Math 4740

8-30-23

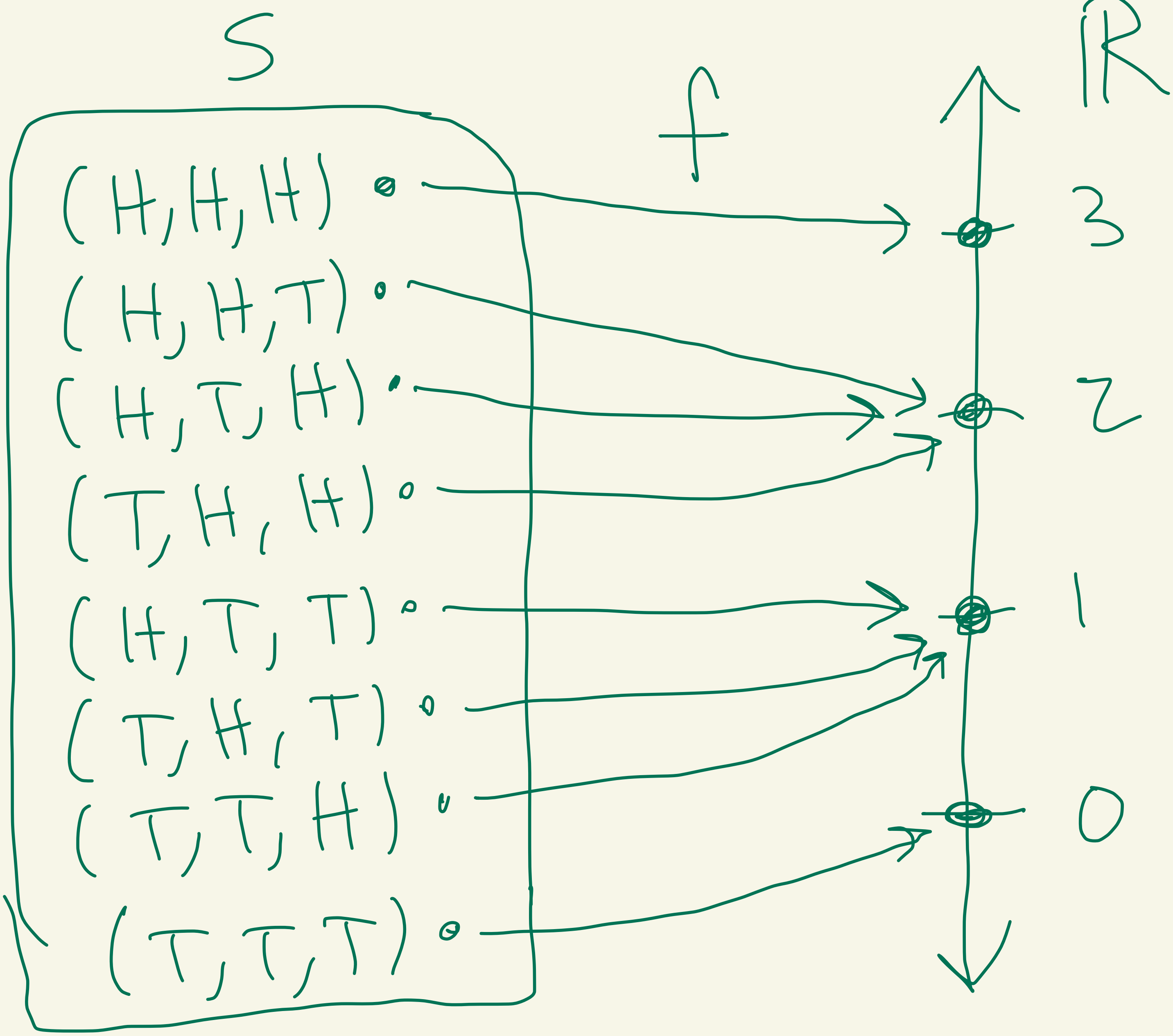
Ex: Let

$$S = \{(H,H,H), (H,T,H), (H,H,T), (H,T,T), (T,H,H), (T,T,H), (T,H,T), (T,T,T)\}$$

(flipping a coin 3 times)

Let $f: S \to \mathbb{R}$ be the number of heads that occur.

$\boxed{\mathbb{R} \text{ is the set of real numbers}}$



For example,

$$f(H,H,T) = 2$$

Later in the class f will be called a random variable.

Example of making a probability space

Suppose we want to model the experiment of rolling/throwing a 4-sided die.

$$S = \{1, 2, 3, 4\}$$

sample space of all possible outcomes of rolling the die

Omega

$$\Omega = \{ \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\},$$

$$\{1, 2, 3, 4\} \; \}$$

Ω is called the set of events. It will have certain properties. Ω contains all the sets that we want to be able to measure the probability of. When S is finite, then Ω usually contains all the subsets of S.

What do these events mean?

$\emptyset \leftarrow$ represents no number came up on the die

$\{2\} \leftarrow$ represents a 2 came up on the die

$\{2,3\} \leftarrow$ represents either a 2 or a 3 came up on the die

$\{1,2,4\} \leftarrow$ represents either a 1 or a 2 or a 4 came up on the die

$\{1,2,3,4\} \leftarrow$ represents either 1, 2, 3, or 4 came up on the die

Now we make a probability function $P: \Omega \to \mathbb{R}$.

On a 4-sided die each side is equally likely to occur.

The first step is to assign the probability to each side/number.

$$\left.\begin{aligned} P(\{1\}) &= \frac{1}{4} \\ P(\{2\}) &= \frac{1}{4} \\ P(\{3\}) &= \frac{1}{4} \\ P(\{4\}) &= \frac{1}{4} \end{aligned}\right\}$$

make equally likely. and adds up to 1.

Now we extend P across all the events by doing disjoint sums. For example,

$$P(\{1,3\}) = P(\{1\}) + P(\{3\})$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(\{1,2,3\}) = P(\{1\}) + P(\{2\}) + P(\{3\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

$$P(\{1,2,3,4\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\})$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$$

We define

$$P(\phi) = 0$$

Def: A probability space consists of two sets and a function (S, Ω, P).

We call S the sample space of our experiment. The elements of S are called outcomes. Ω is a set of subsets of S.

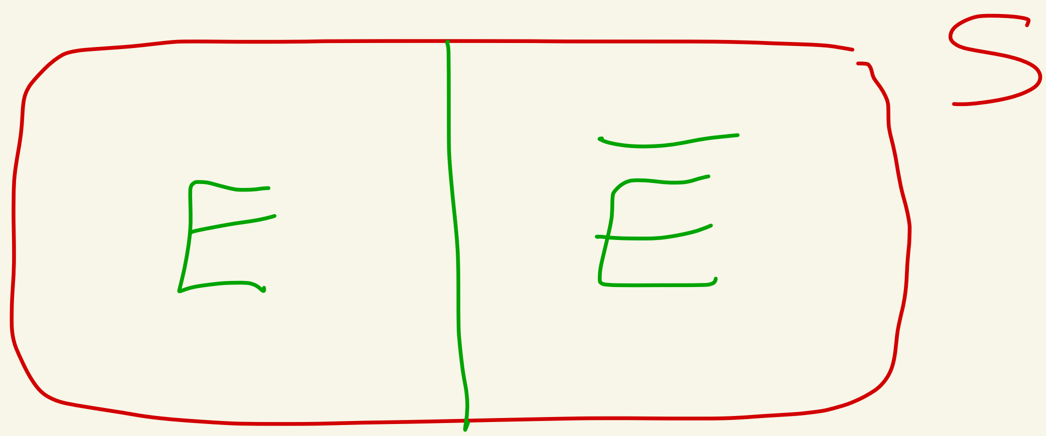
The elements of Ω are called events.

$P : \Omega \to \mathbb{R}$ is a function where for each event E from Ω we get a probability $P(E)$.

Furthermore, the following axioms/properties must hold:

1. S is an event in Ω
   (We must be able to calculate $P(S)$)

2. If E is an event in Ω, then $\overline{E}$ is an event in Ω



3. If $E_1, E_2, E_3, \ldots$ is a finite or infinite sequence of events in Ω,

then $\bigcup_i E_i$ is an event in Ω

④ $0 \le P(E) \le 1$ for any event E in Ω

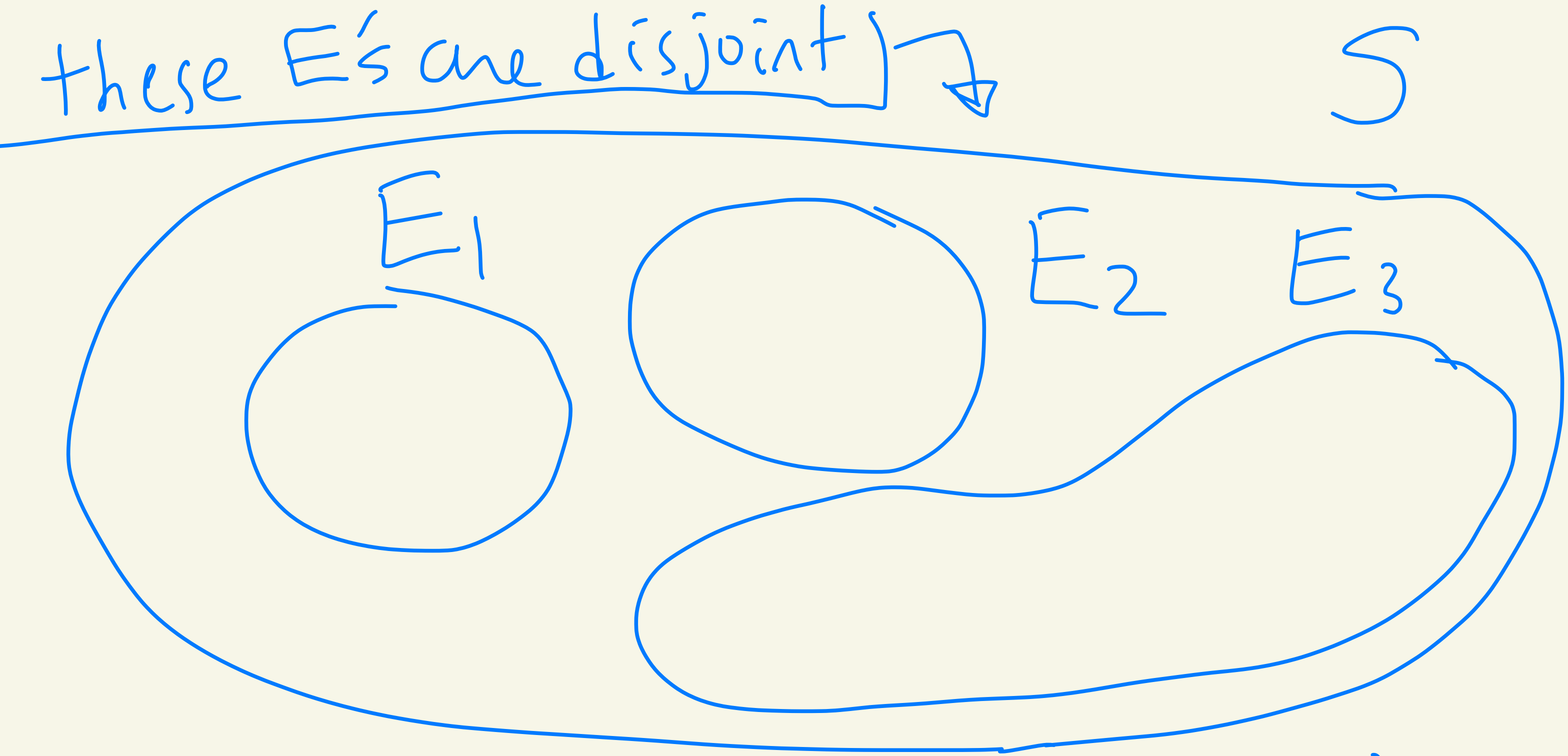
⑤ $P(S) = 1$

⑥ If $E_1, E_2, E_3, \dots$ is a finite or infinite sequence of events in Ω that are pair-wise disjoint $[E_i \cap E_j = \phi$ if $i \neq j]$

then $P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$

(disjoint means no overlap)

these E's are disjoint



$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$

$\boxed{\text{End of def}}$

This def is based on the work of Andrey Kolmogorov 1930s

Remark: A set Ω satisfying ①, ②, and ③ above is called a **σ-algebra** or **σ-field**.

Remark: If Ω is a σ-algebra one can show that:

(a) $\phi \in \Omega$

(b) If $E_1, E_2, E_3, \ldots$ are in Ω, then $\bigcap_i E_i$ is in Ω

Proof: look at 1/31/22 notes from Spring 22 $\square$

How to construct a probability space when S is finite

Suppose S is a finite sample space and we want to make a probability space.

Define Ω to be the set that includes <u>all</u> the subsets of S.

[Ω is the power set of S]

For each outcome $\omega \in S$ pick some real number $0 \leq n_\omega \leq 1$ and define

$$P(\{\omega\}) = n_\omega$$

probability that ω occurs

At the same time pick these numbers so that

$$\sum_{w \in S} n_w = 1$$

$\underbrace{\sum_{w \in S}}$ means sum over w in S

Ex:

$$S = \{\overset{w}{1, 2, 3, 4}\}$$

$$P(\{1\}) = \frac{1}{4} = n_1$$

$$P(\{2\}) = \frac{1}{4} = n_2$$

$$P(\{3\}) = \frac{1}{4} = n_3$$

$$P(\{4\}) = \frac{1}{4} = n_4$$

$$n_1 + n_2 + n_3 + n_4 = 1$$

Now extend P to any set E in Ω.

Suppose

$$E = \{w_1, w_2, \ldots, w_r\}$$

Define

$$P(E) = \sum_{i=1}^{r} P(\{w_i\})$$

$$= P(\{w_1\}) + P(\{w_2\}) + \ldots + P(\{w_r\})$$

$$P(\{1,2\}) = P(\{1\}) + P(\{2\})$$

If $E = \phi$, then define

$$P(\phi) = 0.$$

<u>Theorem:</u> The construction above creates a probability space (S, Ω, P).

<u>proof:</u> See notes from 2/2/22 Spring 22 $\blacksquare$