Math 5401

5/4/20

$\boxed{\text{From last time}}$:

The splitting field of $x^3 - 2$ is $\mathbb{Q}(2^{1/3}, \zeta_3)$ where $\zeta_3 = e^{2\pi i/3}$

$$\begin{array}{ll} & \mathbb{Q}(2^{1/3}, \zeta_3) \\ & \quad \Big|\, 2 \\ 6 & \mathbb{Q}(2^{1/3}) \\ & \quad \Big|\, 3 \\ & \mathbb{Q} \end{array}$$

$2 \leftarrow m_{\zeta_3, \mathbb{Q}(2^{1/3})}(x) = x^2 + x + 1$

basis for $\mathbb{Q}(2^{1/3}, \zeta_3)$ over $\mathbb{Q}(2^{1/3})$ is $1, \zeta_3$

$3 \leftarrow m_{2^{1/3}, \mathbb{Q}}(x) = x^3 - 2$

basis of $\mathbb{Q}(2^{1/3})$ over $\mathbb{Q}$: $1, 2^{1/3}, 2^{2/3}$

So, $[\mathbb{Q}(2^{1/3}, \zeta_3) : \mathbb{Q}] = 6$

$$\mathbb{Q}(2^{1/3}, \zeta_3) = \left\{ a \cdot 1 + b \cdot 2^{1/3} + c\, 2^{2/3} + d\,\zeta_3 + e\, 2^{1/3}\zeta_3 + f\, 2^{2/3}\zeta_3 \;\middle|\; a, \dots, f \in \mathbb{Q} \right\}$$

Let $\sigma \in \mathrm{Aut}(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q})$ (pg 2)

then

$$\sigma\left(a \cdot 1 + b \cdot 2^{1/3} + c(2^{1/3})^2 + d\rho_3 + e\,2^{1/3}\rho_3 + f(2^{1/3})^2\rho_3\right)$$

$$= a + b\,\sigma(2^{1/3}) + c(\sigma(2^{1/3}))^2 + d\,\sigma(\rho_3) + e\,\sigma(2^{1/3})\sigma(\rho_3) + f(\sigma(2^{1/3}))^2\sigma(\rho_3)$$

σ fixes $\mathbb{Q}$

$\sigma(x+y) = \sigma(x) + \sigma(y)$

$\sigma(xy) = \sigma(x)\sigma(y)$

So, σ is determined by $\sigma(2^{1/3})$ and $\sigma(\rho_3)$.

Since $\mathbb{Q}(2^{1/3}, \rho_3)$ is the splitting field of the separable (no multiple roots) polynomial $x^3 - 2$ over $\mathbb{Q}$ we have that $\mathbb{Q}(2^{1/3}, \rho_3)$ is Galois over $\mathbb{Q}$.

Thus, $|\mathrm{Gal}(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q})| = |\mathrm{Aut}(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q})| = [\mathbb{Q}(2^{1/3}, \rho_3) : \mathbb{Q}] = 6$

Let $\sigma \in Gal(\mathbb{Q}(2^{1/3}, \zeta_3)/\mathbb{Q})$.

$\sigma(2^{1/3})$ is a root of $m_{2^{1/3},\mathbb{Q}}(x) = x^3 - 2$

So, $\boxed{\sigma(2^{1/3}) \in \{2^{1/3}, 2^{1/3}\zeta_3, 2^{1/3}\zeta_3^2\}}$.

$\sigma(\zeta_3)$ is a root of $m_{\zeta_3,\mathbb{Q}}(x) = \Phi_3(x)$

$= x^2 + x + 1$.

So, $\boxed{\sigma(\zeta_3) \in \{\zeta_3^1, \zeta_3^2\}}$

2 possibilities

3 possibilities

That gives 6 possibilities.

Since we know there are exactly 6 elements in $Gal(\mathbb{Q}(2^{1/3}, \zeta_3)/\mathbb{Q})$ these 6 possibilities will give us the group.

$$\Phi_n(x) = \prod_{\substack{1 \le a \le n \\ \gcd(a,n)=1}} (x - \zeta_n^a)$$

roots of $x^n - a$, $a \in \mathbb{R}$, $a > 0$

$x = a^{1/n}\zeta_n^k$, $k = 0, 1, \ldots, n-1$.

$\zeta_n = e^{2\pi i/n}$

Elements of $\mathrm{Gal}(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q})$

$i: \begin{cases} 2^{1/3} \longmapsto 2^{1/3} \\ \rho_3 \longmapsto \rho_3 \end{cases}$	$\sigma_3: \begin{cases} 2^{1/3} \longmapsto 2^{1/3} \\ \rho_3 \longmapsto \rho_3^2 \end{cases}$
$\sigma_1: \begin{cases} 2^{1/3} \longmapsto 2^{1/3}\rho_3 \\ \rho_3 \longmapsto \rho_3 \end{cases}$	$\sigma_4: \begin{cases} 2^{1/3} \longmapsto 2^{1/3}\rho_3 \\ \rho_3 \longmapsto \rho_3^2 \end{cases}$
$\sigma_2: \begin{cases} 2^{1/3} \longmapsto 2^{1/3}\rho_3^2 \\ \rho_3 \longmapsto \rho_3 \end{cases}$	$\sigma_5: \begin{cases} 2^{1/3} \longmapsto 2^{1/3}\rho_3^2 \\ \rho_3 \longmapsto \rho_3^2 \end{cases}$

$$\mathrm{Gal}(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q}) = \{i, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$$

Calculations: $\sigma_1 \sigma_4 = \sigma_1 \circ \sigma_4$

$$(\sigma_1 \circ \sigma_4)(2^{1/3}) = \sigma_1(\sigma_4(2^{1/3})) = \sigma_1(2^{1/3}\rho_3)$$
$$= \sigma_1(2^{1/3})\,\sigma_1(\rho_3) = 2^{1/3}\rho_3 \cdot \rho_3 = 2^{1/3}\rho_3^2$$

$$(\sigma_1 \circ \sigma_4)(\rho_3) = \sigma_1(\sigma_4(\rho_3)) = \sigma_1(\rho_3^2) = [\sigma_1(\rho_3)]^2$$
$$= [\rho_3]^2 = \rho_3^2 \qquad \text{So, } \boxed{\sigma_1 \sigma_4 = \sigma_5}$$

What is $\sigma_4 \circ \sigma_1$?

$$\sigma_4(\sigma_1(2^{1/3})) = \sigma_4(2^{1/3}\rho_3) = \sigma_4(2^{1/3})\sigma_4(\rho_3)$$
$$= [2^{1/3}\rho_3][\rho_3^2] = 2^{1/3}\rho_3^3$$
$$= 2^{1/3}$$

$\boxed{\rho_3^3 = 1}$ $\boxed{\rho_n^n = 1}$

$$\sigma_4(\sigma_1(\rho_3)) = \sigma_4(\rho_3) = \rho_3^2$$

So, $\sigma_4 \circ \sigma_1 = \sigma_3$

Thus, $\sigma_1 \sigma_4 = \sigma_5$ and $\sigma_4 \sigma_1 = \sigma_3$.

So, $\sigma_1 \sigma_4 \neq \sigma_4 \sigma_1$.

So, $Gal(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q})$ is a non-abelian group of size 6.

You can show $Gal(\mathbb{Q}(2^{1/3}, \rho_3)/\mathbb{Q}) \cong D_6$
$\cong S_3$

Ex: Consider $f(x) = (x^2-2)(x^2-3)$.

The roots of f are $\pm\sqrt{2}, \pm\sqrt{3}$.

We showed previously that

$$\begin{array}{c} K = \mathbb{Q}(\sqrt{2},\sqrt{3}) \\ \big|\, 2 \\ \mathbb{Q}(\sqrt{2}) \\ \big|\, 2 \\ \mathbb{Q} \end{array} \qquad 4$$

$$\mathbb{Q}(\sqrt{2},\sqrt{3}) = \left\{ a\cdot 1 + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \;\middle|\; a,b,c,d \in \mathbb{Q} \right\}$$

no repeated roots

Since f is seperable and $K = \mathbb{Q}(\sqrt{2},\sqrt{3})$ is the splitting field of f over $\mathbb{Q}$, K is Galois over $\mathbb{Q}$.

Thus, $|\mathrm{Gal}(K/\mathbb{Q})| = [K:\mathbb{Q}] = 4$.

Let $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Then σ is determined by $\sigma(\sqrt{2})$ and $\sigma(\sqrt{3})$. $\sigma(\sqrt{2})$ is a root of $m_{\sqrt{2},\mathbb{Q}}(x) = x^2-2$. So, $\boxed{\sigma(\sqrt{2}) \in \{\sqrt{2}, -\sqrt{2}\}}$

$\sigma(\sqrt{3})$ is a root of $m_{\sqrt{3},\mathbb{Q}}(x) = x^2-3$. So, $\boxed{\sigma(\sqrt{3}) \in \{\sqrt{3}, -\sqrt{3}\}}$

The 4 elements of $Gal(K/\mathbb{Q})$ are (pg 7)

$$i : \begin{cases} \sqrt{2} \longmapsto \sqrt{2} \\ \sqrt{3} \longmapsto \sqrt{3} \end{cases} \qquad \sigma_1 : \begin{cases} \sqrt{2} \longmapsto -\sqrt{2} \\ \sqrt{3} \longmapsto \sqrt{3} \end{cases}$$

$$\sigma_2 : \begin{cases} \sqrt{2} \longmapsto \sqrt{2} \\ \sqrt{3} \longmapsto -\sqrt{3} \end{cases} \qquad \sigma_3 : \begin{cases} \sqrt{2} \longmapsto -\sqrt{2} \\ \sqrt{3} \longmapsto -\sqrt{3} \end{cases}$$

$$Gal(K/\mathbb{Q}) = \{ i, \sigma_1, \sigma_2, \sigma_3 \}$$

$Gal(K/\mathbb{Q})$	i	σ_1	σ_2	σ_3
i	i	σ_1	σ_2	σ_3
σ_1	σ_1	i	σ_3	σ_2
σ_2	σ_2	σ_3	i	σ_1
σ_3	σ_3	σ_2	σ_1	i

This is an abelian group of size 4. No element has order 4, so,

$Gal(K/\mathbb{Q}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$(\sigma_3 \sigma_1)(\sqrt{2}) = \sigma_3(\sigma_1(\sqrt{2})) = \sigma_3(-\sqrt{2}) = -\sigma_3(\sqrt{2}) = -(-\sqrt{2}) = \sqrt{2}$

$(\sigma_3 \sigma_1)(\sqrt{3}) = \sigma_3(\sigma_1(\sqrt{3})) = \sigma_3(\sqrt{3}) = -\sqrt{3}$

$a \in \mathbb{R},\ a > 0$

$$x^n - a = 0$$

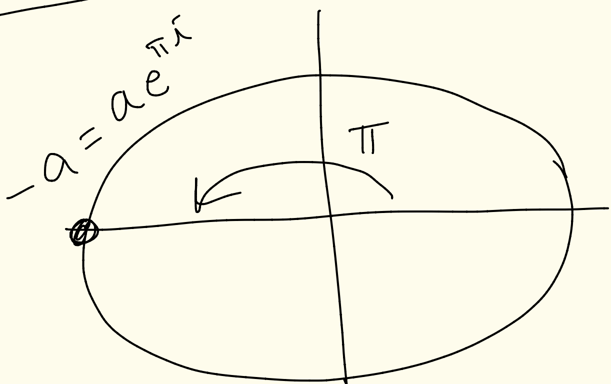
$$x = a^{1/n} \zeta_n^k, \quad k = 0, 1, 2, \ldots, n-1$$

$$\zeta_n = e^{2\pi i/n}$$

$a \in \mathbb{R},\ a > 0$

$$x^n + a = 0$$

$$x = a^{1/n} e^{(\pi + 2\pi k)i/n}, \quad k = 0, 1, \ldots, n-1$$



$$x^n = -a = ae^{\pi i}$$

$$x^n = ae^{\pi i + 2\pi k i}$$

$$x = a^{1/n} e^{(\pi i + 2\pi k i)/n}$$

$$= a^{1/n} e^{\frac{\pi i}{n}} \left(e^{\frac{2\pi i}{n}}\right)^k$$

For final

You can have a piece of paper with:

- any definitions
- any theorems (no proofs)
  corollary/prop statement

- **No** proofs or examples written out.