Math 4740
11/6/23

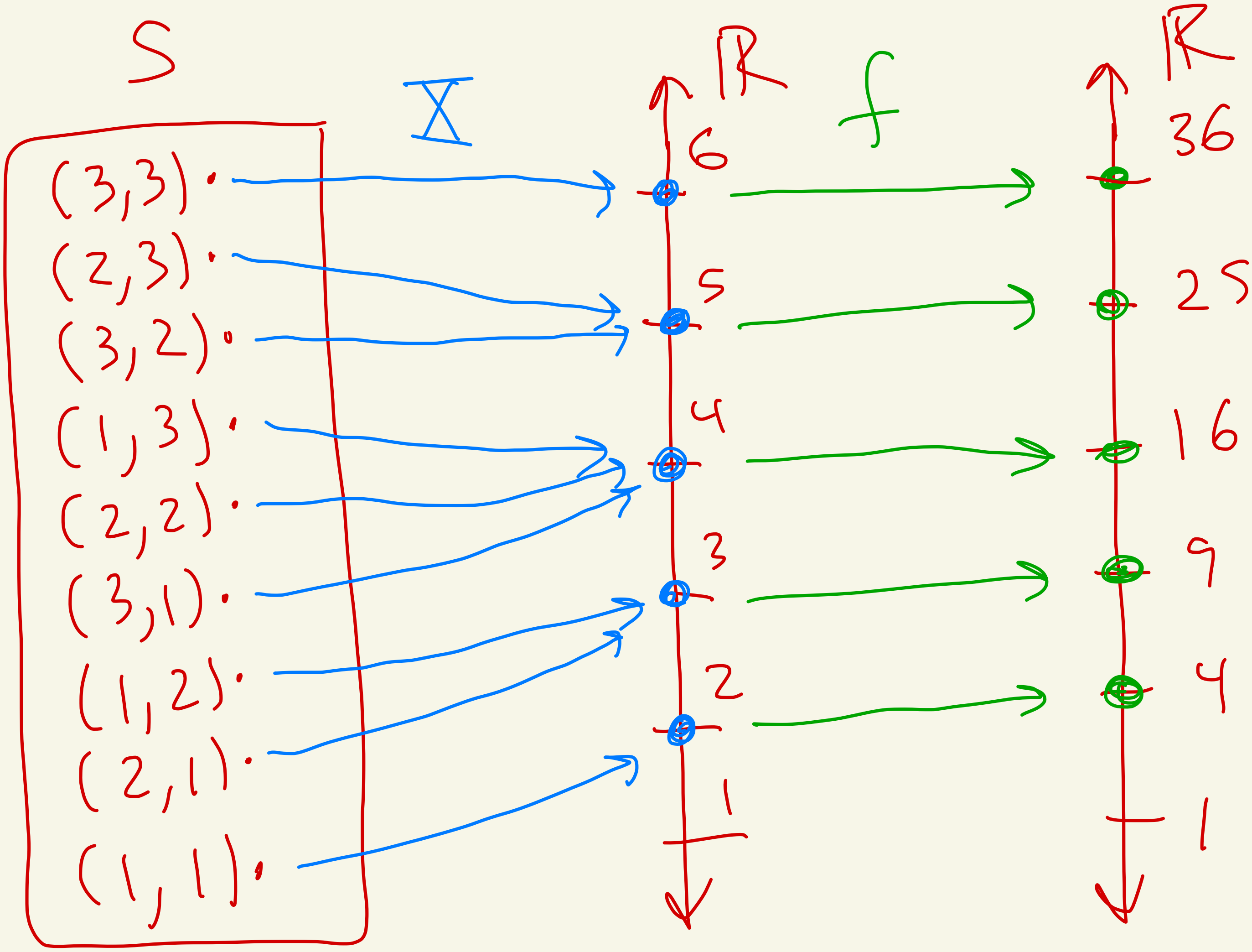
Topic 6 – More on Expected Value, Variance, Standard Deviation

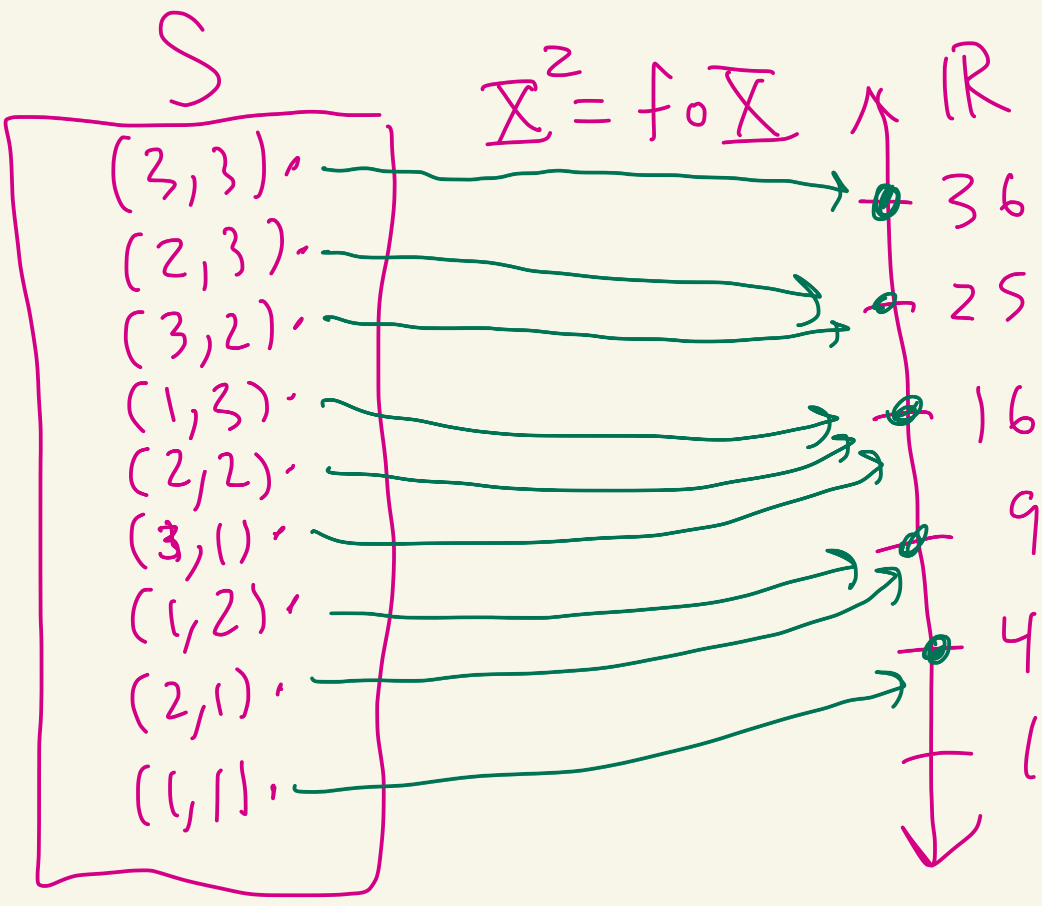
Given a discrete random variable $X : S \to \mathbb{R}$, if you take a function $f : \mathbb{R} \to \mathbb{R}$ and compute the composition $f \circ X$ then you will get a new random variable

under appropriate conditions such as a finite sample space

<u>Ex:</u> Suppose we roll two 3-sided dice, each labeled 1, 2, 3 where each side is equally likely. Let X be the sum of the dice.

Let $f:\mathbb{R}\to\mathbb{R}$ be $f(t)=t^2$.



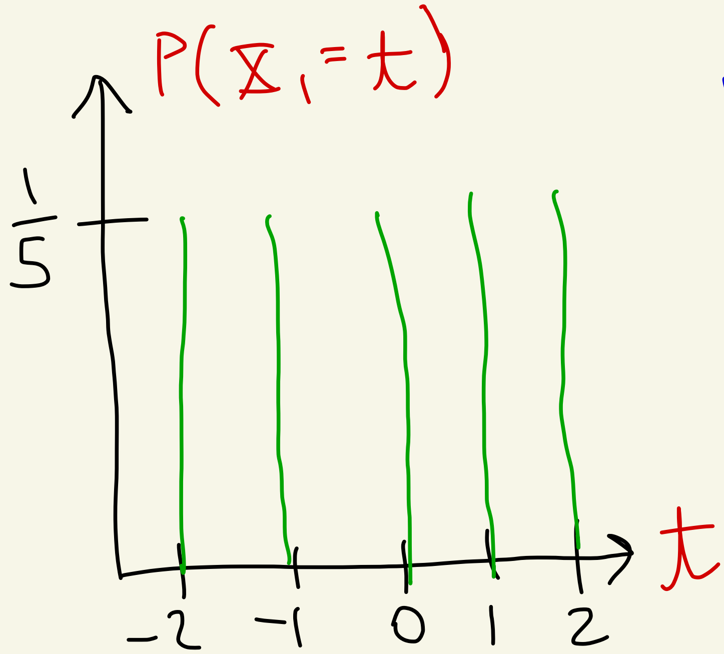


Example

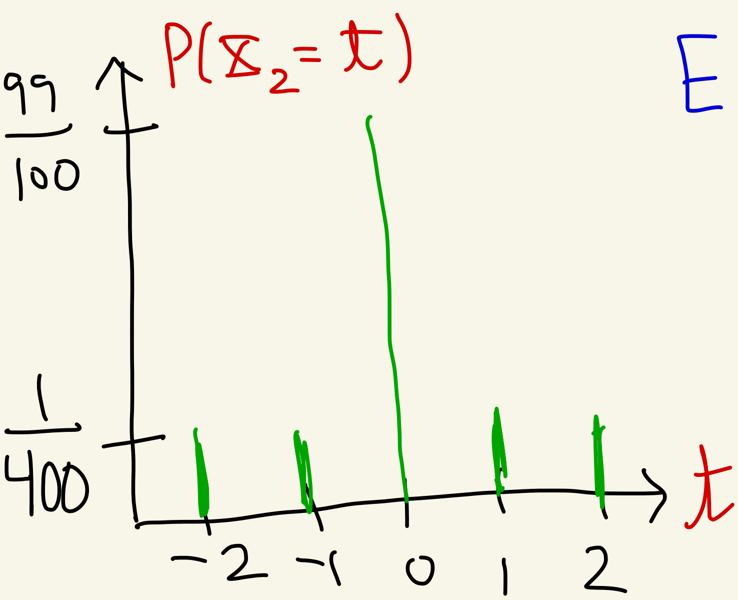
$$(f\circ X)(2,3)$$
$$=f(X(2,3))$$
$$=f(2+3)$$
$$=f(5)$$
$$=5^2=25$$

Expected value doesn't give all the info for a probability function. It can't detect how much the data is spread out or not spread out.

Ex: (Two probability functions w/ same expected value but data spread out differently)



$$E[X_1] = (-2)\left(\frac{1}{5}\right) + (-1)\left(\frac{1}{5}\right) + (0)\left(\frac{1}{5}\right) + (1)\left(\frac{1}{5}\right) + (2)\left(\frac{1}{5}\right) = 0$$



$$E[X_2] = (-2)\left(\frac{1}{400}\right) + (-1)\left(\frac{1}{400}\right) + (0)\left(\frac{99}{100}\right) + (1)\left(\frac{1}{400}\right) + (2)\left(\frac{1}{400}\right) = 0$$

We want a number that measures the average magnitude of the fluctuations of the random variable from it's expected value.

Let $\mu = E[X]$.

One might try to measure the expected value of $|X-\mu|$, ie the expected value of the distance between X's values and μ. This is too hard to use.

So instead we measure

$$E\Big[\underbrace{(X-\mu)^2}_{\text{square of distance between } X \text{ and } \mu}\Big]$$

$$|X-\mu| = \sqrt{(X-\mu)^2}$$
$$(X-\mu)^2 = |X-\mu|^2$$

Def: Let X be a discrete random variable. Define the *variance* of X to be $Var(X) = E[(X-\mu)^2]$.

Define the *standard deviation* of X to be $\sigma_X = \sigma = \sqrt{Var(X)}$

(where $\mu = E[X]$)

Note: One can prove that if $x_1, x_2, x_3, \ldots$ are the outputs of X, and $f: \mathbb{R} \to \mathbb{R}$, then

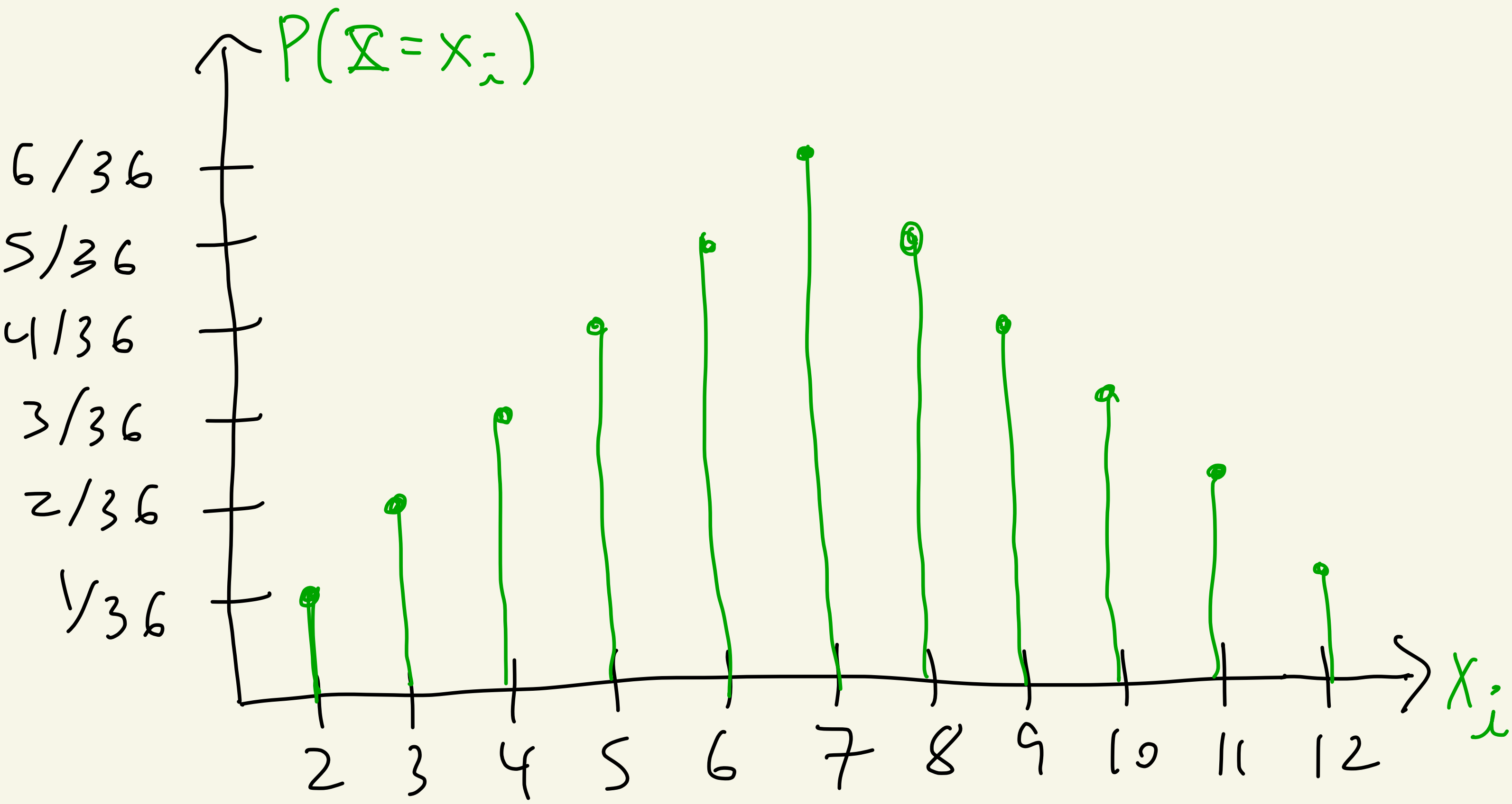
$$E[f(X)] = \sum_i f(x_i) \cdot P(X = x_i)$$

Thus,

$$Var(X) = \sum_i (x_i - \mu)^2 \cdot P(X = x_i)$$

(where $\mu = E[X]$)

Ex: Consider the experiment of rolling two 6-sided dice. Let X be the sum of the dice



Recall that $\mu = E[X] = 7$.

Then,

$$Var(X) = \sum_{x_i} (x_i - 7)^2 \cdot P(X = x_i)$$

$$= (2-7)^2 \cdot \left(\frac{1}{36}\right) + (3-7)^2 \cdot \left(\frac{2}{36}\right)$$

$$+(4-7)^2\left(\frac{3}{36}\right)+(5-7)^2\left(\frac{4}{36}\right)$$

$$+(6-7)^2\cdot\left(\frac{5}{36}\right)+(7-7)^2\left(\frac{6}{36}\right)$$

$$+(8-7)^2\left(\frac{5}{36}\right)+(9-7)^2\cdot\left(\frac{4}{36}\right)$$

$$+(10-7)^2\left(\frac{3}{36}\right)+(11-7)^2\left(\frac{2}{36}\right)$$

$$+(12-7)^2\left(\frac{1}{36}\right)=\boxed{\frac{35}{6}}\approx\boxed{5.83}$$

$$\sigma_X=\sqrt{Var(X)}=\sqrt{\frac{35}{6}}\approx\boxed{2.415}$$

Theorem: Let X be a discrete random variable. Let $\mu = E[X]$.

Then,

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$
$$= E[X^2] - \mu^2$$

Proof: Let $x_1, x_2, x_3, \ldots$ be the values of X. Then

$$\mathrm{Var}(X) = \sum_i (x_i - \mu)^2 \cdot P(X = x_i)$$
$$= \sum_i x_i^2 \cdot P(X = x_i)$$

equals $\mu = E[X]$

$$- 2\mu \boxed{\sum_i x_i \cdot P(X = x_i)}$$
$$+ \mu^2 \boxed{\sum_i P(X = x_i)}$$

equals 1

$$= \boxed{\sum_i x_i^2 \cdot P(X = x_i)} \quad \text{equals } E[X^2]$$

$$- 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

$\blacksquare$