Math 4740
9/6/23

Ex: Suppose you have six-sided die with sides labelled 1, 2, 3, 4, 5, 6.

But through experimentation you realize that each outcome is not equally likely. You estimate the following probabilities.

outcome	probability
1	2/8
2	1/8
3	1/8
4	1/16
5	1/16
6	3/8

Note:

$$\frac{2}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{3}{8} = 1$$

Let's make a probability space.

$S = \{1, 2, 3, 4, 5, 6\}$

$\Omega = \{\text{all subsets of } S\}$

$$= \{ \phi, \{1\}, \{2\}, \ldots, \{6\}, \{1,2\}, \{1,6\}, \ldots, \{2,3,5\}, \ldots \}$$

Make a probability function

$$P : \underbrace{\Omega}_{\text{input}} \longrightarrow \underbrace{\mathbb{R}}_{\text{output}}$$

Define

$$P(\{1\}) = 2/8 \quad P(\{3\}) = 1/8 \quad P(\{5\}) = 1/16$$
$$P(\{2\}) = 1/8 \quad P(\{4\}) = 1/16 \quad P(\{6\}) = 3/8$$

If E is an event from Ω define

$$P(E) = \sum_{w \in E} P(\{w\})$$

For example,

$$\underbrace{P(\{2,4,6\})}_{\text{probability of rolling an even number}} = P(\{2\}) + P(\{4\}) + P(\{6\})$$
$$= \frac{1}{8} + \frac{1}{16} + \frac{3}{8} = \frac{9}{16}$$
$$\approx 0.5625$$
$$\approx 56.25\%$$

And

$$\underbrace{P(\{1,2,3,4,5\})}_{\text{probability of not getting a 6}} = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) + P(\{5\})$$

$$= \frac{2}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$$

$$= \frac{5}{8} \approx 0.625 \approx 62.5\%$$

Note: Suppose (S, Ω, P) is a probability space and S is finite.

Suppose each outcome w in S is equally likely, that is

$$P(\{w\}) = \frac{1}{|S|} \quad \text{for all } w \text{ in } S.$$

In this case if E is an event with n elements, say

$$E = \{w_1, w_2, \ldots, w_n\}$$

then

$$P(E) = P(\{w_1\}) + P(\{w_2\}) + \cdots + P(\{w_n\})$$

$$= \frac{1}{|S|} + \frac{1}{|S|} + \cdots + \frac{1}{|S|}$$

$$= \frac{n}{|S|} = \frac{|E|}{|S|}.$$

So, $\boxed{P(E) = \frac{|E|}{|S|}}$

formula holds only when each outcome is equally likely

Ex: Suppose we do the experiment where we roll two 6-sided dice. These are normal dice, each side is equally likely

$(a,b) \leftarrow$ represents a on die 1, b on die 2

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),$$
$$(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$$
$$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),$$
$$(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

$\Omega = \{\text{all subsets of } S\}$

$|S| = 36$

$$P(\{(a,b)\}) = \frac{1}{|S|} = \frac{1}{36}$$

For example, $P(\{(1,5)\}) = \frac{1}{36}$

die 1 = 1   die 2 = 5

<u>Question:</u> What is the probability that the sum of dice is 7 ?

Let E be the event that the sum of the dice is 7.

Then,

$$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

Since all outcomes are equally weighted,

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6} \approx 0.166$$

$$\approx 16.6\%$$

$\boxed{\text{Note:}}$ You can construct a probability space when S is countably infinite, that is S is infinite and you can list the elements. Suppose

$$S = \{w_1, w_2, w_3, w_4, w_5, \ldots\}$$

infinitely many more

Let Ω be the set of all subsets of S.

Define $P(\{w_i\})$ for each i so that $0 \le P(\{w_i\}) \le 1$ and

$$\sum_{i=1}^{\infty} P(\{w_i\}) = 1$$

If E is an event, define

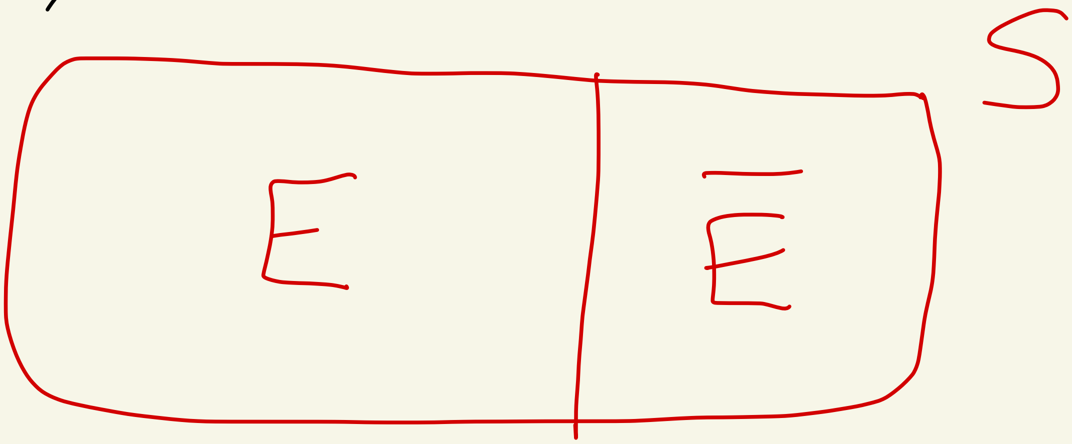
$$P(E) = \sum_{w \in E} P(\{w\}).$$

$\boxed{\text{Theorem:}}$ This will be a probability space.

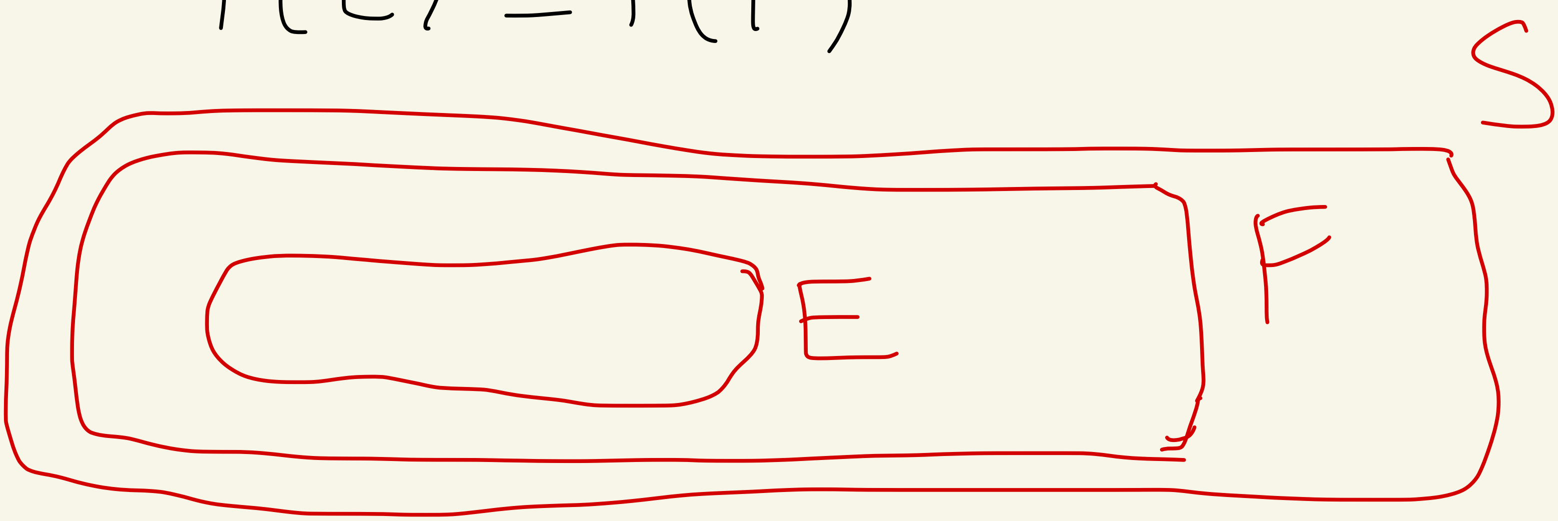
Theorem: Let (S, Ω, P) be a probability space. Let E and F be events.

Then:

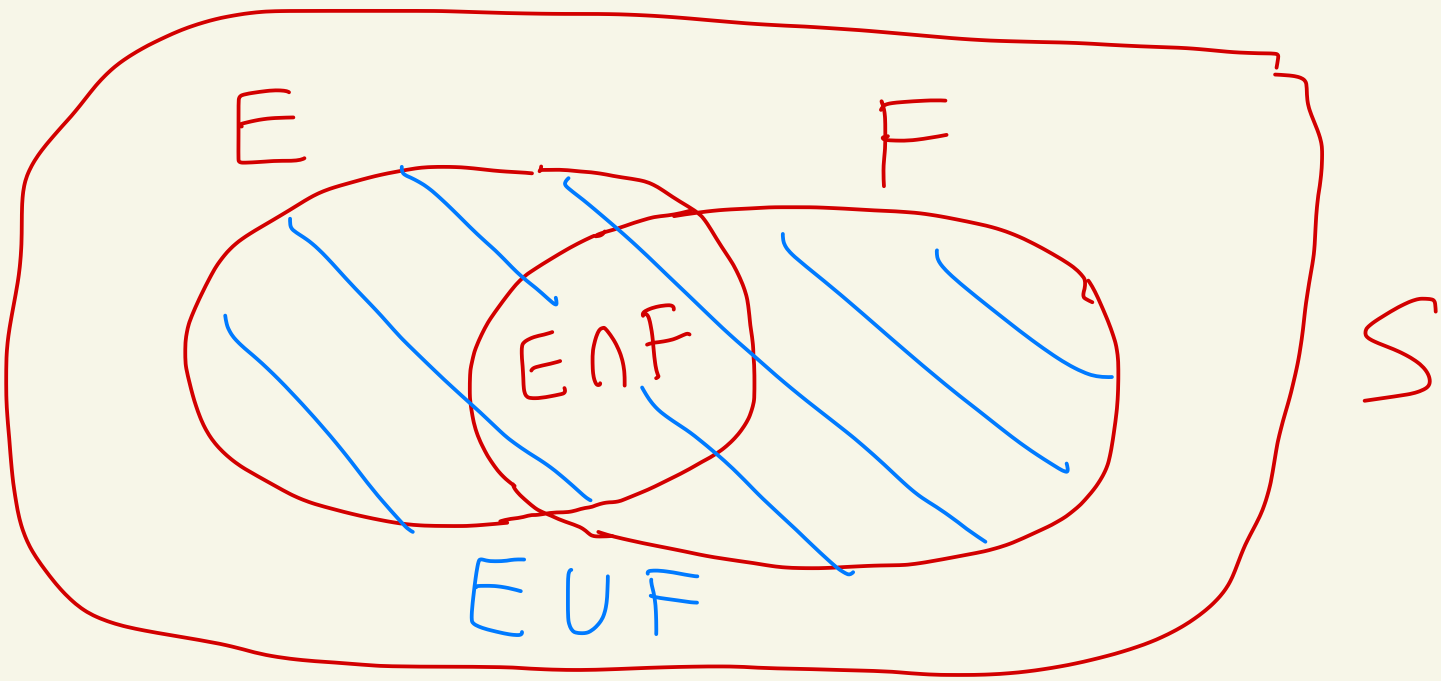
① $P(\overline{E}) = 1 - P(E)$



② If $E \subseteq F$, then $P(E) \leq P(F)$

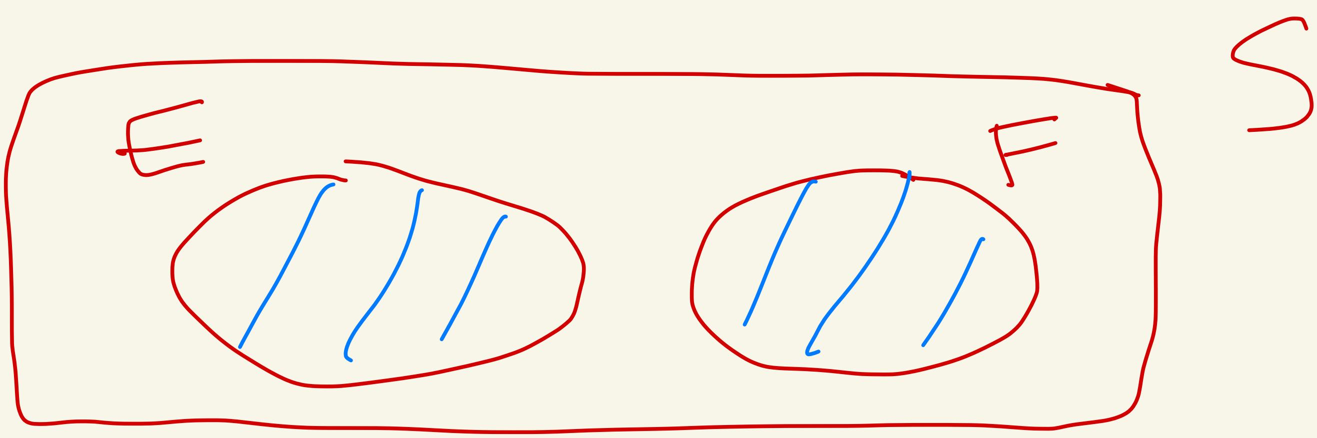


(3) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$



(4) If E and F are disjoint, that is $E \cap F = \phi$, then

$$P(E \cup F) = P(E) + P(F)$$



proof: See 2-7-22 notes from Spring 22. ☒

Ex: Suppose we roll two 12-sided dice. Each die has 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 on it and are equally likely.

What is the probability that at least one of the dice is 4, 5, 6, 7, 8, 9, 10, 11, or 12?

Ex:

die 1	die 2		
$\boxed{3}$	$\boxed{3}$	X	neither die is 4−12
$\boxed{2}$	$\boxed{7}$	✓	at least one die

$\boxed{\boxed{6}\ \boxed{12}}$ ✓ $\boxed{\text{is } 4-12}$

<u>Sample space:</u>

$$S = \left\{ (a,b) \;\middle|\; \begin{matrix} a = 1, 2, 3, \ldots, 12 \\ b = 1, 2, 3, \ldots, 12 \end{matrix} \right\}$$

$$= \{ (1,1), (5,12), (3,4), \ldots \}$$

Then, $|S| = 12 \cdot 12 = 144$

Let E be the event that at least one die is a 4, 5, 6, 7, 8, 9, 10, 11, 12

So,

$$E = \{ (6,12), (4,5), (4,10), \ldots \}$$

too many to count!

Instead we count $\overline{E}$.

We have

$\overline{E}$ is the event that none of the die are $4, 5, \ldots, 12$.

So,

$$\overline{E} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Then,

$$P(\overline{E}) = \frac{|\overline{E}|}{|S|} = \frac{9}{144} = \frac{1}{16}$$

Thus from the previous theorem

$$P(E) = 1 - P(\overline{E}) = 1 - \frac{1}{16}$$

$$= \frac{15}{16} \approx \boxed{0.9375}$$