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A FICTIONALIST ACCOUNT OF THE INDISPENSABLE APPLICATIONS OF MATHEMATICS*

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The main task of this paper is to defend anti-platonism by providing an anti-platonist (in particular, a fictionalist) account of the indispensable applications of mathematics to empirical science. By 'platonism' I mean the view that (a) there exist abstract (i.e., aspatial, atemporal, non-physical, non-mental) mathematical objects, and (b) our mathematical theories describe such objects. Anti-platonism is the view that mathematical theory has to be interpreted in some other way, because there are no such things as abstract objects. I believe that *fictionalism* – the view that our mathematical theories are fictions, that sentences like '3 is prime' have the same status as sentences like 'Oliver Twist lived in London'[1] – is the best version of anti-platonism; thus, I will concentrate on it and ignore other versions of anti-platonism, but nothing important depends upon this choice, because my account of applicability can be adopted by other sorts of anti-platonists, e.g., deductivists.

The most important argument against fictionalism – the *Quine/Putnam indispensability argument*[2] – is that fictionalists cannot account for the fact that mathematics is applicable (and perhaps indispensable) to empirical science; the idea is that if mathematics were really fictitious, we wouldn't expect it to be any more applicable to physics than *Oliver Twist* is. Field has tried to respond to this by arguing that (a) there *are* no indispensable applications of mathematics to empirical science, and (b) the *dispensable* applications of mathematics can be accounted for. The hard part is establishing (a); to do this, Field has to show how physics can be *nominalized*, i.e., restated so that it doesn't refer to, or quantify over, mathematical objects. Unfortunately, there are numerous problems with Field's proposed method of nominalization. I am not convinced that these problems are insurmountable,[3] but for the purposes of this paper, I will assume that they are. Thus, what I want to do in this paper is provide a fictionalist response to the Quine/Putnam argument by (a) assuming that there *do* exist indispensable applications of mathematics to physics, and (b) *accounting* for these applications in fictionalist terms. (To make the discussion concrete, I will assume that quantum mechanics – or QM – makes indispensable use of mathematical objects, e.g., Hilbert spaces, and I will concentrate on that example.)

1. PROBLEMS FOR PLATONISM

Before I provide my account of applicability, I want to address the question of whether *platonists* can adequately account for the fact that mathematics is applicable to physics. The traditional wisdom is that they can, that the Quine/Putnam argument raises a problem for anti-platonism *only*, i.e., that it is an argument *for* platonism.[4] In this section, I will argue that the traditional wisdom is confused here; in particular, I will argue that there is as much (or almost as much) reason to doubt that platonists can account for applicability as there is to doubt that anti-platonists can.[5]

Many platonists write as if they can explain the applicability of our mathematical theories by merely pointing out that, on their view, these theories are *true*. But this is wrong. To account for applicability, we have to account for *relevance*, and a mere appeal to truth doesn't do this. If I have a theory of Mars which makes indispensable use of facts about Charles Manson, I cannot account for this by merely pointing out that all my claims about Manson are true. I have to say what Manson has to do with Mars. Likewise, platonists have to say what mathematical objects have to do with the physical world; that is, they have to account for the *relevance* of mathematical theory to physical theory. But there is a *prima facie* reason for thinking that it will be very difficult for platonists to do this; for since platonists maintain that mathematical objects exist outside of spacetime, they endorse what we might call the *principle of causal isolation*, or PCI, which says that *there are no causal interactions between mathematical and physical objects*; but this gives rise to the following question: if there are no mathematical *facts* which are causally relevant to physical *facts*, why is mathematical *theory* relevant to physical *theory*?

It is worth noting here that PCI is behind not just the *platonist's* problem of applicability, but the *anti-platonist's* problem of applicability as well. This can be appreciated by noting that (a) anti-platonists who *endorse* PCI – viz., those (e.g., fictionalists and deductivists) who deny that there exist any mathematical objects[6] – *encounter* the worry about the relevance of mathematical theory expressed at the end of the last paragraph, whereas (b) anti-platonists who *reject* PCI – viz., those (e.g., Mill) who think that there *are* mathematical objects but maintain that these objects are ordinary physical objects – *do not encounter* this worry (although they *may* encounter some *other* problem having to do with applicability). Moreover, the same goes for *platonists* here: those who take mathematical objects to exist *within* spacetime, e.g., Maddy,[7] will *reject* PCI and, for this reason, will *not* encounter the worry of the last paragraph (although, again, they may encounter some other problem with respect to applicability). The upshot of all of this is that the Quine/Putnam argument should not be construed as an argument for platonism or for the truth of mathematics; *it should, rather, be construed as an argument against PCI* – it is best understood as a challenge to people who deny that mathematics is about a collection of causally efficacious objects to account for the relevance of mathematics to physics.[8] And I should note that it is very important that we find a response to this challenge to PCI, for there are very good reasons to believe that PCI is *true*, i.e., that no anti-PCI philosophy of mathematics (whether it is platonistic or anti-platonistic) can work.[9]

So let us turn to the question of whether platonists can *answer* this challenge, i.e., whether they can account for the applicability of mathematics. It seems to me that the acceptance of PCI already rules out one strategy that platonists might be inclined to attempt, namely, the strategy of claiming that the reason mathematics is relevant to physics is that many of the facts with which physics is concerned are not purely physical facts, but rather, *mathematico-physical* facts. (To give a bit more detail, one might claim that when we say, for instance, that the physical system S is forty degrees Celsius, we are expressing a mathematico-physical fact, namely, the fact that S stands in a certain relation (viz., the Celsius relation) to the number 40. The problem with this account of applicability is that while this relational fact – let's calle it C(S,40) – *is* a mixed fact, i.e., a *mathematico-physical* fact, PCI entails that it is *not* a *bottom-level* mixed fact, i.e., that it supervenes on a purely physical fact about S and a purely mathematical fact about 40.[10] Thus, it seems that this account of applicability just moves the problem back a step; the challenge now becomes that of explaining what C(S,40) has to do with the purely physical fact of S's temperature state.)

But while this account of applicability fails, there is *another* account which platonists might offer here, one which is, I think, a bit more subtle. The account I have in mind proceeds as follows. "We *admit* that facts like C(S,40) are not bottom-level facts; the reason the real number line is useful in making temperature ascriptions is not that numbers are causally relevant to the temperatures of physical systems, but that they provide a convenient way of *representing*, or *expressing*, purely physical temperature facts. To give a bit more detail, what we do here is define a function Φ which maps physical objects into real numbers, so that for any physical object x and any real number r, if Φ(x) = r, then x is r degrees Celsius. Thus, what's going on here is that we use the numbers to *represent* (or to *name*) the purely physical temperature states. The reason this is convenient is that (a) the various temperature states are related to one another in a way that is analogous to the way in which the real numbers are related to one another,[11] and (b) it is simply *easier* to say that Φ(x) = 40 – or more colloquially, that x is forty degrees Celsius – than it is to describe (in purely physical terms) the purely physical bottom-level fact of x's temperature."

This account of applicability – let's call it the *representational account* – seems to be exactly what PCI-platonists need: it explains how talk of mathematical objects is relevant to describing the physical world without violating PCI. (PCI is not violated because the only relations between physical and mathematical objects that the representational account alludes to – e.g., the relation corresponding to the fact that Φ(x) = 40 – are *non-causal* relations.) But there is a problem with the representational account: it might not be able to explain *all* applications of mathematics to physics. The easiest way to appreciate this is to notice that any application of mathematics which can be explained by the representational account can be dispensed with. Thus, insofar as we're not sure that all uses of mathematics

in physics are dispensable, we're not sure that the representational account can explain all uses of mathematics.

But why should we believe that any application of mathematics that can be explained by the representational account can be dispensed with? Well, whenever the representational account can be used to explain a given application of mathematics, we will be able to define a function Φ, of the sort discussed above, from the physical objects that the assertions in question (i.e., the assertions of the physical theory which refer to mathematical objects) can be taken to be *about* to the mathematical objects of the mathematical theory being applied. But to define such a Φ is just to prove a *representation theorem* for the given use of mathematics; that is, the Φ here is going to be exactly the sort of function constructed by a representation theorem: it is going to be an appropriate sort of homomorphism mapping an appropriate sort of empirical structure (which the assertions in question can be taken to be about) into the mathematical structure that we're applying. But once we've proved a representation theorem, it is not hard to nominalize the assertions in question. All we have to do is restate these assertions solely in terms of the empirical structure that we defined along with our homomorphism Φ. The idea is to give the temperatures of physical objects by specifying relations that they bear not to *numbers*, but to *other physical objects*, viz., objects which are slightly warmer and slightly cooler. We will already have the nominalistic predicates – e.g., 'cooler than' – that will enable us to do this, for they will have been defined in the process of defining the empirical structure.[12]

Now, it *might* be that platonists can use the representational account of applicability, because it might be that all applications of mathematics to physics can be dispensed with. But (a) if this *is* the case, then platonists cannot maintain that fictionalists have a problem with respect to indispensability, because in this case, Field's response to the problem will succeed, and (b) I am assuming in this paper that some applications of mathematics to physics *cannot* be dispensed with. Thus, we must also assume that some applications cannot be explained by the representational account; thus, platonists need to find another explanation of these indispensable applications.

What I want to do at this point is return to the fictionalist's point of view. I will try to meet the Quine/Putnam challenge to fictionalism by assuming that some applications of mathematics are indispensable to physics and accounting for these applications in fictionalist terms. Then at the end of the paper, I will argue (very briefly) that PCI-platonists can account for indispensable applications in the same sort of way.

2. WHAT EXACTLY NEEDS TO BE ACCOUNTED FOR?

I want to guard against a possible confusion. One might think that fictionalists have to account for the very fact of indispensability. But this is wrong; all that's needed is an account of applicability. The argument against fictionalism is that it leaves mysterious the fact that mathematical theory is *relevant* to physical theory. To eliminate this mystery, it would be sufficient to account for the mere applicability of mathematics. It is not required that fictionalists provide an account of indispensability. Now, this is *not* to say that indispensability is unimportant in this connection; on the contrary, it is crucially important: by pointing out that mathematics is indispensable to some of our physical theories, we block a certain sort of account of applicability, namely, the sort of account which Field has tried to give, i.e., the sort of account which relies upon the claim that physics can be nominalized. Thus, while indispensability is certainly relevant to the challenge facing fictionalists, it is not indispensability that they need to explain. The point is that fictionalists have to account for *all* applications of mathematics, *including* those which seem indispensable; but they *only* have to account for the *usefulness* of the mathematics in these cases; they do *not* have to account for its (seeming) indispensability.[13]

(This is not to say that indispensability *never* needs to be explained. My point is simply that our *prima facie* worry about fictionalism is a worry about mere applicability and *not* about indispensability. In other words, if we assume that everyone can account for the applicability of mathematics to empirical science – i.e., that they can account for *all* applications of mathematics, including those that seem indispensable – then there is absolutely no reason to think that fictionalists will have more difficulty than others in explaining the *further* fact that some uses of mathematics seem to be indispensable to empirical science.)

3. A FICTIONALIST ACCOUNT OF THE APPLICABILITY OF MATHEMATICS

In section 3.2, I will provide an account of the (dispensable and indispensable) applications of mathematics, and in section 3.3, I will argue that fictionalists can use this account to solve their problem of applicability. But before I do any of this, I want to briefly describe a strategy which fictionalists might try to employ here but which I think is misguided.

3.1. *A Misguided Strategy: Instrumentalism*

Fictionalist might try to solve their problem by adopting a *general instrumentalism*, i.e., by claiming that our empirical theories are not true, but merely empirically adequate. If fictionalists make this move, they will not have a problem with respect to applicability; for (a) on this line, empirical theories are every bit as fictitious as mathematical theories, and (b) it's entirely obvious how one fiction could be applicable to another. (All you have to do is make up the two fictions in the right way; thus, within a general instrumentalism, the fact that mathematical theories are applicable to physical theories is no more surprising than is the fact that "Rambo II" is applicable to "Rambo III".)

The problem with this view is that general instrumentalism is implausible. I will not attempt to justify this claim here; I merely note that I am assuming that fictionalists have to be realists about empirical science. Nonetheless, it is clear that they *cannot* be *standard* realists, because standard realists claim the QM is *true*, and one cannot make this claim without admitting that Hilbert spaces, real numbers, and all sorts of other mathematical objects exist. Thus, fictionalists have to find an alternative form of scientific realism, one which is consistent with their anti-realism about mathematics.[14] For purely rhetorical reasons, I will wait to formulate the version of scientific realism that I have in mind for fictionalists until after I provide my account of applicability.

3.2. *An Account of the Applicability of Mathematics*

I will not attempt to provide a *complete* account of applicability here; my intention is to (a) suggest a general way of viewing applicability, (b) provide some motivation for it, and (c) show that if this view is correct, then the worry about fictionalism (and PCI-platonism) evaporates.

We are led to an account of the applicability of mathematics by understanding the problem with the representational account of applicability discussed in section 1. The general intuition behind the representational account – viz., that mathematics is relevant to physics because (a) certain *non-causal* relations hold between mathematical and physical objects, and (b) we use these relations to help us *express*, or *represent*, purely physical facts – is essentially correct. The representational account only goes wrong when it tries to pin down exactly how this works. It accurately describes how it works for some applications, but not all. (In particular, it cannot handle applications which are indispensable.) Thus, what I want to do is give an account of applicability which is more general than the representational account and then claim that there are numerous ways in which this general account can be realized, i.e., that the representational account has located just *one* of these ways.

My general position on the applicability of mathematics to empirical science can be summed up in the following claim.

(APP) All mathematics ever does in empirical science is provide theoretical apparatuses (or, in other words, conceptual frameworks) in which to make assertions about the physical world.

Another way of saying this is that mathematics is relevant not to the *operation* of the physical world, but only to *our understanding* of the physical world. Physical theories *never* make claims of the form: 'physical phenomenon X occurs *because* the mathematical realm has nature Y'; rather, they make claims of the form: 'the behavior (or state) of physical system S can be understood in terms of the mathematical structure M by . . .'

Now, we need to emphasize that (APP) is true of indispensable as well as dispensable applications. Hilbert spaces are no more causally relevant to quantum phenomena than real numbers are to the temperatures of physical objects. Just as the real number line merely provides us with a convenient way of stating temperature facts about physical objects, so Hilbert spaces merely provide us with a convenient way of stating facts about quantum phenomena. The fact that

in the one case, we *also* know how to describe these facts *without* referring to mathematical entities, while in the other case we do not, is wholly irrelevant. In short, the point is that from a broad perspective, dispensable and indispensable uses of mathematics play the exact same role in empirical science; the only differences lie in the details, i.e., in the *manner* in which they enable us to describe physical phenomena.

One might object here as follows. "You have to say what these details *are*. That is, you have to say exactly *how* Hilbert spaces enable us to describe quantum phenomena. We know how real numbers enable us to describe the temperatures of physical objects, and so (APP) is acceptable there; but insofar as we don't know how Hilbert spaces enable us to describe quantum facts, we don't know if (APP) is acceptable here."

My response to this is that we *do* know how Hilbert spaces enable us to describe quantum phenomena. We don't need to nominalize QM, or set up homomorphisms from empirical structures to mathematical structures, in order to explain the role that Hilbert spaces play in QM. All we have to do is go through the theory and explain the various uses of Hilbert spaces and how they help us state facts about the quantum world. This would surely be a time-consuming thing to do; but it would *not* be difficult, because it would never *seem* that our view was wrong; that is, the fact that Hilbert spaces are merely theoretical devices, used only to help us say what we want to say about purely physical phenomena, would always be right there on the surface. All we would have to do is explain how it works.

Let me give an example. The main use of Hilbert spaces in QM is for *representation*. For example, we represent the possible pure states of quantum systems with vectors in Hilbert spaces, and we represent observable quantities of quantum systems (e.g., position and spin) with Hermitian operators defined on Hilbert spaces. But most important is the representation of *quantum events* (or equivalently, *propositions*) with closed subspaces of Hilbert spaces: if we let 'A' denote some observable, 'Δ' denote some Borel set of real numbers that can be values of A, and '(A, Δ)' denote the quantum event of a measurement of A yielding a value in Δ (or equivalently, the proposition which asserts that this event has occurred, or perhaps, will occur) then we can represent (A, Δ) with the closed subspace

L(A, Δ) of the Hilbert space H in which A is represented, where L(A, Δ) is defined as follows: a vector v of H is in L(A, Δ) iff there is a probability of 1 that a measurement of A, for a quantum system in the state represented by v, will yield a value in Δ.

To go a bit deeper into the example, let me explain the use of probabilities here. In classical mechanics, we can think of a state as a function from propositions of the above sort to truth values. QM, however, is a probabilistic theory: it does not (in general) predict with certainly how a quantum system in some given state will behave when we measure it. Thus, instead of thinking of quantum states as functions from propositions to truth values, we think of them as functions from propositions to *probabilities*, i.e., to [0,1]. Thus, a given quantum state Ψ will assign to each proposition (A, Δ) a real number r in [0,1]; r is the probability that the event (A, Δ) will occur if a state-Ψ system is measured for A (or that the corresponding proposition will be *true*). Thus, each quantum state determines a probability function from a lattice of quantum propositions to [0,1].

Finally, there is a rule for calculating the exact probability r that a particular state Ψ assigns to a particular proposition (A, Δ) from the vector v that's used to represent Ψ and the closed subspace L(A, Δ) that's used to represent (A, Δ). (The rule won't concern us, but for the curious, it tells us that r is equal to the inner product of v and the projection of v onto L(A, Δ).) Applying all of this to a concrete example, if we let 'z+' denote the spin-up-in-the-z-direction state, 'Sx' denote the spin-in-the-x-direction observable, and '+' denote the spin-up value, then (assuming that z is orthogonal to x) z+ determines a probability function p_{z+} such that $p_{z+}(Sx, +) = 0.5$.

Now, by merely pointing out that QM uses Hilbert spaces (or more aptly, vectors, operators, and subspaces) mainly for representation, we have *not* shown the way to nominalizing QM. For aside from the fact that QM refers to *other* kinds of mathematical objects (e.g., real numbers and probability functions) some of the things being represented here – e.g., quantum propositions – are *themselves* abstract objects.[15] But nevertheless, it is not hard to see that all the mathematical baggage here (and we can take this baggage to include abstract objects which wouldn't ordinarily be called 'mathematical objects,' e.g., propositions) is doing nothing but providing a convenient and precise way of describing purely nominalistic facts about the quantum world. For instance, one such fact – associated with the fact stated in platonistic terms at the end of the last paragraph, viz., the fact that $p_{z+}(Sx, +) = 0.5$ – is that if we take an ensemble of electrons which are spin-up in the z direction and measure them for spin in the x direction, then half of them will be spin-up. (Actually, it won't always be the case that *exactly* half are spin up, just as it's not always the case that *exactly* half the tosses of a fair coin are heads. But this problem can be solved by speaking in terms of relative frequencies or propensities; thus, for instance, we might say that the larger the ensemble of z+ electrons being measured for Sx, the closer we will get to the result that for every electron which is measured spin-up, there corresponds a unique electron which is measured spin-down, and vice-versa.)

QM expresses a bunch of facts like this, i.e., a bunch of facts about quantum phenomena which seem to be purely physical facts. Now, QM uses Hilbert spaces to express these facts in a theoretically attractive way, but there's nothing mysterious going on here. To explain *how* Hilbert spaces enable us to describe quantum phenomena would be to do nothing more than to run through the theory and do what I just did for this one fact (i.e., the fact that half of the z+ electrons will, if measured for Sx, be found to be spin-up); that is, it would be to run through the theory and state the purely physical facts being described and explain how the theory uses mathematical objects to describe these facts. This would be boring and time-consuming, but it would not be difficult, for it would all be rather like the above case. (All uses of Hilbert spaces in QM are rather similar; in a nutshell, what we do is use lattices of closed subspaces of Hilbert spaces as *event spaces*.)

One might object here as follows. "What you're talking about doing here – namely, explaining how QM uses Hilbert spaces to express purely physical facts about the quantum world – is going to involve expressing these facts in purely physicalist, i.e., nominalist, terms. But to express in nominalist terms the facts about the quantum world which QM expresses in platonist terms is just to *nominalize* QM. Thus, what you've got in mind *is* difficult to do. Indeed, we're presently assuming that it can't be done."

This objection is misguided. Let us say that to express in nominalist terms the physical facts that QM expresses in platonist terms is to state the *nominalistic content* of QM. This is *different* from nominalizing QM; among other differences, a nominalization of QM would have to be stated in a *theoretically attractive way*,[16] but a statement of the nominalistic content of QM would *not* have to be so stated. In claiming that it would be difficult to nominalize QM, I mean to claim that it would be difficult to *replace* QM with a nominalistic theory which we would be proud to present as *our theory of the quantum world*. What I'm saying would be *easy* to accomplish, on the other hand, is the project of running through QM and listing off – in an ugly, hodgepodge way, but also in purely nominalist terms – what the theory is saying about the quantum world. Now, of course, I am *not* going to do this here; I will simply assume that it *can* be done.

I should also note that even if it were *difficult* to state the nominalistic content of QM, we should not doubt that it can be done; that is, we should not doubt that QM *has* a nominalistic content. (I suppose that one might doubt that QM has a nominalistic content which captures its '*complete* picture of the physical world,' i.e., which isn't *missing* important features of what QM tells us about the physical world; and one might motivate this doubt by claiming that (a) part of what QM tells us about the physical world is that certain mathematico-physical facts obtain, whereas (b) the nominalistic content of QM won't do this. The problem with this argument is that the mathematico-physical claims of empirical science *do* have nominalistic contents which capture their *complete* pictures of the physical world. This can be seen as follows. Let 'R(p,m)' denote some mathematico-physical fact, where p is a physical object, m is a mathematical object, and R is a non-causal relation. Now, we saw in section 1 that facts like this are not *bottom-level mixed facts*, i.e., that they supervene on purely physical facts about the physical objects in question and purely mathematical facts about the mathematical objects in question. But this means that the sentence 'R(p,m) obtains' has a nominalistic content which captures its *complete* picture of the physical world; that nominalistic content says, in a nutshell, that p holds up *its end* of the 'R(p,m) bargain,' i.e., that p is such that it would stand in R to m if m existed (and if m had the properties ascribed to it by the relevant mathematical theory).)[17]

My point here can be better appreciated if we take a step back and recall what started this whole discussion. I began by stating a certain view of the applicability of mathematics, namely, the theoretical-apparatus view of (APP). I then raised an objection to this view, namely, that since we don't know *how* QM uses Hilbert spaces to describe purely physical facts, we can't be sure that it *does*. My response was that we *do* know how QM does this, that any decent presentation of QM *shows* how it does this. But what I'm now pointing out is that the objection was misguided from the start, because we cannot seriously doubt that QM does this; that is, QM *obviously* uses Hilbert spaces to describe purely physical quantum phenomena; this is, on the very *face* of it, just what the theory *does*.[18]

I conclude, then, that the objection has been met and that the way has been cleared to accepting (APP). What remains is to argue that fictionalists can use (APP) to solve their problem of applicability.

3.3. *Nominalistic Realism and the Fictionalist's Use of (APP)*

Before arguing that fictionalists can meet the Quine/Putnam challenge by appealing to (APP), I want to return to the topic of section 3.1 and formulate the version of scientific realism that I claim that fictionalists can adopt. Recall that the challenge to fictionalists is to adopt a version of scientific realism (i.e., to maintain that our empirical theories are not merely empirically adequate) without adopting a *full-blown* scientific realism (i.e., without admitting that our empirical theories are true). My claim is that fictionalists can do this by maintaining that *the nominalistic content of our empirical theories is true*. I will call this form of realism *nominalistic realism*.

It should be clear that nominalistic realism is a *genuine* form of scientific realism. For on this view, *everything* our empirical theories say about the physical world – including their claims about 'theoretical entities' like electrons – is true; that is, according to nominalistic realism, our empirical theories (e.g., QM) provide us with a *true* picture of the physical world. (That this is consistent with fictionalism – i.e., that nominalistic realists can maintain that everything our empirical theories say about the mathematical realm is *false* – will become clear below.) For now, I merely note that nominalistic realism (the belief in the truth of the nominalistic content of our empirical theories) lies precisely between full-blown scientific realism (the belief in the truth of these theories) and instrumentalism (the belief that these theories are merely empirically adequate).

Now, I suppose that one might object that nominalistic realism is not strong enough. That is, one might claim that fictionalists need to be full-blown scientific realists and, thus, either (a) accept QM as true and, hence, abandon fictionalism, or else (b) replace QM with another theory – presumably a nominalistic version of QM – which they take to be true. There is a sense in which my fictionalists *are* replacing QM with a true nominalistic theory; for the nominalistic content of QM is just such a theory. But this is not really an accurate way of putting things; for insofar as the nominalistic content of QM needn't be stated in a theoretically attractive way, my fictionalists are not going to want to replace QM with it. According to them, QM should still be accepted as our 'official theory of the quantum world'. This is perhaps the most important difference between my brand of fictionalism and Field's. On my view, *there is no need to replace QM*, because (a) the nominalistic content of QM is true, and (b) this is all we need to demand of our empirical theories. To the charge that fictionalists have to endorse a full-blown scientific realism, i.e., to the charge that fictionalists need there to be a (theoretically attractive) quantum theory which is *true*, I will simply ask, '*Why*?' I see why fictionalists cannot adopt a general instrumentalism, but I simply don't see the argument for the claim that they cannot adopt nominalistic realism.[19]

(I should point out here that there is no guarantee that there even *is* a true and attractive theory of the physical world. If mathematics is indispensable to the very project of doing physics (in a theoretically attractive way) and if there are no mathematical objects, then there is no true and attractive physics. In such a case, the best we could hope for would be true nominalistic content. Thus, even if mathematics is indispensable to the very project of doing physics, we cannot infer that platonism is true; we can only infer that either platonism is true or there is no true and attractive theory of the physical world. Now, of course, my fictionalists are not *committed* to there being no such theory, because they're not committed to indispensability; they can hope that, someday, we come up with an attractive nominalistic alternative to QM.)

Let me turn now to the main thesis of this section, i.e., that fictionalists can solve their problem of applicability by adopting the theoretical-apparatus view of applicability expressed in (APP). What I need to show is that a fictional story about a non-existent mathematical realm can do the work that mathematics does in empirical science; thus, taking (APP) for granted, I need to show that a mathematical fiction can provide a theoretical apparatus (or conceptual framework) for empirical science. Putting the point in terms of our example, I need to show that Hilbert space theory can do the work it's supposed to do in QM even if it is fictitious, i.e., that *QM's picture of the physical world can be true even if there are no such things as Hilbert spaces*. I have two arguments for this claim; one relies upon PCI, and the other upon (APP).

The first argument is this: QM's picture of the physical world can be accurate, even if there are no Hilbert spaces, because QM never entails that Hilbert spaces are causally relevant to the physical world. The argument can be stated more slowly as follows: if there exist any mathematical objects, then they are not causally relevant to the physical world (and QM doesn't entail otherwise); thus, the behavior of the physical world will be the same whether or not there exist mathematical objects (and QM doesn't entail otherwise); thus, QM's picture of the physical world will have the same degree of accuracy whether or not there exist mathematical objects; that is, QM's picture of the physical world can be true, even if there are no mathematical objects, e.g., Hilbert spaces.[20]

My second argument proceeds as follows. According to (APP), QM merely uses Hilbert spaces as a means of expressing certain facts about quantum phenomena. Thus, talk of Hilbert spaces (and vectors and subspaces) appears in QM as a sort of *heuristic device*. Thus, such talk will serve its purpose if it succeeds in lending to the understanding of quantum phenomena. But such talk can succeed in this task even if there are no Hilbert spaces; this is simply because, in *general*, talk intended as a heuristic device needn't be true (or genuinely referential) in order to be successful. Thus, the conclusion is that QM can succeed in providing an accurate picture of the physical world even if there are no Hilbert spaces.

That successful heuristic devices don't have to be true – i.e., that falsehoods can aid our understanding – can be appreciated by merely changing the example. The challenge to fictionalism is often presented as follows: if fictionalism were correct, then we shouldn't expect mathematics to be any more applicable to physics than *Oliver Twist* is. My response is that, in principle, it's *not*! A novel *can* provide a theoretical apparatus for a description of some part of physical reality. An historical description of the years surrounding the Russian revolution, for instance, could very easily use talk of the novel *Animal Farm* as a theoretical apparatus, or heuristic device. But the teacher who enlightens a pupil by claiming that Stalin was like the pig Napoleon does *not* commit to the *existence* of Napoleon; the talk of Napoleon serves as a mere heuristic device; that is, the *historical content* of the teacher's description can be true even if there is no Napoleon. And this, of course, is analogous to our case: the nominalistic content of QM can be true (i.e., its picture of the physical world can be accurate) even if there are no Hilbert spaces, because talk of Hilbert spaces appears in QM as a mere heuristic device.

Now, one might object that there is a disanalogy between the two cases, because QM makes *indispensable* use of mathematical entities, while our history teacher's appeal to Napoleon is dispensable. I have two responses to this. First, and most important, the objection is irrelevant. My claim is that it doesn't matter to QM's picture of the physical world whether any mathematical objects exist, i.e., that that picture could be accurate even if there are no mathematical objects; I have justified this by pointing out that mathematical objects play a merely *non-causal* and *heuristic* role in QM, i.e., that PCI and (APP) are true. I never claimed that QM's use of mathematics is dispensable; thus, the whole issue of indispensability is irrelevant here. A second response is that we can imagine a team of historians who lacked the wherewithal to describe the Russian revolution in anything but *Animal-Farm*-theoretic terms. We might think these historians exceedingly stupid, but regardless, we would like to say that they should not infer that Napoleon exists from their inability to do history without referring to him; for (a) they know that Napoleon is a fictional character, and (b) their picture of the Russian revolution could be accurate even if he didn't exist. Likewise, we should not infer that mathematical objects exist from our inability to do physics without referring to them; for our picture of the physical world could be accurate even if they don't exist.

My conclusion, then, is that fictionalists can solve their problem of applicability by adopting (APP) and nominalistic realism.

I would like to close this section by emphasizing that my account of applicability is entirely different from that of Terence Horgan and Geoffrey Hellman.[21] On their view, our empirical theories are *replaced* with counterfactual versions of these theories. I do not want to go into the various problems that this view encounters; I merely wish to emphasize that, on my view, we have no need to replace QM; we can accept it *as is* and still maintain mathematical fictionalism. The reason for this can be stated in terms of nominalistic realism or in terms of (APP) and PCI. In connection with the former, the reason is that when we 'accept' QM, we need only commit to the truth of its nominalistic content. In connection with the latter, the reason is that the fictional mathematical claims of QM are simply *not* a part of what's being said about the physical world; they are, rather, part of the apparatus which enables us to say what's being said. Moreover, the inclusion of the fiction does not *infect* what's being said about the physical world, because the fictional entities are not taken to be causally relevant to the physical world. This is why fictionalism and nominalistic realism are consistent with one another. And it is also why nominalistic realism is a legitimate form of scientific realism, whereas *macro-level realism* (i.e., the view that everything physics tells us about the macro-level is true) is *not* a legitimate form of realism: taking electron talk as fictitious would wreak havoc on our picture of the physical world, because electrons play a *causal role* in that picture.

Here's a metaphor which, although probably more vivid than accurate, may help: QM's nominalistic content is its picture of the physical world, whereas its mathematical content is the canvas on which this picture is painted. Thus, physicists should no more care if their mathematical apparatus is true than portrait artists should care if the canvases under their portraits resemble anything in the world.[22]

4. PCI-PLATONISM REVISITED AND SOME LOOSE ENDS

If the above remarks adequately solve the fictionalist's problem of applicability, then PCI-platonists can solve their problem of applicability in an analogous manner. Indeed, they will have an even easier time than fictionalists, for they can claim that theories like QM are true. But PCI-platonists *do* run into a problem that doesn't seem to arise for fictionalists. Even if (APP) is correct and mathematics is only useful in providing theoretical apparatuses in which to make claims about the physical world, one might find it surprising that it can be put to even *this* use. In other words, it's not obvious why theories about a causally inaccessible realm should be useful in providing a conceptual framework in which to do physics; thus, the fact that it *can* be put to such a use needs to be explained.

PCI-platonists can solve this problem by adopting a very special sort of platonism, which I will call *full-blooded platonism*, or FBP. FBP can be intuitively but sloppily expressed with the slogan, 'All possible mathematical objects exist'; more precisely (i.e., without the *de re* modality) the view is that there exist as many mathematical objects as there possibly could, or perhaps that there exist mathematical objects of all kinds. (I have argued elsewhere that FBP is the only tenable version of platonism; the reason is that it is the only one that avoids Benacerraf's epistemological objection.[23]) Now, given FBP, it should not be surprising that mathematics can be used for setting up a theoretical apparatus in which to do physics. The only reason it might *seem* surprising that mathematics could be used to this end is that this seems to suggest that there is an inexplicable *correlation* between the mathematical realm and the physical world. But within FBP, this illusion evaporates: the mathematical realm is so robust that it provides an apparatus for *all* situations; that is, no matter how the physical world worked, there would be a (true) mathematical theory which could be used to help us do physics.

We might try to generate a similar problem for fictionalists, but they can solve it in the same sort of way. The upshot of the platonist's appeal to FBP is that for any physical set-up, there is a way to use mathematics to do physics. What enables FBP-ists to make this claim is that, on their view, any consistent purely mathematical theory is, from a purely ontological point of view, as good as any other. But fictionalists are already committed to this last claim, because they think that all such theories are false; thus, they can solve the above problem just as FBP-ists do.

Now, one might try to press the above objection in the following way. "Even if we grant that (APP) is correct, and even if we grant that PCI-platonists and fictionalists have access to *all* consistent purely mathematical theories, there is still a problem. For what accounts for the fact that *any* of these theories are applicable to physics? In other words, what accounts for the fact that any mathematical theory can be used to provide a theoretical apparatus in which to make assertions about the physical world?"

The main thing to be noted here is that this is a problem for *every* philosophy of mathematics; more precisely, there's no reason to think that by adopting PCI-platonism or fictionalism, we make this problem worse. Indeed, the whole point of this paper has been to establish that this problem is *not* worse for PCI-platonism or fictionalism, because mathematics could be used in the way that we use it, even if it were false or about a causally isolated mathematical realm. In fact, it seems to me that fictionalism and FBP are *better off* with respect to this general problem of applicability than other philosophies of mathematics; for these views save the intuition that mathematics would be applicable to physics even if the physical world were utterly different.

I should note as an aside that in claiming that the problem of the paragraph before last is a problem for *everyone*, I do not mean to suggest that I think it is a *serious* problem. Many people – e.g. Wigner and Steiner[24] – have thought that this general problem of applicability is a very difficult problem to solve, but it seems to me that for many cases of applicability, the requested explanation has *already* been given. For instance, Krantz, Luce, Suppes, and Tversky have shown that the real number line is relevant to length assertions, because physical objects can be ordered with respect to length just as the real numbers are ordered with respect to 'greater than'. More generally, the point is that when a mathematical structure M is useful for making assertions about a collection of physical objects, it is often because we can build a physicalistic structure – whose domain consists of the given collection of physical objects – which is embeddable in M. But regardless of whether it's *easy* to solve this general problem of applicability, the main point is, again, that it's not relevant to *our* problem, i.e., the problem of whether platonism is true, i.e., whether mathematical objects exist.[25]

NOTES

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[1] The important point here is that according to fictionalism, mathematical singular terms are *vacuous*, i.e., do not refer to anything. Now, I will assume a view of vacuity which takes sentences with vacuous singular terms to be *false*, but (a) this is not essential to fictionalism (i.e., the general fictionalist view of mathematics is consistent with other theories of the semantics of vacuity) and (b) in the present context, nothing important depends upon my adoption of this particular theory of vacuity. (I should also note that according to this particular version of fictionalism, not all mathematical sentences are false; some are vacuously true. I will ignore this complication.) Fictionalism is developed by Hartry Field in his (1980).

Note that while '3 is prime' is taken by fictionalists to be false, it can be distinguished from '4 is prime' by noting that it is *true in the story of mathematics*, just as 'Oliver lived in London' (as opposed to, say, 'Oliver lived in Paris') is *true in the story of Oliver Twist*. Note, too, that fictionalists don't take *all* mathematical sentences to be false; some (e.g., 'if 3 is prime, then 3 is prime') are *vacuously true*; I will ignore this complication. Fictionalism is developed by Harty Field in his (1980).

[2] See Putnam (1971, chapters V–VIII) and Quine (1951, pp. 44–45).

[3] David Malament discusses many of these problems in his (1982). The most pressing of these, in my opinion, is that Field's method of nominalization might not work for quantum mechanics. But my (1996) shows how this problem can be solved.

[4] Actually, the traditional argument for platonism has the form of a *pair* of inferences to the best explanation. The Quine/Putnam argument is the first; it claims that we can only account for the indispensability of mathematics by endorsing the face-value truth of mathematics. It is then argued that we can only account for such truth by adopting platonism.

[5] I am not the first to detect a problem for platonism here. See Shapiro (1983, section II.3) and Kitcher (1984, pp. 104–105).

[6] It should not be surprising that fictionalists endorse PCI: since they don't believe in mathematical objects, they cannot believe in causal relations between mathematical objects and physical objects.

[7] Given that platonism is the belief in abstract objects, it might seem that platonists *cannot* claim that mathematical objects exist within spacetime, and hence, cannot reject PCI; for on the traditional usage (and the usage of this paper) 'abstract' *entails* 'outside of spacetime'. But Maddy (1990) has formulated a non-traditional version of platonism according to which mathematical objects are both abstract (in a certain non-traditional sense) *and* spatio-temporal. I discuss all of this – and argue that views of this sort are untenable – in my (1994).

[8] Of course, platonism is still relevant here; fictionalists, for instance, accept PCI *because* they reject platonism.

[9] I argue this point in my (1994).

[10] That PCI entails this claim can be seen in the following way: insofar as S's temperature plays a causal role in determining its behavior, if the bottom-level fact of S's temperature is a fact partly about the number 40, it would follow that 40 plays a causal role in determining S's behavior, which contradicts PCI. Now, I suppose that one might try to maintain that it could be that both (a) C(S,40) is bottom-level and causally efficacious and (b), 40 is *not* causally efficacious. But this is impossible; for if we assume that C(S,40) is causally efficacious and that 40 is not, then it would seem to follow that there is some purely physical fact F about S *alone* which is responsible for the causal efficacy in question here; but this is just to say that C(S,40) is *not* bottom-level, that it supervenes on F (and some purely mathematical fact about 40). After all, F will be the *complete* fact of S's temperature state, because it will be doing *all* the causal work that the fact of S's temperature state is supposed to do.

(Note that I am not making any *general* claim here about extrinsic explanations, or about non-causal explanations. All I am saying is that platonists cannot take facts like C(S,40) to be bottom-level facts, because that would contradict PCI.)

[11] For example, the temperature states are ordered with respect to 'cooler than' as the real numbers are ordered with respect to 'less than'. Thus, what we want when we define our temperature function Φ is a function which preserves these structural similarities. The Fahrenheit, Kelvin, and Celsius scales are three such functions, but of course, there are infinitely many other functions from physical objects to real numbers which would do just as well. To use the terminology of measurement theory, the functions that will be acceptable here are the *interval scales*; for a definition of this term, see Krantz, Luce, Suppes, and Tversky (1971, p. 10).

[12] This is essentially how Field (1980, chapter 7) nominalizes temperature assertions.

[13] Implicit here is another assumption: fictionalists only have to account for indispensable applications of mathematics to *our* physical theories. They do *not* have to account for any alleged fact of *absolute* indispensability, i.e., the indispensability of mathematics to the very project of doing physics. For, among other reasons, we simply have no reason to believe that the absolute claim is true. Even if we grant that QM is true and not nominalizable, physics could still be nominalizable, because there could still be a nominalistic theory N, which we haven't thought of, but which is empirically equivalent to (and just as attractive as) QM. (Why wouldn't N just *be* a nominalization of QM? Because it might look nothing like QM. Thus, to generalize, for T′ to be a nominalization of T, it must be nominalistic, empirically equivalent to T, *and* have the 'look and feel' of T. This third constraint will be satisfied if (and perhaps only if) representation theorems hold between the nominalistic structures of T′ and the platonistic structures of T.)

[14] Field's nominalization programme offers one such sort of scientific realism; but we are presently assuming that that programme is unworkable.

[15] In my (1996) I seek to replace these propositions with (nominalistically kosher) *propensities*.

[16] For instance, the Craigian reaxiomatization of the nominalistic consequences of

a platonistic theory would *not* be an acceptable nominalization of that theory.

[17] This suggests that it is very *easy* to state the nominalistic content of QM; that content is simply that the physical world holds up *its end* of the 'QM bargain'. Now, one might object that we are not speaking here in purely nominalist terms, because when we unpack the expression 'QM bargain', we will encounter all of the platonistic lingo of QM. To block this worry, I need to clarify my remark that to express in nominalistic terms the physical facts that QM expresses in platonistic terms is to state the nominalistic content of QM. In making this claim, I did not mean to say what the nominalistic content of a theory *is*; all I meant to say was that this is one way to express the nominalistic content of a theory. As for what the nominalistic content of a theory *is*, it is simply what the theory entails about the physical world in *particular*, i.e., as opposed to what it entails about platonic heaven. Given this, the claim that the physical world holds up its end of the 'QM bargain' does seem to capture the nominalistic content of QM, because it seems to capture what QM is saying about the physical world in *particular*.

This definition of 'nominalistic content' also helps to explain and motivate my claim that we cannot seriously doubt that QM (and the rest of empirical science) *has* a nominalistic content. For consider: it is *conceivable* (barely) that it is impossible to express in nominalist terms the physical facts that empirical science expresses in platonist terms, because it is conceivable that there are purely physical facts that are *ineffable* in the sense that they cannot be expressed in purely nominalistic terms – even in an ugly, hodgepodge way. But even if mathematics were *radically indispensable* to empirical science in this way, empirical science would still have a nominalistic content. For since mathematical objects are causally inert, the truth of empirical science would still supervene on two independent sets of facts – one purely nominalistic and one purely platonistic – and, therefore, there would still be something that empirical science entails about the physical world in *particular*, namely, that it holds up *its end* of the 'empirical-science bargain'. The appeal to causal inertness is supposed to show not just that empirical science *has* a nominalistic content, but that this content captures everything important that empirical science has to say about the physical world and that it could be true even if the platonistic content of empirical science is fictitious. All of this will be reinforced and clarified in the next section, where I argue that fictionalists can use these considerations to solve the problem of applicability.

[18] There is also a *third* response to the objection: in general, lacking an account of how something works does *not* provide a reason for doubting that it does. For instance, we are confident that perceptions give rise to beliefs, but we can't say exactly how.

[19] One reason for adopting scientific realism, rather than instrumentalism, is that by claiming that our empirical theories are true, we *explain* their empirical adequacy. But this is no argument against nominalistic realism, because (a) the truth of the nominalistic content of our empirical theories explains their empirical adequacy as well as full-blown truth does, and (b) full-blown truth doesn't explain truth of nominalistic content, except in a trivial sort of way. (The trivial explanation will simply be an instance of the rule that 'P & Q' explains 'P'; I call it trivial because it won't appeal to any causal processes.)

[20] This argument assumes that QM *has* a picture of the physical world that doesn't involve any claims about mathematical objects. But I've already justified this assumption, because I've already argued that QM has a nominalistic content which captures its *complete* picture of the physical world.

[21] See Horgan (1984) and Hellman (1989).

[22] This analogy seems to extend to the point about indispensability: if we understand the term 'canvas' very broadly, so that it applies to any sort of surface that one could cover with paint, e.g., a wall, then it may well be that canvases are indispensable to the project of painting portraits; but this does *not* mean that artists need their canvases to be accurate representations of things in order to maintain that their *portraits* are accurate representations of things. Likewise, even if mathematics is indispensable to the project of doing physics, it does *not* follow that we need to believe that our mathematical theories are true in order to believe that the nominalistic content of empirical science is true.

[23] Obviously, I haven't the space to justify this claim here, but I can at least mention the strategy to be used in arguing that FBP avoids the problem raised by Benacerraf (1973). Since according to FBP, all possible mathematical objects exist, it follows that in order to attain knowledge of such objects, it is sufficient to come up with a possible (i.e., consistent) purely mathematical theory. For according to FBP, every such theory will accurately describe some collection of actually existing mathematical objects. Thus, to attain knowledge of mathematical objects, one only has to attain knowledge of mathematical consistency. But knowledge of the consistency of a theory does not depend upon any sort of *access* to the objects that the theory is about. Thus, the epistemological problem of a 'lack of access' has been solved. For a full justification of all of this, see my (1995).

[24] See Wigner (1960) and Steiner (1989).

[25] I would like to emphasize that the point of this paper is merely to defend fictionalism against a certain objection, i.e., that the point is *not* to argue that fictionalism is *true*. Most anti-platonists argue *for* their view by showing that the alternative, viz., platonism, is false. But I do not think there are any good arguments against platonism – I hinted above at how I think the best candidate for such an argument, viz., the Benacerrafian epistemological argument, can be answered – and so my view is that *there are no good arguments on either side of this dispute*. Now, it might seem that if this is right, then Ockham's razor dictates that we ought to accept anti-platonism. But it seems to me that anti-platonists who reject the face-value truth of mathematics, e.g., fictionalists, cannot use Ockham's razor against platonism. There are at least two arguments for this claim. I do not have the space to properly develop them here, but the main idea is as follows. First, fictionalists cannot account for everything that platonists can account for; in particular, they cannot account for facts such as that $2 + 2 = 4$. In other words, the dispute here is not between two explanations of an already-agreed-upon collection of facts; it is, rather, a dispute about what the facts which require explanation *are*; thus, Ockham's razor is simply not applicable. And second, while fictionalism is simpler than platonism in terms of ontological parsimony, the added ontology does not seem to add complexity (in the sense of extra 'loops' and 'cogs') to the overall 'feel' of the theory, and moreover, platonism is simpler than fictionalism in various non-ontological ways (e.g., in enabling us to say that our mathematical and physical theories are straightforwardly true). Thus, it is not at all clear which view is simpler *overall*.

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