-objectivism*

I begin with a quick argument against the traditional anti-objectivism described in the indented passage three paragraphs back. According to this view, there is no objectively correct answer to questions like the CH question, because anti-realism is true and, hence, there are no mathematical objects for sentences like CH and ~CH to be right or wrong *about*. The problem here is that if we apply this reasoning in other cases, we are led to an extreme version of anti-objectivism that is wildly implausible and that flies in the face of mathematical practice. Consider, for instance, the question of whether 4 is even or odd. Anti-objectivists of the above sort seem committed to saying that there is no correct answer to this question because there are no mathematical objects for sentences like '4 is even' and '4 is odd' to be right or wrong *about*. But this seems implausible, and it seems inconsistent with mathematical practice. Surely, there is some important sense in which '4 is even' is correct and '4 is odd' is incorrect.

The same sort of argument can be constructed in connection with undecidable sentences as well. Let S be a mathematical sentence that is undecidable in all of our current axiomatic theories. Suppose that some mathematician M discovers a new axiom candidate AX that is *extremely obvious* to everyone in the mathematical community, intuitively speaking, and suppose that M proves S from AX (and other already-accepted sentences). Given this, what we *want* to say, intuitively, is that M has shown that S is correct and ~S is incorrect. Moreover, this is what mathematicians *would* say. But anti-objectivists of the above sort cannot say this, for the argument they use to arrive at anti-objectivism about CH applies straightforwardly to S, and so they are committed to taking an anti-objectivist line on S. Thus, we have reason to be dissatisfied with the above sort of anti-objectivism: it has consequences that (a) are counterintuitive and (b) fly in the face of mathematical practice. Or to put point (b) differently, the problem is that this sort of anti-objectivism is revisionistic and, hence, violates certain widely accepted principles of naturalism.

(In section 6, we will encounter more reasons for being dissatisfied with traditional anti-objectivism, but this is good enough for now.)

3. *A Theory of Objective Mathematical Correctness*

I now turn to the heart of the paper. In this section, I will present a theory of objective mathematical correctness, and in subsequent sections, I will argue that this theory is right, that it's consistent with both realism and anti-realism, and indeed, that the best versions of realism and anti-realism involve a commitment to this theory of correctness. For reasons that will emerge below, I will call my theory *Intention-Based Partial Objectivism*, or for short, IBPO. The centerpiece of IBPO is the following principle:

(COR) A mathematical sentence is *objectively correct* just in case it is "built into", or follows from, the notions, conceptions, intuitions, and so on that we have in connection with the given branch of mathematics.[5]

Thus, for instance, an arithmetical sentence is correct, on this view, if and only if it is built into our *full conception of the natural numbers* (FCNN), where FCNN is the sum total of all of our "natural-number thoughts" and everything that follows from these thoughts. Likewise, a set-theoretic sentence is correct if and only if it is built into our *full conception of the universe of sets* (FCUS).

(*Prima facie*, it might seem that (COR) provides a subjectivist, or psychologistic, account of mathematical correctness; but we will see below that this is not the case; indeed, we will see that mathematical realists can (and should) endorse (COR) and that, within a realist view, the notion of correctness delivered by (COR) is a standard notion of correspondence truth.)

The reader might be wondering what the phrase 'built into' means here. It's very simple: to take the case of arithmetic, we can say that a sentence is built into FCNN just in case it is one of our (explicit or implicit) beliefs about the natural numbers or follows from these beliefs. I should also note that in using 'follows' here, I do not have in mind first-order or second-order entailment; I am thinking, rather, of our intuitive, pre-theoretic conception of entailment.[6] A similar point holds for FCNN – it is not a precise formal theory. For insofar as our thoughts aren't always perfectly precise, and insofar as there isn't perfect agreement in the mathematical community, the FCs in the various branches of mathematics can also be imprecise. Thus, for instance, there may be some sentences for which it is not clear whether they are built into FCNN or, indeed, for which there is no fact of the matter as to whether they are built into FCNN. But

for *most* sentences, it is perfectly clear – e.g., '4 is even' and '4 is not identical to any emperor of ancient Rome' are clearly in FCNN, and '4 is odd' and '4 is Julius Caesar' are clearly not in FCNN – and (as we'll see below) the fact that there may be some sentences for which it's not clear is not a problem.[7] Moreover, it should be noted that for those sentences that are clearly in FCNN (or clearly not in FCNN), it is an *objective fact* that they are (or aren't) in FCNN. E.g., it is an objective fact that '4 is even' is in FCNN and '4 is Julius Caesar' is not. The reason these facts are objective is that (a) they're facts about what beliefs we have and what follows from these beliefs, and (b) facts of this sort are objective. (I assume here that facts about what follows from what are objective; that is, I assume that objectivism about logic is true.)

It should be noted that it follows from (COR) that a sentence could be undecidable in all of our currently accepted axiomatic theories and still be objectively correct, because it could be built into the intuitions, notions, conceptions, and so on that we have in the given branch of mathematics. For instance, an arithmetical sentence could be undecidable in theories like Peano Arithmetic (PA) but still follow from FCNN. That this is possible is clear from the fact that FCNN is strictly stronger than any theory like PA. All of the axioms and theorems of PA are clearly built into FCNN, but there are also sentences in FCNN that are not theorems of any of our axiomatic theories – e.g., 'The number 4 is not identical to any emperor of ancient Rome'. And there are also purely mathematical sentences of this sort; for it is surely built into our notions, conceptions, intuitions, and so on that the Gödel sentences of our axiomatic theories are true. Also, if we restrict our attention to first-order theories, then another example will be the sentence 'Every natural number has only finitely many predecessors'.[8]

But while it follows from (COR) that open questions about undecidable sentences *can* have objectively correct answers, it does not follow that *all* such questions do have objectively correct answers. For it may be that some of these questions are such that none of their answers are built into the intuitions, notions, conceptions, and so on that we have in the given branch of mathematics. Consider, for instance, the CH question. It may be that neither CH nor ~CH is built into FCUS. Or equivalently, it may be that CH and ~CH are both perfectly consistent with FCUS. Thus, to make room for possibilities like this, I add the following two principles to IBPO:

(INCOR) A mathematical sentence is *objectively incorrect* just in case it is inconsistent with the notions, conceptions, intuitions, and so on that we have in connection with the given branch of mathematics (i.e., just in case its negation is "built into", or follows from, these notions, conceptions, and so on).

(NEUT) Many mathematical sentences are objectively correct, and many are objectively incorrect, but it *may* be that there are some mathematical sentences that are neither objectively correct nor objectively incorrect, because (a) they do not follow from the notions, conceptions, intuitions, and so on that we have in the given branch of mathematics, and (b) they are not inconsistent with these notions, etc.[9]

So IBPO is at least *partially* objectivist. In other words, it is not consistent with a full-blown anti-objectivism, although it does allow for the possibility of a sort of "weak anti-objectivism" according to which all of the "important" open questions about undecidable sentences that mathematicians are actually concerned with lack objectively correct answers. On the other hand, IBPO also allows for the possibility of a rather strong sort of objectivism being true; for it allows that it *may* be that all of the open questions of mathematics have objectively correct answers. It also allows for the possibility of objectivism being true in some branches of mathematics (e.g., arithmetic) but false in others (e.g., set theory); in other words, it could be that all arithmetical questions have objectively correct answers but that some set-theoretic questions do not. Indeed, we'll see in section 6 that this is fairly likely. But for now, the main point I want to emphasize is that IBPO is neutral on all these questions: as far as IBPO is concerned, it could be that objectivism is false in all branches of mathematics, or that it's true in some branches and false in others, or that it's true in all branches. (It is worth noting, however, that even if objectivism were true in all *current* branches of mathematics, this would not entail a full-blown mathematical objectivism; for at any time, we could develop a new branch of mathematics, and it could contain sentences that neither followed from, nor were inconsistent with, the notions, conceptions, intuitions, and so on that people had in this new branch of mathematics.)

Before going on, I would like to point out that our various FCs – that is, FCNN, FCUS, and so on – can evolve. Because of this, a sentence that was once neither correct nor incorrect could slowly *become* correct. To see this, suppose that S is undecidable in ZF and that we are trying to figure out whether it is true or false. Suppose also that some mathematician comes up with a new axiom candidate – call it A – and proves S from ZF+A. And finally, suppose that A, ~A, S, and ~S are all consistent with FCUS, so that according to IBPO, none of these sentences are objectively correct or incorrect. (The mathematical community may or may not know whether these sentences are consistent with FCUS – for the present purposes, it doesn't matter.) Even if all this were the case, the mathematical community might slowly come to "embrace" ZF+A, and indeed, they might have very good reasons for doing this – e.g., it might be a highly useful and aesthetically pleasing theory. If this happened, then FCUS could slowly change, so that, eventually, it included A. (Note, too, that this could happen even if A never became intuitive. And more generally, it is important to note that FCUS is not wholly intuitive or pre-theoretic; it certainly includes some intuitive and pre-theoretic maxims, but it's also highly influenced by set theory and includes some sentences that are not at all intuitive. Indeed, some of the axioms of ZF, which are all pretty clearly built into FCUS, are not intuitive.)

4. *From Anti-Realism to IBPO*

I think that both realists and anti-realists should endorse IBPO – i.e., that the best versions of these views involve a commitment to IBPO – and I want to begin my argument for IBPO by explaining how an anti-realist might be led to endorse this view and how a realist might be led to endorse it. I begin in this section with anti-realism and turn, in section 5, to realism.

There are a number of different versions of anti-realism. The most traditional is probably formalism, but I think that the best one is *fictionalism*, and so I am going to start out by concentrating on it; but we will see at the end of this section that everything I say here about fictionalism applies to other versions of anti-realism as well. Fictionalism is the view that (a) our mathematical theories do *purport* to be descriptions of abstract mathematical objects (as realists assert) but (b) there are no such things as mathematical objects (because there are no such things as abstract objects) and so our mathematical theories are not true. They are not true for the same reason that fictional tales are not true: *The Lion, the Witch, and the Wardrobe* isn't true because there is no such place as Narnia, and likewise, our mathematical theories aren't true because there is no such thing as the mathematical realm, or Platonic Heaven.[10]

Now, we saw in section 1 that, *prima facie*, anti-realist views like fictionalism seem to lead to a rather strong version of anti-objectivism. But we also saw (section 2) that anti-realists would be wise to try to avoid this result (because this strong sort of anti-objectivism is counter-intuitive and at odds with mathematical practice). Thus, fictionalists need to find some way to salvage some measure of objectivity in mathematics; that is, they need to account for the fact that certain mathematical sentences and theories are, in some relevant sense, objectively correct, whereas others are incorrect. But in order to do this, fictionalists need to come up with a notion of mathematical correctness that is distinct from the standard notion of mathematical truth. For according to their view, all mathematical theories are fictional, and hence, according to the standard view of truth, none of them is true (except for those that are vacuously true). Fictionalists might begin here by saying something like the following:

> It is true that there are no such things as mathematical objects, and so it follows that, e.g., '4 is even' and '4 is odd' are both untrue. But there is still an important difference between these two sentences that is analogous to the difference between, say, 'Santa Claus lives at the North Pole' and 'Santa Claus lives in Tel Aviv'. The difference is that '4 is even' is part of a certain well-known mathematical story, whereas '4 is odd' is not. Field has expressed this idea by saying that while neither '4 is even' nor '4 is odd' is true *simpliciter*, there is another truth predicate – viz., '. . . is true in the story of mathematics' – that applies to '4 is even' but not to '4 is odd'.[11] Thus, we can say that a mathematical sentence is *objectively correct* just in case it is true in the story of mathematics.

But by itself, this does not solve the problem. Fictionalists cannot simply claim that '4 is even' is correct because it is part of the story of mathematics and leave it at that, because there are *alternative* mathematical "stories" consisting of sentences that are not part of standard mathematics, e.g., sentences like '4 is odd'. To block this objection, fictionalists need to point out that when they say that '4 is even' is part of *the* story of mathematics, what they really mean is that it is part of *our* story of mathematics. But what does it *mean* to say that '4 is even' is part of our story of mathematics? Well, one might think it means that '4 is even' is a theorem of our arithmetical theories, e.g., PA. But if fictionalists take this stand, they won't be able to salvage a strong enough version of objectivism. For on this view, any sentence that's undecidable in our axiomatic theories will be neither objectively correct nor objectively incorrect; but in section 2, we saw that this is unacceptable, that undecidable sentences can be objectively correct because they can follow from undiscovered sentences – *potential axioms*, we might call them – that are intuitively obvious to everyone.

So fictionalists need to maintain that there is more to "our story of arithmetic" than our axiomatic arithmetical theories. How can they do this? Well, it seems to me that the obvious thing for fictionalists to say here is that "our story of arithmetic" is determined by what we (as a community) *have in mind* when we're doing arithmetic. In other words, it's determined by the intuitions, notions, conceptions, and so on that we have in connection with arithmetic. In other words, it's determined by our *full conception of the natural numbers*, i.e., by FCNN. It should be clear that this gives fictionalists exactly what they need. If someone discovered a new arithmetical axiom candidate that was intuitively obvious to all of us, we would want to say that this new axiom (and everything it entailed) was correct (and fictionalists would want to say that it was part of the story of arithmetic). But on the present view, fictionalists would be able to say these things. On the other hand, if some mathematician tried to settle some open question by introducing a new axiom that didn't

seem so obvious to people, we would want to say that it wasn't clear that he or she had really settled the question. But again, on the present view, fictionalists would be able to say this. Indeed, this is more or less entailed by the present view.

All of this suggests that fictionalists ought to endorse (COR). And from this, it follows pretty trivially that they also ought to endorse (INCOR) and (NEUT). Indeed, all we need to add here to arrive at this result is that fictionalists ought to allow that it *may* be that in at least *one* branch of mathematics, the collection of notions, conceptions, intuitions, and so on that we have in that branch of mathematics is not strong enough to settle *every* question in that branch of mathematics. But it seems clear that fictionalists ought to allow for this possibility, for there is simply no good reason for them to insist that it is not possible.

(It is worth noting, however, that while fictionalists who endorse (COR) *should* endorse (NEUT), they don't *have* to. If fictionalists wanted to, they could maintain that our mathematical notions, conceptions, intuitions, and so on are strong enough to settle all mathematical questions and, hence, that all mathematical sentences are either objectively correct or objectively incorrect. The reason this is important is that it shows that fictionalism is compatible with a full-blown objectivism. So much for the myth that anti-realism entails anti-objectivism. It doesn't even entail a *partial* anti-objectivism.)

One might object to the argument of this section by claiming that IBPO-fictionalist correctness isn't a *genuine* sort of correctness because it's determined by factors *within us*. The full response to this worry cannot be given until we see how IBPO dovetails with realism and how realists can maintain that, contrary to first appearances, IBPO-correctness is a standard realist notion of correspondence truth. But I can say a few words now about how fictionalists can respond to this worry. They might say something like this:

> IBPO-fictionalist correctness isn't correspondence truth (though as we'll see, IBPO-realist correctness *is* correspondence truth), so if that's what you mean by "genuine" correctness, then you won't be happy with our view. But of course, if that's your position, then you won't be happy with fictionalism anyway. But fictionalists think that genuine mathematical correctness – i.e., the sort of correctness that mathematicians are actually after – is *not* correspondence truth, and they do not think that correspondence truth is a mathematical virtue. The question is whether IBPO-fictionalism provides an acceptable analysis of the sort of correctness that mathematicians actually seek. But it seems that it does, for it seems that IBPO-fictionalist correctness has all the features that mathematical correctness has. Most notably, it is objective and it captures the right extension, i.e., it applies to the sentences it should apply to (e.g., '4 is even') and not to the sentences it shouldn't apply to (e.g., '4 is odd'). So fictionalists think that this is exactly the sort of correctness that is at work in mathematical practice, and to claim that it's "non-genuine" simply because it's not correspondence truth is to beg the question.

Again, a more complete response to this worry will emerge below, but for now, this is good enough.

Finally, I would like to point out that everything I have said in this section about fictionalism carries over to other versions of anti-realism. All anti-realists need to find some way of avoiding the extreme anti-objectivism discussed in section 2. I have argued here that one plausible way for fictionalists to do this is to adopt IBPO. But it should be clear that this argument could be extended to cover other versions of anti-realism, because IBPO can do all the same work for every version of anti-realism that it can do for fictionalism. Non-fictionalistic anti-realists would simply have to phrase things differently; e.g., they wouldn't say that '4 is even' is correct because it's part of our *story* of arithmetic, but they could all say something more or less equivalent, because any anti-realist could say something to the effect that '4 is even' is correct because it follows from our arithmetical intentions, or from *what we have in mind* in connection with arithmetic. But if anti-realists say anything along these lines, they will be led to IBPO in essentially the same way that fictionalists are.

5. *From Realism to IBPO*

The arguments of the last section are not supposed to show that anti-realists *should* endorse IBPO. I will argue that later. The point of the last section was just to explain how an anti-realist might plausibly be led to endorse IBPO. The same goes for realism and the arguments of the present section: I will argue later that the *best* realist view is an IBPO-ist view, but in this section, I merely want to explain how a realist might plausibly be led to endorse IBPO. (*Prima facie*, it might seem surprising that a realist could endorse IBPO, for it might seem that IBPO is an anti-realist view; thus, I will go through this very slowly.)

Mathematical realists think that our mathematical theories are about abstract objects, or a non-spatio-temporal mathematical realm. We can divide realists into two camps, depending on whether they think the mathematical realm is plenitudinous or sparse. To say that the mathematical realm is *plenitudinous* is to say (roughly) that all the mathematical objects that (logically) possibly could exist actually do exist; and to say that the mathematical realm is *sparse* is to say that it's not plenitudinous. I will use the term *full-blooded platonism*, or FBP, to refer to plenitudinous versions of realism. Now, FBP-ist views are relative newcomers to the realist landscape, but they have been much discussed in the recent literature.[12] At any rate, in this section, I will explain how realists who endorse FBP-ist views might very plausibly be led to IBPO. In section 6, I will go on to argue that realists *should* endorse this line – i.e., that IBPO–FBP-ist versions of realism are superior to other versions of realism – but for now, I just want to concentrate on FBP and explain how it leads naturally to IBPO.

Let's begin by asking what FBP-ists ought to say about undecidable propositions like CH. The first point to note here is that it follows from FBP that every consistent purely mathematical theory truly describes some collection of mathematical objects. This is not just to say that FBP entails that every such theory has a model, for theories can have very *unnatural* models, i.e., models that are such that we would say that the given theory was not *about* that model. FBP entails that every consistent purely mathematical theory truly describes some collection of mathematical objects that is such that we would say that the given theory was straightforwardly *about* that collection of objects. For if there were such a theory that didn't do this, then we would have a counterexample to the FBP-ist claim that all the mathematical objects that possibly could exist actually do exist.[13] Now, given this result, and given that ZF+CH and ZF+~CH are both consistent purely mathematical theories, it follows that according to FBP, both ZF+CH and ZF+~CH truly describe collections of mathematical objects, or more specifically, set-theoretic hierarchies. (Of course, FBP-ists don't think these two theories truly describe one and the same hierarchy; they think they describe *different* hierarchies.)

Now, given all of this, an FBP-ist might be inclined to reason as follows: "Since ZF+CH and ZF+~CH both truly describe collections of mathematical objects, it follows that they are both *true* and, hence, that there is no objectively correct answer to the CH question." But this remark is confused, for as we will see below, FBP-ists cannot move so quickly from the claim that a mathematical theory truly describes a collection of mathematical objects to the claim that it's true. Moreover, if FBP-ists did endorse this stance, then they would be committed to a strong sort of anti-objectivism, and we have already seen (section 2) that this sort of anti-objectivism flies in the face of both intuition and mathematical practice. Thus, I do not think that FBP-ists should endorse this stance. (It is worth noting, however, that FBP-ists *could* take this line if they wanted to. And this means that mathematical realism – the view that our mathematical theories are descriptions of objectively existing abstract objects – is consistent with a full-blown anti-objectivism. So much for the myth that realism entails objectivism. It doesn't even entail a *partial* objectivism. But again, I do not think that FBP-ists should endorse this sort of anti-objectivism.)

Thus, as was the case with fictionalists, FBP-ists need to find some way of avoiding anti-objectivism, i.e., of accounting for the fact that among the consistent purely mathematical sentences and theories, some are correct and others are incorrect. FBP-ists might start out here by saying something like the following:

It is true that every consistent purely mathematical theory truly describes some collection of mathematical objects. But when we are doing mathematics, we are always talking about some *particular* collection of mathematical objects, or some particular structure. Thus, we can say that a mathematical sentence is *objectively correct* just in case it is true of the mathematical structure that we're talking about, or that we have in mind, in the given branch of mathematics. (Or using a more standard terminology, we might say that a mathematical sentence is *objectively correct* iff it's true in the *intended structure*, or the *standard model*, for the given branch of mathematics.[14]) By taking this line, we can salvage the fact that some mathematical sentences are correct and others are incorrect; indeed, we can salvage the fact that some *undecidable* sentences are correct whereas others are incorrect.

I think this idea is on the right track, but it is not quite right. One problem is the underlying assumption that in every branch of mathematics, there is a *unique* intended structure. FBP-ists cannot assume this. Consider, for instance, set theory. FBP-ists cannot assume that our set-theoretic intentions are strong enough to pick out a unique structure, or even a unique structure up to isomorphism, i.e., a unique collection of structures that are all isomorphic to one another. It may be that there are numerous hierarchies of sets that are not isomorphic to one another but are all perfectly consistent with all of our set-theoretic intentions and in which all of these intentions come out true. If this is indeed the case, then on the present view, set-theoretic correctness will have to be decided relative to *all* of these hierarchies, for they will all count as intended, i.e., as hierarchies that we're talking about in set theory. Thus, FBP-ists might be inclined to say that a mathematical sentence is objectively correct just in case (a) it's true in all of the intended structures for the given branch of mathematics and (b) there is at least one such intended structure.

But this isn't quite right either. For if the word 'structure' is understood in the usual model-theoretic way, then FBP-ists would be wise to avoid it here, because there may be mathematical theories that truly describe intended parts of the mathematical realm that don't qualify as structures. For instance, one might argue that since the intended part of the mathematical realm for set theory would presumably include all sets, and since there is no set of all sets, and since model-theoretic structures have sets as their domains,[15] there is no intended model-theoretic structure for set theory. Thus, to allow for this, FBP-ists might say something like this:

(A) A mathematical sentence is *objectively correct* if and only if (a) it is true in all the parts of the mathematical realm that count as intended for the given branch of mathematics and (b) there is at least one such part of the mathematical realm.[16]

Now, one might wonder what determines which parts of the mathematical realm count as intended for a given branch of mathematics. Not surprisingly, this is determined by the intentions that we have in that branch of mathematics.[17] And these intentions are, in turn, determined by the notions, conceptions, intuitions, and so on that we have in that branch of mathematics; that is, they are determined by the FC for that branch of mathematics. In short (and somewhat roughly[18]) we can say that a given part of the mathematical realm counts as intended, for a given branch of mathematics, just in case everything that's built into, or follows from, the FC for that branch of mathematics is true in that part of the mathematical realm. Thus, according to this view, (A) is equivalent to

(A′) A mathematical sentence is *objectively correct* if and only if (a) it is true in all the parts of the mathematical realm in which everything that's built into, or follows from, the FC for the given branch of mathematics is true; and (b) there is at least one such part of the mathematical realm.

Finally, to allow for the possibility that it may be that our mathematical intentions do not pick out unique structures up to isomorphism (while also allowing for the possibility that they do), FBP-ists should add the following two principles to (A):

(B) A mathematical sentence is *objectively incorrect* if and only if (a) it is false in all the parts of the mathematical realm that count as intended for the given branch of mathematics or (b) there is no such part of the mathematical realm;[19]

and

(C) Many mathematical sentences are objectively correct, and many are objectively incorrect, but it *may* be that there are some mathematical sentences that are neither objectively correct nor objectively incorrect because they are true in some parts of the mathematical realm that count as intended for the given branch of mathematics and false in others.[20]

It is important to note that the view of mathematical correctness that's inherent in (A)–(C) dovetails perfectly with standard realist ways of thinking about truth and also with mathematical practice, i.e., with the way that mathematicians speak of truth. As for the latter point, what mathematicians very often mean when they say that a mathematical sentence is true is that it's true in the standard model, or the intended structure (or the class of intended structures) for the given branch of mathematics; but the theory of correctness inherent in (A)–(C) is very much in this spirit. As for the former point – that (A)–(C) dovetails with standard realist ways of thinking about truth – I can motivate this by simply describing (very briefly and roughly) one view of truth that I take to be a standard realist view and pointing out that this view fits perfectly with (A)–(C).

The view of truth I have in mind can be summarized, very roughly, as follows. Consider the question of whether or not the sentence 'Snow is white' is true. If we are speaking of the sentence type, as opposed to any particular token of it, then all we can say is that it is true in some languages and false in others. For instance, if Schmenglish is a language in which 'Snow is white' means that grass is orange, then 'Snow is white' is *true-in-English* but *false-in-Schmenglish*. But for sentence *tokens*, we can define a notion of truth *simpliciter*. To state the view very crudely, we can say that a sentence token is *true simpliciter* if and only if it is true in the *intended* language (or rather, iff it's a token of a type t that is true in the intended language). This is obviously very rough, but for the present purposes, it is good enough; my claim is simply that this conception of truth *simpliciter* captures the basic idea behind one standard realist way of thinking about truth.

Now, if this view of truth *simpliciter* is at least roughly right, then the notion of truth in the standard model *is* a notion of truth *simpliciter*. For as we've seen, to say that a model is standard is just to say that it's *intended*. But to say that a sentence is true in the intended model is essentially equivalent to saying that it's true in the intended *interpretation*, and so it is also more or less equivalent to saying that it's true in the intended *language*. But if the notion of truth in the standard model, or intended structure, is a notion of truth *simpliciter*, then so is the notion of truth in the intended part (or parts) of the mathematical realm.[21] Thus, I think it is fair to say that the view of correctness inherent in (A)–(C) dovetails perfectly with the above conception of truth *simpliciter*, and hence also that it dovetails with at least one standard realist way of thinking about truth.

It should now be clear how FBP-ists can respond to a worry that the reader might have had about FBP-ists endorsing (A)–(C). One might have objected as follows:

> *Prima facie*, it seems that (A)–(C) is incompatible with FBP. For (a) FBP entails that all consistent purely mathematical theories truly describe some collection of mathematical objects, and so it would seem that, on this view, all such theories are *true*; but (b) it is not the case that all such theories are true in a standard model, or an intended part of the mathematical realm.

But FBP-ists can respond to this worry by maintaining that not all theories that truly describe some collection of mathematical objects are true. And they can do this very easily, for it fits perfectly with the above view of truth *simpliciter*. The fact that every consistent purely mathematical theory truly describes some collection of mathematical objects shows that every such theory is true in some language L. (Actually, FBP entails more than this; as we saw above, it entails that every such theory is true in a language that interprets the given theory in a "natural way", that is, a way that takes the given theory to be about the objects that, intuitively, it *is* about.) But none of this shows that all consistent purely mathematical theories are true *simpliciter*. Indeed, insofar as many of these theories have never been tokened, the notion of truth *simpliciter* does not even make sense in connection with them. For on the present view, the notion of truth *simpliciter* is defined only for sentence tokens (and collections of sentence tokens) and not for sentence types.

Now, it is important to note here that, according to FBP, there is nothing *metaphysically* special about the mathematical theories that happen to be true *simpliciter*. Take any consistent purely mathematical theory T. If mathematicians became interested in the structure (or structures) that T describes and formulated T in an effort to describe that structure (or those structures), then according to FBP, T would be true *simpliciter*, i.e., correct. But it doesn't follow from this that T is correct, or true *simpliciter*, right now. Thus, whether a mathematical theory is correct, or true *simpliciter*, depends partially upon facts about us. But this is just what we *want*. For in general, whether our utterances are true, or correct, depends partially upon our intentions, upon what we intend these utterances to mean. For instance, part of the reason that our utterances of 'Snow is white' are true is that when we utter this sentence, we intend to be saying that snow is white, as opposed to, say, that grass is orange.

So far so good. But I still need to explain how all of this suggests that FBP-ists ought to endorse IBPO. The main point that needs to be made here is that, given FBP, (A), (B), and (C) lead directly to (COR), (INCOR), and (NEUT). We saw above that according to FBP, (A) is equivalent to (A′). But any FBP-ist who accepted (A′) would also accept (COR). Let me illustrate this point with the case of arithmetic. (A′) tells us that an arithmetical sentence S is objectively correct if and only if (a) S is true in every part of the mathematical realm in which FCNN is true and (b) there is at least one such part of the mathematical realm. And (COR) tells us that S is true if and only if it is built into, or follows from, FCNN. But first of all, if S is true in every part of the mathematical realm in which FCNN is true (and if, as FBP asserts, all the parts of the mathematical realm that possibly could exist actually do exist, and if, as FBP asserts, mathematical truth has to do with truth in the mathematical realm), then if FCNN is true then S must also be true, i.e., it couldn't be that FCNN is true and S is false, and so S follows from FCNN. And second of all, if S follows from FCNN, then it will be true in every part of the mathematical realm in which FCNN is true (and if FBP is true, then there will be at least one such part of the mathematical realm, so long as FCNN is consistent, which we are assuming here). Thus, it seems that, given FBP, (A) leads directly to (COR). Likewise, there are analogous FBP-ist routes from (B) and (C) to (INCOR) and (NEUT), respectively. Thus, it seems that FBP-ists of the sort that I am imagining here endorse IBPO.

Now, it is important to note that unlike fictionalists, FBP-ists do not think that (COR) provides a *definition* of 'correct'. Rather, they think that 'correct' is defined by (A) and that (COR) tells us something about what the phenomenon of mathematical correctness ultimately depends upon. The reason this is important is that it enables FBP-ists to hang onto the result argued for above that their notion of correctness is a standard, realist notion of truth *simpliciter*. Now, *prima facie*, it might seem surprising that FBP-ists could endorse (COR) and still maintain that they have a realist view of correctness; for (COR) tells us that mathematical correctness depends upon facts about *us*, and so it might seem like a subjectivist view, or an anti-realist view. But we've already seen that it is perfectly consistent with realism to maintain that correctness depends partly upon facts about us; again, most realists would say that our tokens of 'snow is white' are correct, or true, partly because we intend them to mean that snow is white.

At this point, one might object as follows:

> According to realism, truth *simpliciter* depends *partly* upon facts about us. But (COR) seems to say that mathematical truth, or correctness, depends *solely* on facts about us. It tells us, for instance, that a set-theoretic sentence is correct if and only if it is built into FCUS. But this is not analogous to what standard realists say about sentences like 'snow is white'. They think this sentence is true partly because of our intentions and partly because real snow is really white. And moreover, they think there are sentences about snow that are true but not built into the very concept of snow – e.g., 'snow is something in which many children like to play'.

This objection is based on a mistake. FBP-ists simply don't say that mathematical correctness depends solely on facts about us. They maintain, in standard realist fashion, that our mathematical utterances are true partly because of what we intend them to mean and partly because there really do exist objects that are accurately described by these utterances. What the author of the above objection fails to appreciate is that FBP-ists think that (COR) is true *only because FBP is true*. They think that because the mathematical realm is plenitudinous, there will always be objects corresponding to our mathematical intentions (as long as these intentions are consistent), and because of this, facts about mathematical correctness turn on facts about our intentions. Thus, FBP-ists think that there is a sort of *confluence* of mathematical intentions and mathematical accuracy, but only because FBP is true. They do not think that mathematical accuracy (or correctness or truth) is *defined in terms of* our intentions. Put differently, they think that (COR) is true, but they do not think it provides a definition of correctness.

One more point that's worth making here is that the FBP-ist's claim that there may be some mathematical sentences that are neither correct nor incorrect is also perfectly consistent with a standard realist view of correctness. Suppose that a small child starts using the word 'macky' to refer to the fish that live in his family's tank and that, somehow or other, this usage spreads, so that everyone in the child's linguistic community uses 'macky' to refer to various different kinds of fish. Now suppose that a slightly older child asks his teacher whether any mackies are mammals. The teacher (if she is a wise realist) might say something like this: "I don't know. It depends on whether 'macky' means *fish* or *animal that lives in water*, because fish aren't mammals, but whales are." Finally, suppose that the teacher – being a folk semanticist in addition to a wise realist – asks all the members of the linguistic community whether 'macky' means *fish* or *animal that lives in water*, and all of these people respond by saying something like, "I don't know; I never thought about it." It seems to me that standard realists ought to say that in this case, the sentence 'Some mackies are mammals' is neither correct nor incorrect because the community's concept of a macky is too weak to determine whether it's correct, and nothing else could determine whether it's correct because the word 'macky' is *their* word. FBP-ists think that mathematical correctness works in the same way: if FCUS isn't strong enough to pick out a unique hierarchy of sets (up to isomorphism), then *nothing* could do this, because 'set' is *our* word and so its extension depends upon our intentions. (And note, too, that 'Some mackies are mammals' could *become* correct, on the standard realist view, because the meaning of 'macky' could evolve. And, again, this is analogous to the IBPO-ist view of mathematical correctness.)

6. *The Case for IBPO*

So far, I have explained how realists and anti-realists might be led to endorse IBPO. In the remaining two sections of the paper, I will argue that they *should* endorse IBPO, that IBPO-realism and IBPO-anti-realism are superior to non-IBPO-ist versions of realism and anti-realism. Now, I think there are several reasons for thinking that IBPO-FBP and IBPO-fictionalism are the best versions of realism and anti-realism, respectively, and I have given arguments for this elsewhere.[22] But what I want to argue here is that IBPO-realism and IBPO-anti-realism are superior to non-IBPO-ist versions of realism and anti-realism, *because IBPO is superior to non-IBPO-ist views of mathematical correctness.* I will begin in this section by arguing that IBPO is superior to the views of mathematical correctness inherent in the two traditional views discussed in section 1, i.e., traditional realism–objectivism and traditional anti-realism–anti-objectivism. (Of course, I already argued against traditional anti-objectivism in section 2; thus, in connection with that view, the arguments I give here will just add to what we have already seen.) Then in section 7, I will say something about non-traditional alternatives to IBPO. (Actually, we'll see that there is only one such view in the literature; part of what I'll be doing in section 7 is explaining why this is so.)

In any event, I now turn to my argument. I will list several facts about mathematical practice and argue that (a) IBPO fits perfectly with these facts and accounts for them very well and (b) the non-IBPO-ist views of correctness inherent in traditional realism–objectivism and traditional anti-realism–anti-objectivism cannot account for these facts. (I should note that I do not have the space to argue that the points that I list about mathematical practice here *really are facts*; now, I think that all of these points are more or less uncontroversial, but this is an empirical claim, and I do not motivate it here, so this has to be acknowledged as an assumption of the paper.)

1. *Intuitiveness as a Sign of Correctness*: When mathematicians are presented with an open question, they usually try to answer it by proving that some particular answer to the question follows from already-accepted axioms and theorems. But what happens when it is shown that no specific answer to the question can be proven from our current theories? Well, ideally, what mathematicians like to do in such cases is find new axioms that (a) are intuitively obvious and (b) entail answers to the given open questions. If an axiom is intuitively obvious to everyone, mathematicians take this as a sign that it is correct. Why? How can we account for this? Well, the first point to note is that IBPO accounts for it very easily: IBPO-realists and IBPO-anti-realists would both say that intuitiveness is a sign of correctness because correctness is determined by our intentions and our intentions are determined by the notions, conceptions, intuitions, and so on that we have in the given branch of mathematics.[23] Very surprisingly, however, traditional realism–objectivism cannot account for this basic fact about mathematical practice. That is, it cannot account for why mathematicians take intuitiveness to be a sign of correctness. If there were a single set-theoretic hierarchy in the mathematical realm and our set theories were attempts to describe that hierarchy, why would intuitiveness be a sign of correctness? Just because a sentence seems obvious to *us*, it doesn't mean that it is true of some mathematical structure that exists outside of spacetime. But if the mathematical realm is plenitudinous and the structures that we are talking about in mathematics are determined by our intentions in the way that IBPO-realism suggests, then intuitiveness *will* be a sign of correctness, for our intuitions are wrapped up with our intentions in a single sociopsychological package.

Traditional anti-realism–anti-objectivism is even more incapable of explaining the fact that mathematicians take intuitiveness to be a sign of correctness. Indeed, as we have already seen, this view is incompatible with the bare fact that mathematicians take some of their sentences to be objectively correct and others to be objectively incorrect.

Finally, the problem here seems to extend beyond just the two traditional views. It seems that any non-IBPO-ist view will be in trouble here, because it's hard to see how else we could account for the fact that intuitiveness is a sign of correctness in mathematics except by embracing the IBPO-ist claim that mathematical correctness is determined by our intuitions, intentions, and so on.

2. *Non-Intuitive Axioms*: What happens when mathematicians cannot find an axiom that is intuitive and that settles the open question they are trying to answer? Well, what we find is that different mathematicians have different reactions in situations like this. But it sometimes happens that various mathematicians start proposing axioms that are *not* intuitive. (Of course, they don't propose axioms that are *counter*intuitive, but they do propose axioms about which we have no real intuition either way.) This has happened in the case of CH; set theorists have come up with all sorts of non-intuitive axioms – e.g., axioms of determinacy – in an attempt to answer this open question. But what usually happens when an axiom of this sort is proposed is that other set theorists complain about it. They doubt that it's really correct. Moreover, arguments over axiom candidates often end up centering on *our notion of set*. When an axiom candidate is being discussed, everyone agrees that it's true in some hierarchies, but there is disagreement as to whether it's true in the *right* hierarchy (or hierarchies) – i.e., the ones that correspond to our notion of set. Again, IBPO accounts for all of this beautifully, but as far as I can see, no other view does. Indeed, it seems to me that any view that did account for this would *ipso facto* be an IBPO-ist view, for what's going on here is that arguments over the acceptability of axiom candidates are turning on whether or not these sentences are built into our notion of set.

3. *When Mathematicians Start Doubting That There's a Correct Answer*: Another reaction that some mathematicians have to open questions like the CH one is to start doubting that there is really an objectively correct answer to be discovered. We have already seen that IBPO can account

for this. For if FCUS isn't strong enough to settle the CH question, then according to IBPO, there simply isn't any correct answer to that question. But again, it's not clear that there is any other plausible view that can account for this. Traditional realist–objectivist views clearly cannot account for it, because these views entail that all mathematical questions have correct answers. What might be more surprising, however, is that traditional anti-objectivist views can't account for it either. If traditional anti-objectivism were true, then it would be entirely *trivial* that the CH question doesn't have a correct answer; but this just doesn't fit with the facts about the way mathematicians have treated the CH question. Of course, there are a few mathematicians who think this is trivial, but that is because they are anti-realist–anti-objectivists and their views here are determined by their philosophical commitments. Among set theorists who do not have any strong philosophical views here, i.e., who are interested only in the mathematics, the claim that CH is neither correct nor incorrect is controversial. It may well be the dominant view among set theorists today, but the bulk of those who accept it do not do so because it is a trivial result of anti-realism. That this is so is clear from the fact that these same mathematicians think that other undecidable sentences are correct. The fact of the matter is that these mathematicians have *mathematical* reasons for doubting that the CH question has a correct answer, and this is what traditional anti-realism–anti-objectivism cannot account for. The mathematical reason can be put in a few different ways. One way to put it is as follows: people doubt that there's a unique intended structure in set theory, i.e., they suspect that there might be multiple structures that fit with all of our set-theoretic intentions, and they think that it may be that in some of these structures, CH is true, whereas in others, CH is false. But IBPO accounts for this beautifully, for it entails that if this is indeed the case, then there won't be any fact of the matter as to whether CH is true or false, because CH won't follow from, or be inconsistent with, FCUS.

It is also worth noting the flip side of this point: if mathematicians were convinced that there was a unique intended structure for set theory (up to isomorphism), then they wouldn't doubt that the CH question had a unique correct answer. Again, IBPO accounts for this beautifully, but it is hard to see how non-IBPO-ist views could account for it.

4. *Different Attitudes Toward Different Branches of Mathematics*: Mathematicians are almost universally convinced that objectivism is true in arithmetic, i.e., that all arithmetical sentences are either correct or incorrect. But many of these same mathematicians – indeed, I think we can say most – harbor doubts about whether this is the case in set theory. What accounts for this? How could objectivism be true in arithmetic and false in set theory? Well, the first point to note here is that IBPO accounts

for this very easily. According to IBPO, objectivism is true in arithmetic if and only if FCNN is strong enough to settle all arithmetical questions (i.e., iff every arithmetical sentence either follows from, or is inconsistent with, FCNN). Likewise, objectivism is true in set theory if and only if FCUS is strong enough to settle all set-theoretic questions. Thus, according to IBPO, it could very easily be that objectivism is true in arithmetic but false in set theory. And it's worth noting that this explanation of how objectivism could be true in arithmetic but false in set theory fits perfectly with what mathematicians would say about this. For what mathematicians would say here, I think, is that in arithmetic, it's obvious that there is a unique intended structure (up to isomorphism), but that in set theory, this isn't so obvious.

(By the way, it should be kept in mind that IBPO itself does not commit to the thesis that objectivism is true in arithmetic. It tells us that objectivism is true in arithmetic if and only if FCNN is strong enough to settle all arithmetical questions, but it is officially neutral as to whether or not this is actually the case. Now, I think it's pretty obvious that it *is* the case, because I think it's obvious that FCNN determines a unique structure up to isomorphism – again, this is why mathematicians are convinced of it – but this is not part of IBPO, and so I do not need to defend it here.[24])

In any event, the other point that needs to be made here is that traditional non-IBPO-ist views do not have any explanation at all of how objectivism could be true in arithmetic but false in set theory. According to traditional objectivism, we ought to be objectivists across the board, and according to traditional anti-objectivism, we ought to be anti-objectivists across the board. Thus, advocates of both of these views have to claim that mathematicians are simply confused in this connection.

5. *Revisionism*: This leads us naturally into the next point I want to make, namely, that IBPO enables us to avoid a sort of revisionism that traditional non-IBPO-ist views seem to lead to. According to both traditional realism–objectivism and traditional anti-realism–anti-objectivism, what mathematicians ought to say about open questions like the CH question is determined by metaphysical principles – namely, realism and anti-realism. Traditional realism *commits* us to the view that the CH question has an objectively correct answer, and traditional anti-realism commits us to the view that it does not. But this seems to me an undesirable result. It seems to me that a good philosophy of mathematics should not *dictate* answers to questions like "Does the CH question have an objectively correct answer?" It should leave questions like this open, so that mathematicians can answer them on their own. But IBPO does allow mathematicians to answer such questions on their own. Indeed, it even explains *why* a good philosophy of mathematics should allow mathematicians to answer

such questions on their own, for it follows from IBPO that questions of the form

(F) Does open mathematical question Q have an objectively correct answer?

are paradigmatically *mathematical* questions. This is because (a) according to IBPO, whether some mathematical question Q has an objectively correct answer depends on whether there is a set of axioms that entails an answer to Q and is built into the intentions (i.e., notions, conceptions, etc.) that we have in the given branch of mathematics; and (b) whether there is such a set of axioms is a paradigmatically mathematical question. (It is also worth noting that, historically, when (F)-type questions have been settled, it has always been for mathematical reasons and not for metaphysical reasons. But I cannot argue this point here.[25])

6. *Discovery*: I end with a point that relates only to realism and for which I cannot state the full argument but can only wave my hands at an argument. By adopting IBPO, or IBPO–FBP, realists enable themselves to account for certain epistemological facts about mathematicians that non-IBPO realists cannot account for. Suppose that a sentence S is undecidable in the best axiomatic theory T in a given branch of mathematics. Then one might wonder how realists could ever hope to explain how human beings could discover whether T+S or T+~S was correct.[26] By adopting IBPO, however, realists can easily explain this. If correctness is determined by our intentions, then there is no mystery as to how human beings could discover whether or not S was correct, because we clearly have epistemic access to our own intentions, because our intentions are determined by our notions, conceptions, intuitions, and so on. More specifically, IBPO-ists can maintain that mathematicians could discover that S was correct by coming up with a new axiom A that entailed S and seemed intuitive to us. The intuitiveness of A would show that it was built into our notions, etc. for the given branch of mathematics, and so according to IBPO, it would follow that A was correct and, hence, that S was too. And of course, this fits perfectly with mathematical practice: this is precisely how mathematicians try to settle open questions about undecidable propositions, and when we do succeed in finding sentences like A, mathematicians do accept them (and the sentences they prove from them).

The second half of the present argument is based on the claim that non-IBPO realists cannot account for the epistemological facts in question here, e.g., the fact that mathematicians often do discover that sentences like S are correct. But I do not have the space to argue this point here.[27]

7. *Non-traditional Alternatives to IBPO?*

So far, I have argued that IBPO is superior to the views of mathematical correctness inherent in traditional realism–objectivism and traditional anti-realism–anti-objectivism. I now need to consider the question of whether there are any plausible non-traditional alternatives to IBPO.

The first point to be made here is that (as far as I know) the philosophy-of-mathematics literature contains only one non–traditional anti-realist alternative to IBPO, and it doesn't contain *any* non-traditional realist alternatives to IBPO. (One might wonder *why* there are no realist alternatives to traditional objectivism and IBPO in the literature. The reason, I think, is that realists have been more or less happy with traditional objectivism. And the reason for this, I think, is that, (a) they have usually endorsed *sparse* versions of realism – i.e., they have taken the mathematical realm to be sparse rather than plenitudinous – and (b) sparse realism seems to lead inevitably to traditional objectivism.)

In any event, the non-traditional anti-realist alternative to IBPO that I have in mind is due to Hartry Field. His view is (roughly) that a mathematical sentence is *correct* just in case it is "a *consequence* of accepted axioms [in a] . . . sense of consequence that goes a bit beyond first-order consequence in including the logic of the quantifier 'only finitely many'".[28] This view salvages the objectivity of arithmetic but not set theory. More specifically, it entails that undecidable set-theoretic sentences are neither correct nor incorrect. But we have already seen that this is unacceptable, that undecidable set-theoretic sentences can be objectively correct. Suppose, for instance, that S is undecidable in ZF and that some mathematician discovered a new axiom candidate A that was intuitively obvious to everyone and entailed S (when combined with ZF). Then we would want to say – and mathematicians would say – that we had discovered that S was correct. But Field's view would not allow us to say this. Field allows that if we accepted A, then S would *become* correct, but that seems to get things wrong. If A was built into FCUS before anyone thought of it, then we would want to say – and again, mathematicians would say – that in discovering A (and the fact that ZF+A entails S), we *discovered* that S was correct and, indeed, that it was correct before we ever thought of A. So it seems to me that IBPO is superior to Field's view. (I should note here that I agree with Field that set-theoretic sentences can become correct, but on my view, this happens when FCUS evolves, not when we adopt a new axiom that was, unbeknownst to us, built into FCUS all along.)

Aside from Field's view, I do not know of any non-traditional, non-IBPO-ist view of correctness that realists or anti-realists might adopt.[29] Indeed, realists can't even adopt Field's view, because it's not really consistent with the spirit of realism. And this seems to provide another reason for favoring IBPO over Field's view and, indeed, for thinking IBPO very attractive. For the fact that FBP-ists and anti-realists are both led to IBPO, and from such different starting points, lends a good deal of support to IBPO. Fictionalists start out with the idea of defining 'correct' in terms of inclusion in the "story of mathematics", whereas FBP-ists start out with the idea of defining it in terms of truth in the intended structure. But they both end up with IBPO. And this, I think, adds quite a bit of motivation to IBPO, especially when coupled with the fact that IBPO fits so well with mathematical practice.

But in any event, my first point here is simply that there are no non-traditional, non-IBPO-ist views of mathematical correctness in the literature that are as appealing as IBPO. My second point is that right now, we have reason to think it very unlikely that someone could come up with such a view. For we seemed to find in section 6 that IBPO is precisely what we need in trying to account for various facts about mathematical practice; we seemed to find that any theory that accounted for these facts would *ipso facto* be an IBPO-ist theory. And I might add here that, as far as I can see, there don't seem to be any problems with IBPO that suggest that we ought to be looking for a better theory. And so at this point, I think it is rational to conclude that IBPO is right and that IBPO-realism and IBPO-anti-realism are superior to non-IBPO-ist versions of realism and anti-realism.

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NOTES

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[2] For the purposes of this paper, it won't matter what CH actually says, but for the curious, it says that the cardinal number of R (the set of real numbers) is the immediate successor of the cardinal number of N (the set of natural numbers).

[3] As I've defined realism and anti-realism here, they are not exhaustive, because they do not cover empiricist views that take mathematics to be about physical objects or psychologistic views that take mathematics to be about mental objects. But it is widely believed that all such views are untenable, and I think this is right (see my (1998), chapter 5, for arguments). In any event, I am going to ignore such views here; we can consider it a working assumption of this paper that they are untenable.

[4] See Putnam (1980), Field (1998a), and Maddy (1997).

[5] Actually, this isn't quite right. The problem is that when the notions, conceptions, intuitions, and so on that we have in a given branch of mathematics are *inconsistent*, (COR) – as it's stated here – entails that all sentences in the given branch of mathematics are

correct. But this problem can be solved by merely reformulating (COR) in something like the following way:

(COR′) A mathematical sentence S is *objectively correct* iff either (a) the notions, conceptions, intuitions, and so on that we have in the given branch of mathematics are consistent, and S is "built into", or follows from, these notions, conceptions, and so on; or else (b) these notions, conceptions, and so on are inconsistent, but S has "nothing to do with the contradiction in our thinking" and for all of the intuitively and theoretically attractive ways of eliminating the contradiction, S would turn out correct by clause (a), if we eliminated the contradiction in the given way.

Clause (b) here is obviously very rough and imprecise, and before I would claim to have a really "polished" version of (COR′), (b) would need to be cleaned up considerably. But I don't have the space to do this here; the real *heart* of (COR′) – and of IBPO – is clause (a), and that is all I will be able to discuss here. Thus, for the sake of brevity and rhetorical elegance, I'm simply going to work with (COR) instead of (COR′). That is, I will be making a simplifying assumption here to the effect that there are no contradictions in our mathematical thoughts. But no harm will come from this, for all the points that I make here in arguing for (COR) and IBPO could also be made in terms of (COR′), or a more polished version of (COR′), without making any substantive changes.

[6] For more on the intuitive notion of entailment, and the relations it bears to the various formal notions of entailment, see Kreisel (1967), Field (1991), and my (1998), chapter 3. To say just a few words about this here, it is built into our intuitive notion of entailment that (a) if a sentence P can be derived from a set S of sentences in a sound derivation system, then S intuitively entails P; and (b) if S intuitively entails P, then there is no structure in which S is true and P is false. Thus, in the first-order case, the intuitive notion of entailment is coextensive with the syntactic and semantic notions of entailment.

[7] Perhaps the easiest way to envision an unclear case, or a no-fact-of-the-matter case, is to think of a case in which different members of the mathematical community have differing conceptions. Now, there doesn't need to be perfect agreement here: if a set-theoretic sentence S was inherent in the way that everyone thought about sets except for one person P, then S would still be built into FCUS; in such a case, we would simply say that P was idiosyncratic. But there does need to be something like *wide-spread agreement* here: if S was inherent in the conception of sets of fifty percent of set theorists, then it would not be built into FCUS, because it would be too controversial. But insofar as there are no non-arbitrary definitions of terms like 'wide-spread agreement' and 'idiosyncratic', there may be some sentences for which there is no fact of the matter as to whether they are in FCUS – and hence, according to IBPO, no fact of the matter as to whether they are correct. But again, we'll see below that there's no problem with this; for we will see that the same sort of thing happens with ordinary empirical concepts and the ordinary intuitive notion of truth, or correctness.

[8] One might argue that these sentences are, in fact, *not* built into FCNN. I think this is wrong, but we will see below that it doesn't matter, that my view is consistent with the claim that they are not.

[9] As was the case with (COR), these formulations of (INCOR) and (NEUT) assume that our mathematical notions, conceptions, and so on are consistent. Eventually, we will have to drop this assumption, and at that point, we will have to formulate (INCOR) and (NEUT) in slightly more complicated ways. But again, I am ignoring this complication in this paper. See note 5.

[10] See Field (1989) and my (1998), chapters 5–7, for defenses of fictionalism.

[11] See Field (1989), pp. 2–3.

[12] I discuss and defend FBP in my (1992), (1995), and (1998). Zalta and Linsky defend an FBP-ist view, and try to argue that it's true, in their (1995). Also, while neither Resnik (1997) nor Shapiro (1997) explicitly goes into this, their views are most naturally read as structuralist versions of FBP. (Shapiro would readily accept this characterization; Resnik might resist certain formulations of the plenitude thesis, because he has worries about the use of modalities, but in the end, his view is more or less FBP-ist in nature.) FBP-ist views have also been discussed by Field (1998b), Maddy (1997), and Beall (1999). Finally, FBP has been attacked by Cheyne (1999), and my defense of the view has been attacked by Azzouni (2000).

[13] One might object that this argument will not go through unless the term 'possible' that appears in the definition of FBP is at least as broad as the term 'consistent' that I'm using here. But it is. Indeed, these two terms are equally broad, because I am using them as synonyms. They are both being used to express a primitive, intuitive notion of possibility that corresponds to the intuitive notion of entailment discussed in note 6.

[14] On the present usage, a model is *standard* just in case it's intended. Thus, standard models aren't metaphysically special in any way; they're only psychologically, or perhaps sociologically, special. I will say more about this below.

[15] One might try to avoid this argument by allowing models to have proper classes as their domains; but then the problem will arise for the theory consisting of all the truths about sets and proper classes. Field makes a point similar to this in his (1991), pp. 3–4.

[16] This is a bit simplified; see notes 20 and 5.

[17] Again, we needn't assume here that *every* intention that *anyone* has in this context be included in "the intentions that we have". See the remarks on FCUS in note 7.

[18] I say that this definition of 'intended' is "somewhat rough" because – like the above formulation of (A) – it is a bit simplified; in particular, it assumes that our FCs are consistent. See notes 20 and 5.

[19] Actually, we don't need clause (b) here, because it entails (a).

[20] The formulations given here of (A)–(C), as well as (A′), are a bit simplified. In the end, they will need to be modified in a couple of ways. First of all, for technical reasons, we will want to reword (A), (A′), and (B) so that they are about atomic mathematical sentences only and then let the correctness or incorrectness of molecular sentences be determined in the usual Tarskian way from the correctness or incorrectness of their atomic parts. Second, as was the case with (COR), (INCOR), and (NEUT), the formulations of (A)–(C) and (A′) given here assume that our mathematical intentions are consistent. Eventually, we will have to drop this assumption, and at that point, we will have to formulate these theses in slightly more complicated ways. (Actually, it may turn out that the best way to do this is to leave (A)–(C) alone and to alter the definition of 'intended' given above; but (A′) would still need to be altered.) In any event, I'm not going to address any of this here, because I'm assuming in this paper that our FCs are consistent. See note 5.

[21] One might think that the fact that there might not be a *uniquely* intended part of the mathematical realm will create problems for the view of correctness inherent in (A)–(C). I haven't the space here to respond adequately to this worry, but I should note that elsewhere (1998, chapter 4) I have argued that there is no problem here, that even if it turns out that our mathematical theories are not about unique parts of the mathematical realm, this is not a problem for mathematical realism or, in particular, for FBP. Now, I might add here that there *are* ways for FBP-ists to argue that our mathematical theories are, in fact, always about unique parts of the mathematical realm. E.g., they could maintain that mathematical objects can be "incomplete" and, in particular, that

(*) The intended objects of our mathematical theories have no properties aside from those that they need to have in order to satisfy our FCs.

But I don't think FBP-ists should commit to (*), because (among other reasons) it involves a highly controversial empirical claim about our intentions. And I would say the same thing about any attempt to salvage uniqueness here, because it seems to me very controversial to claim that it's built into our intentions that our mathematical theories are about unique parts of the mathematical realm. I don't think FBP-ists should commit to this for the simple reason that they don't *need* to – because even if our FCs don't pick out unique parts of the mathematical realm, this is not a problem for FBP. But in any event, none of this is relevant to the present paper; for whatever FBP-ists say about all of this, they will still be led to IBPO (or for FBP-ists who seek to salvage uniqueness, a slightly rephrased but essentially equivalent view) in essentially the same way.

[22] See my (1998).

[23] To say that intuitiveness is a sign of correctness is *not* to say that our intuitions are infallible. Indeed, it's clear that they're not infallible, because they sometimes lead to contradiction. But IBPO is consistent with this. See note 5.

[24] Thus, I don't need to consider any arguments suggesting that FCNN doesn't pick out a unique structure up to isomorphism, e.g., the argument in Putnam (1980).

[25] Maddy argues this point; see her (1997), sections III.4–III.6.

[26] One might also wonder how realists could explain how human beings could know whether T+S structures or T+~S structures were *standard*. But this worry is essentially equivalent to the worry in the text, and I would respond to it in essentially the same way.

[27] In my (1998), chapters 2–3, I argue at length that FBP-ists can answer the epistemological worries about mathematical platonism whereas non-plenitudinous platonists cannot.

[28] Field (1998a), p. 391.

[29] There is another view in the literature that is available to both FBP-ists and anti-realists, but it is consistent with – indeed, dovetails with – IBPO. The view I have in mind is Maddy's mathematical naturalism (see her (1997), sections III.4–III.6), which seems to entail that a mathematical sentence is correct iff there are good mathematical reasons for endorsing it, i.e., iff it can be proven from a set of appropriate mathematical assumptions. I think this is right, but I also think it can't be all there is to say about mathematical correctness. There must be a deeper analysis of what correctness ultimately consists in that explains *why* Maddy is right (and I should note here that I think Maddy would probably agree with this – she doesn't claim to be providing a bottom-level analysis). In any event, it seems to me that IBPO fits in perfectly here, for it does explain why mathematical correctness is a mathematical issue in the way that Maddy suggests. I do not have the space to explicitly argue this point here, but a good deal of justification for it is already inherent in the arguments of this paper.

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