

Math 4740

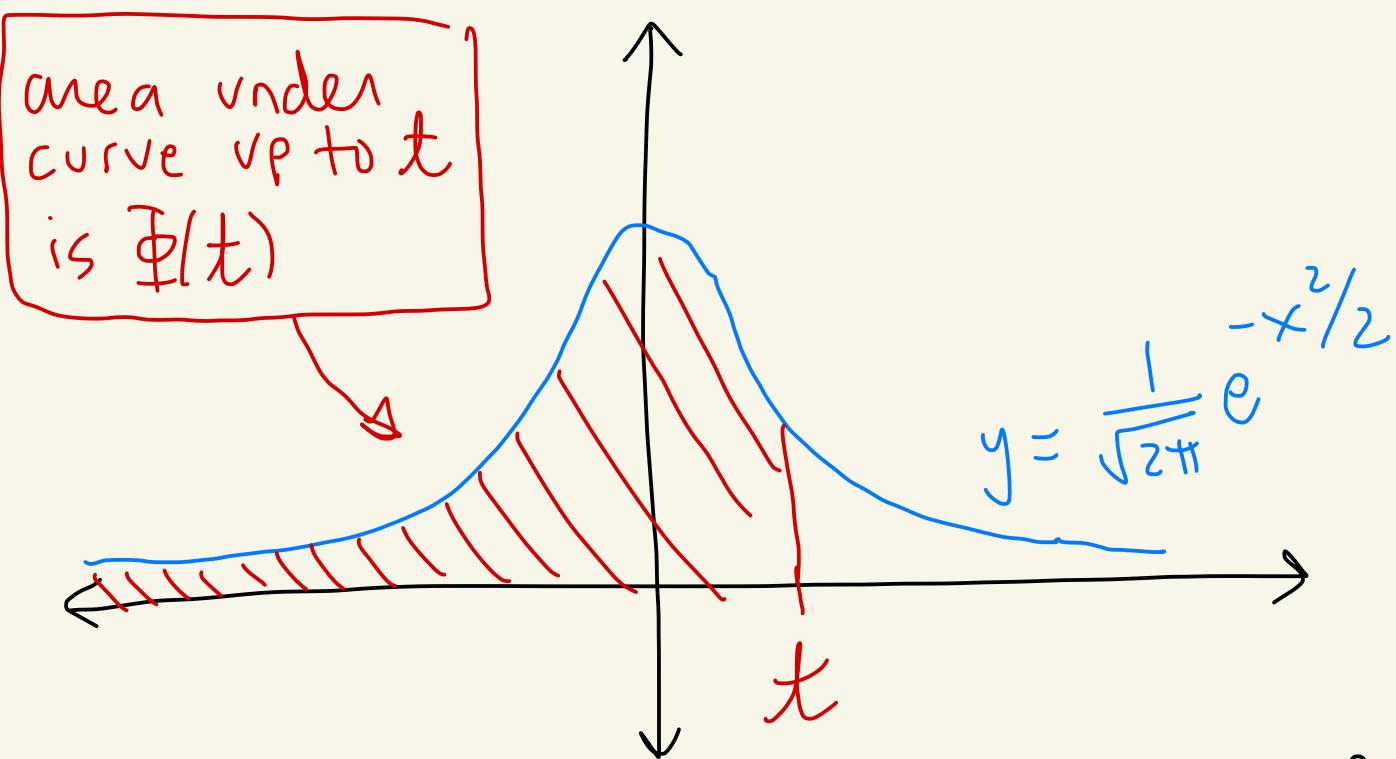
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Topic 7 - Normal approx. of binomial Random Variables

Def. Let

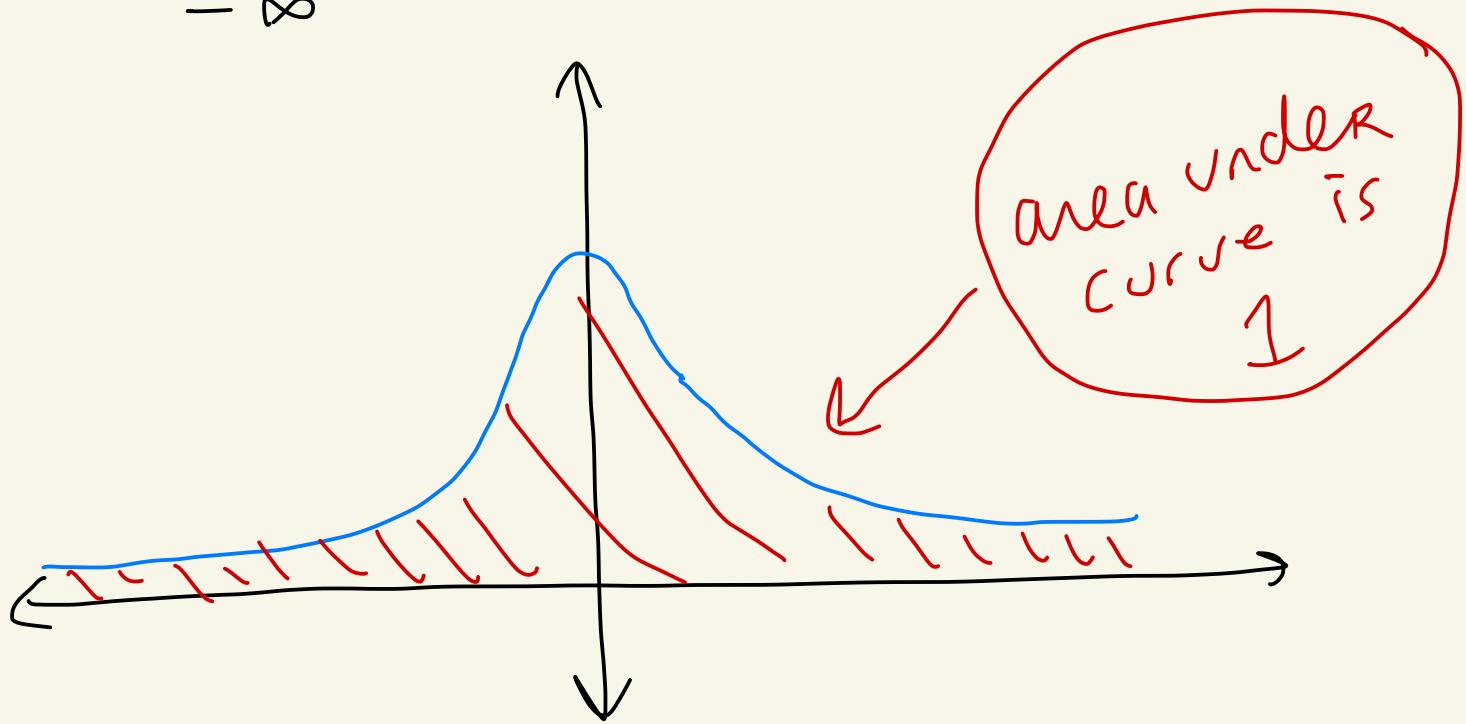
$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx$$



Φ is called the probability density function of the standard normal random variable (topic 8)

In topic 8, we will see
that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1$$



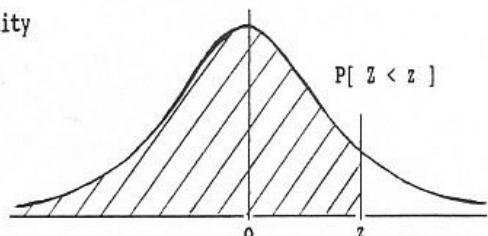
STANDARD STATISTICAL TABLES

1. Areas under the Normal Distribution

The table gives the cumulative probability up to the standardised normal value z

i.e.

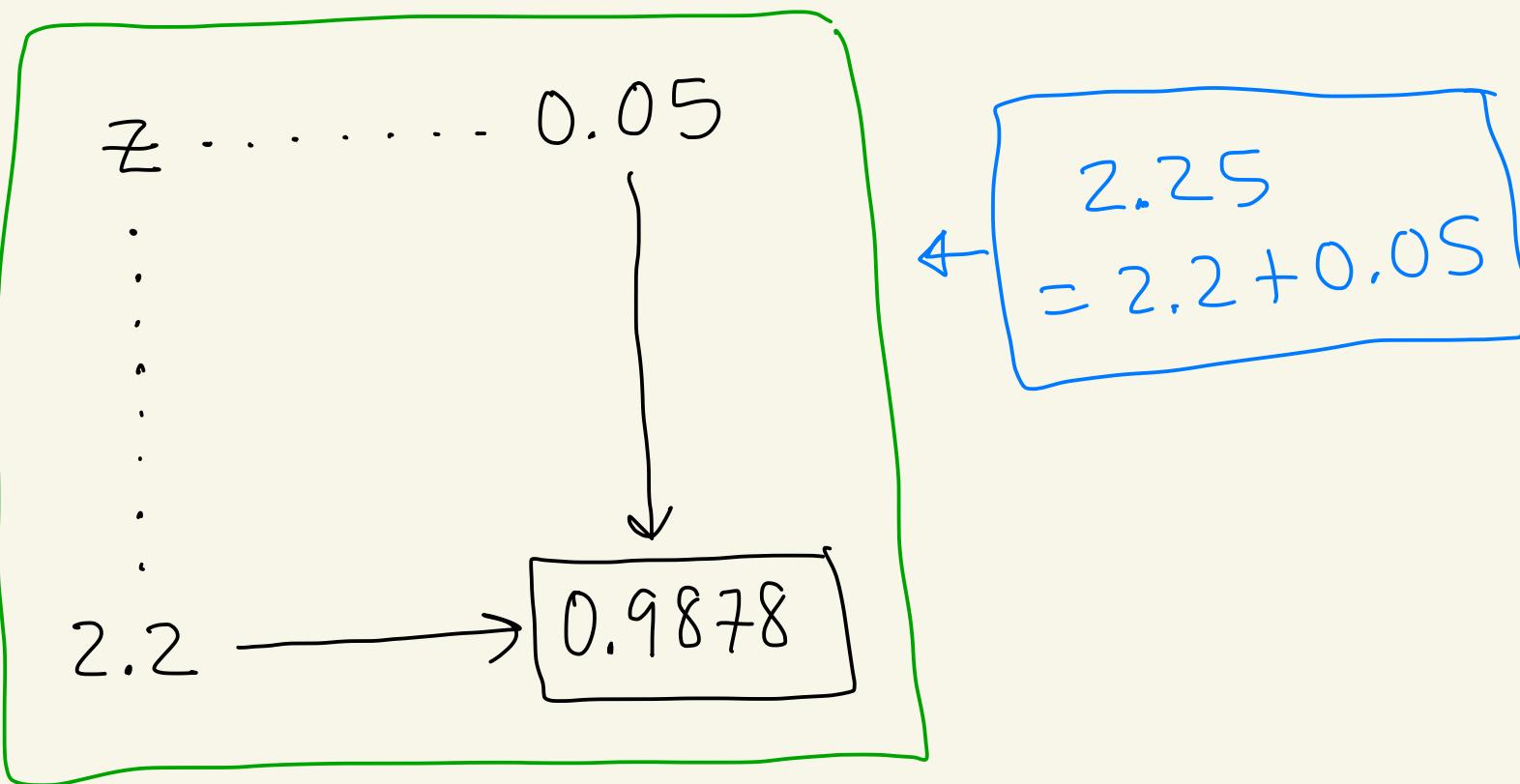
$$P[Z < z] = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5159	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7854
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8804	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9773	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9865	0.9868	0.9871	0.9874	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9924	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9980	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
z	3.00	3.10	3.20	3.30	3.40	3.50	3.60	3.70	3.80	3.90
P	0.9986	0.9990	0.9993	0.9995	0.9997	0.9998	0.9998	0.9999	0.9999	1.0000

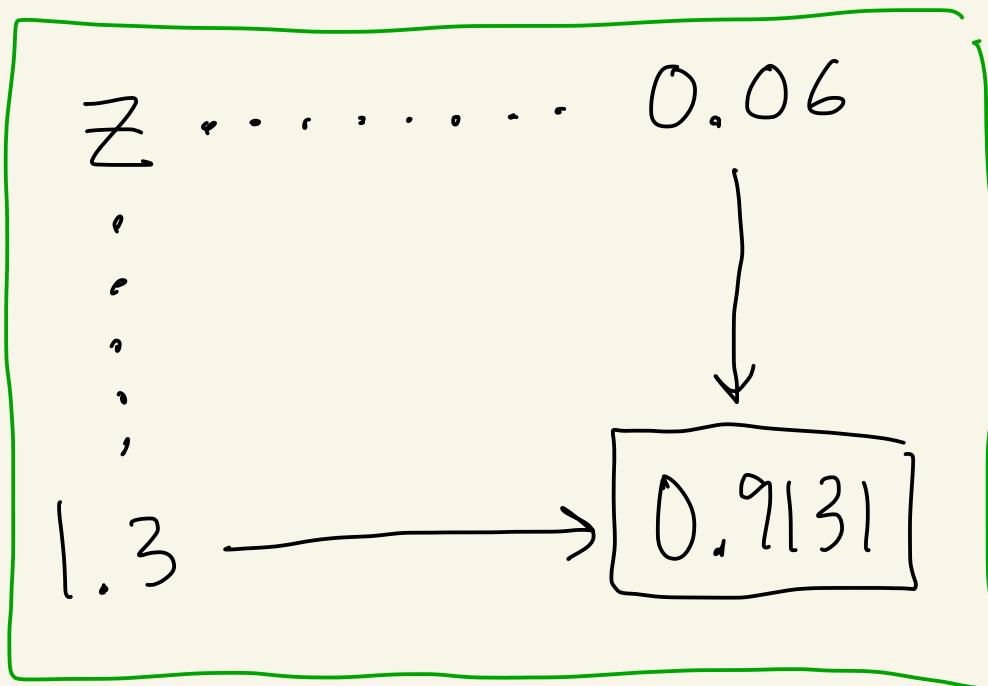
Let's see how to calculate
 $\Phi(t)$ when $t \geq 0$

Ex: Calculate $\Phi(2.25)$



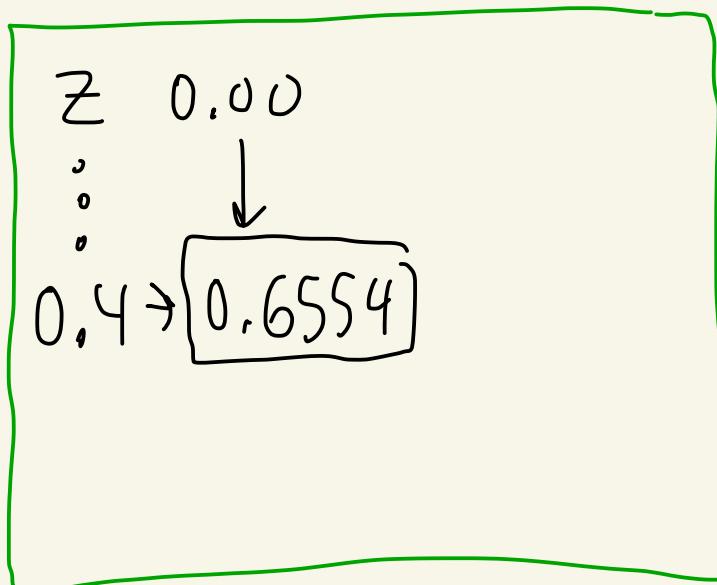
$$\text{So, } \Phi(2.25) \approx 0.9878$$

Ex: Calculate $\Phi(1.36)$



$$S_0, \Phi(1.36) \approx 0.9131$$

Ex: Calculate $\Phi(0.4)$



$$S_0, \Phi(0.4) \approx 0.6554$$

In the table they have

$$\Phi(3.9) \approx 1$$

but it's smaller than 1.

Would need a better table to get
a better approximation,

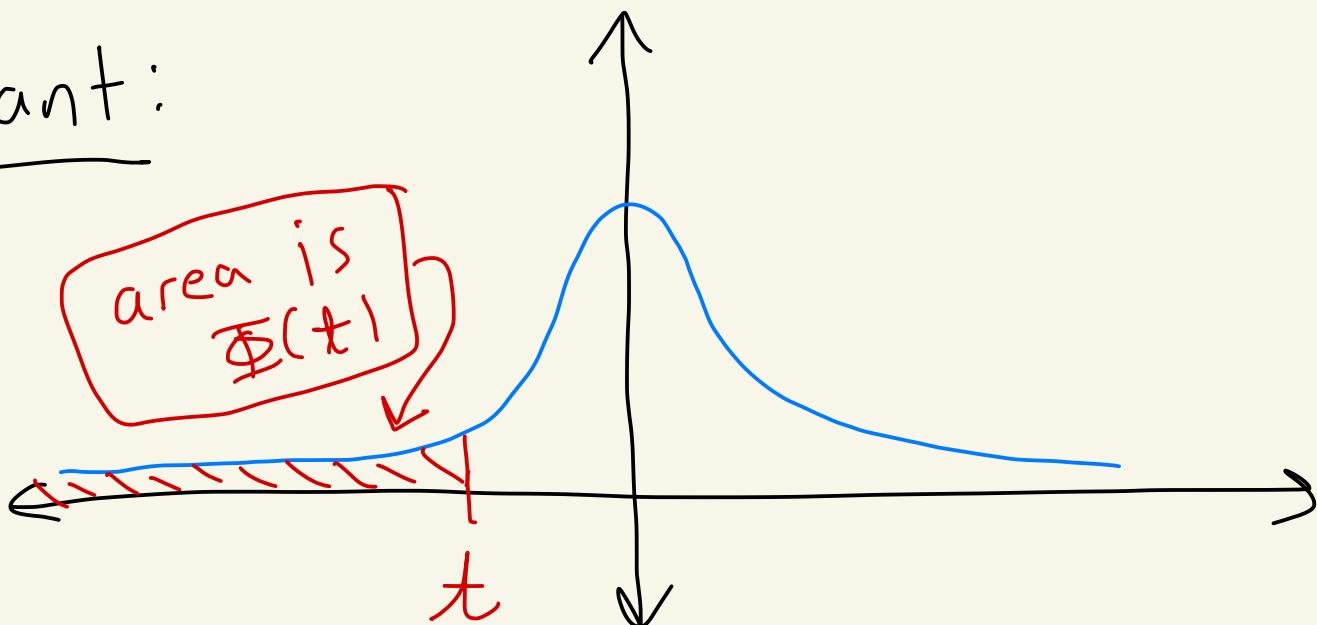
If $t \geq 3.9$ you could approx.

$$\Phi(t) \approx 1$$

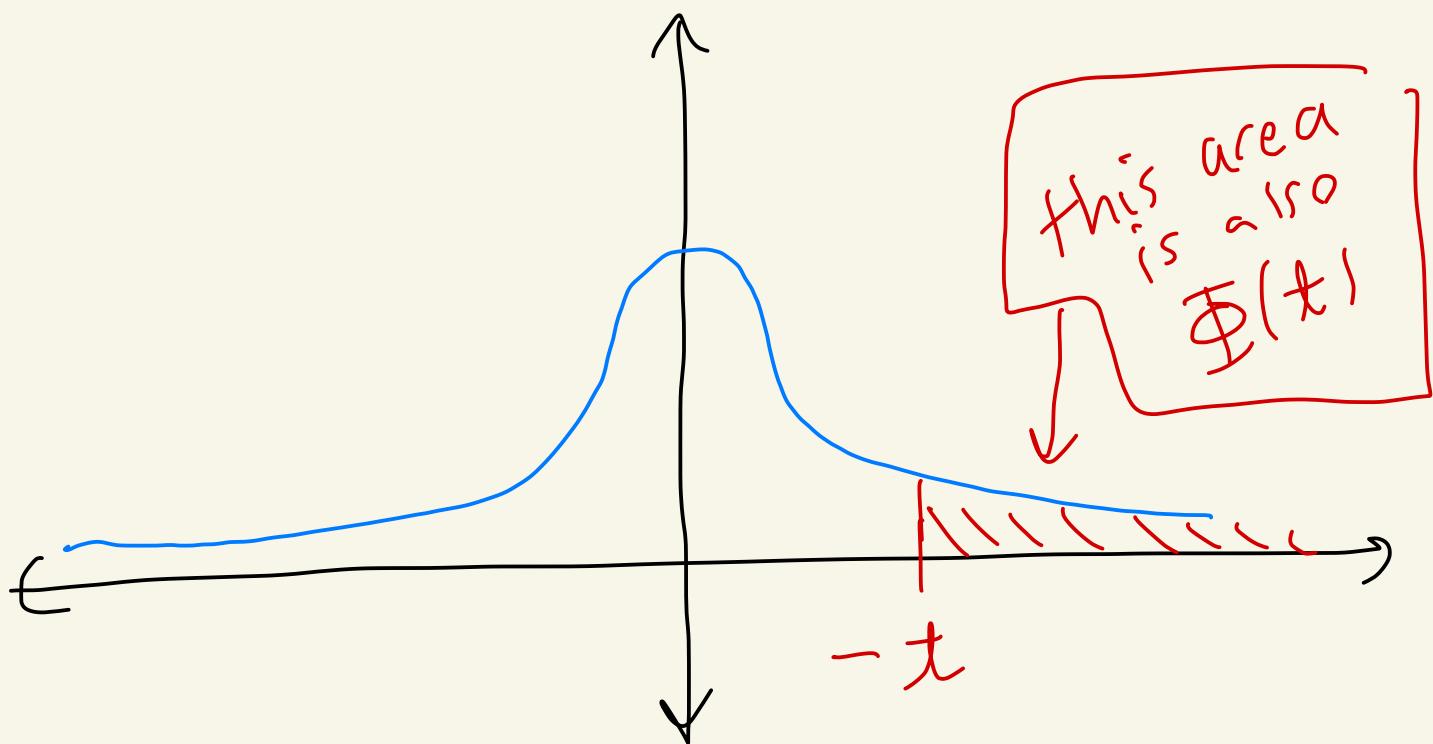
It's close to but less than 1.

How to calculate $\Phi(t)$ when $t < 0$

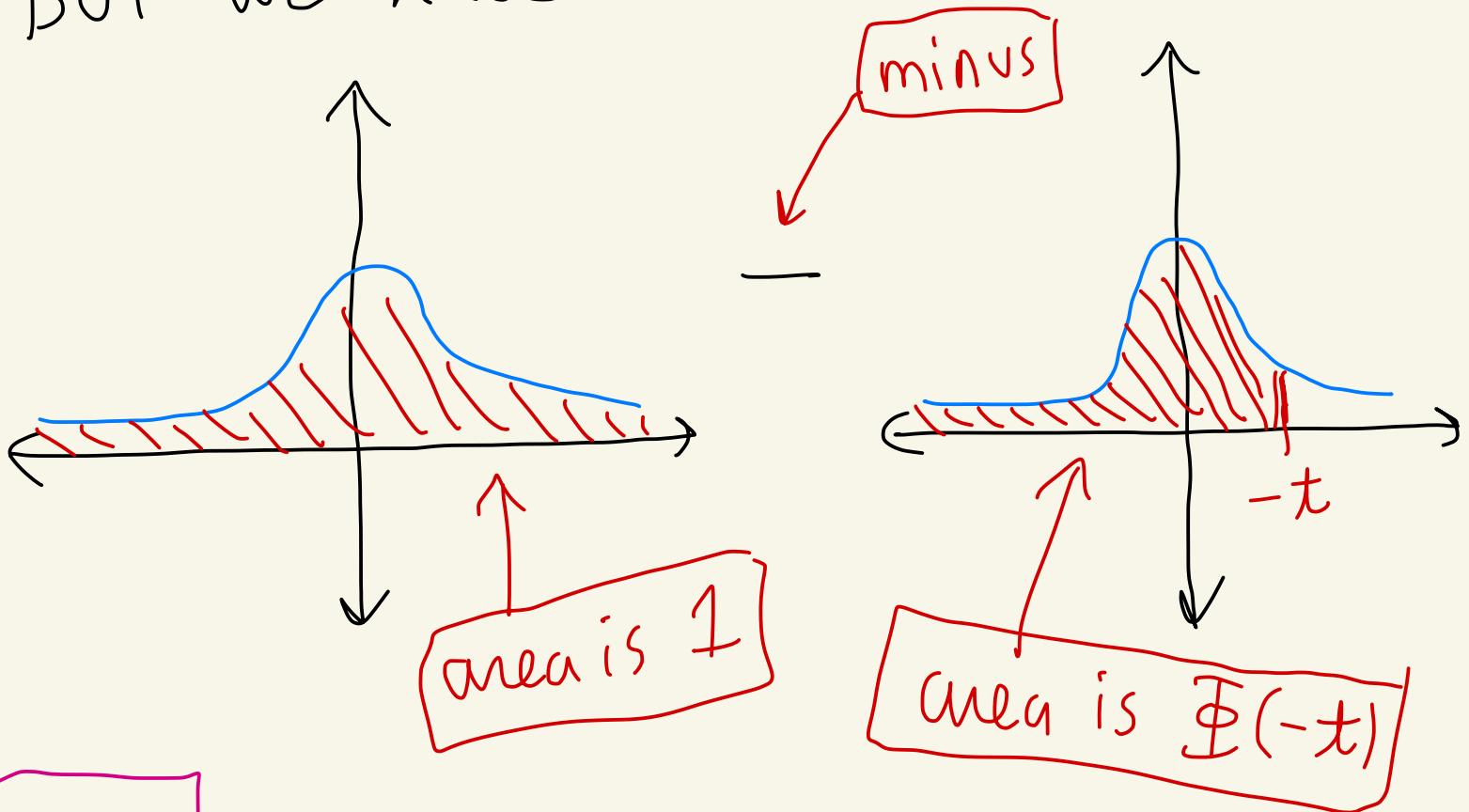
want:



by symmetry (since $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is an even function)



But we have the above area is



So,

$$\Phi(t) = 1 - \Phi(-t) \text{ if } t < 0$$

Ex:

$$\Phi(-2.68) = 1 - \Phi(2.68)$$

$$\approx 1 - 0.9963 \approx 0.0037$$

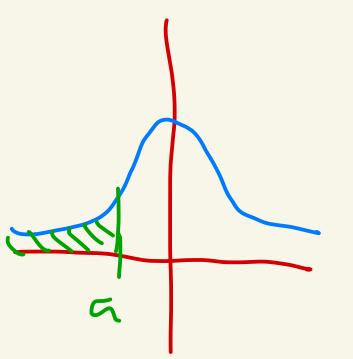
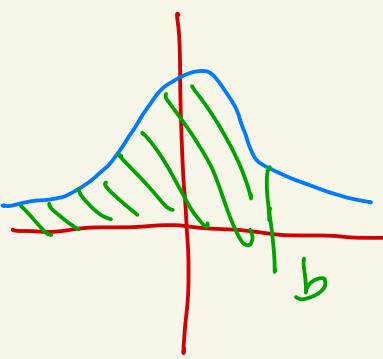
Theorem: (De Moivre - Laplace Theorem)

Let \bar{X} be a binomial random variable with parameters n and p . Then for any real numbers a and b with $a < b$ we have that

$$\lim_{n \rightarrow \infty} P\left(a \leq \frac{\bar{X} - np}{\sqrt{np(1-p)}} \leq b\right) = \Phi(b) - \Phi(a)$$
$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx$$

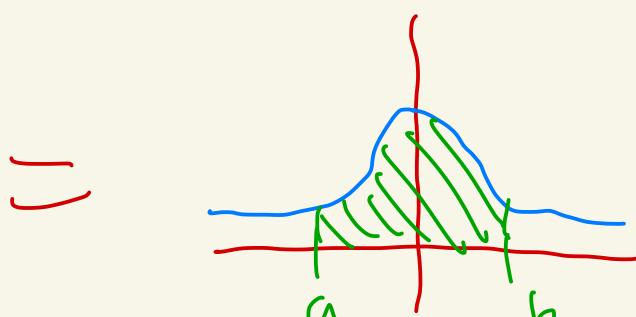
$$E[\bar{X}] = np$$

$$\sigma_{\bar{X}} = \sqrt{np(1-p)}$$



area is $\Phi(b)$

area is $\Phi(a)$



=

You can also do:

$$\lim_{n \rightarrow \infty} P\left(\frac{\bar{X} - np}{\sqrt{np(1-p)}} \leq b\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b e^{-x^2/2} dx$$

"a = -\infty"

$$= \Phi(b)$$

Ex: Suppose we flip a coin 10,000 times. Let \bar{X} be the number of heads that occur.

Approximate the probability that $5000 \leq \bar{X} \leq 5002$.

Here \bar{X} is a binomial random variable with $n = 10,000$ and $p = \frac{1}{2}$.

$$\text{So, } np = 5000 \text{ and } \sqrt{np(1-p)} = \sqrt{2500} = 50$$

Thus,

$$P(5000 \leq \bar{X} \leq 5002)$$

$$= P\left(\frac{\bar{X} - 5000}{50} \leq \frac{\bar{X} - 5000}{50} \leq \frac{5002 - 5000}{50}\right) \approx 0.04$$

$$\approx P\left(0 \leq \frac{\bar{X} - 5000}{50} \leq 0.04\right)$$

$$\approx \Phi(0.04) - \Phi(-0.04) \approx 0.5159 - 0.5$$

$n = 10,000$
is a big #

DeMoivre
Laplace

$$\approx 0.0159$$

$\approx 1.59\%$

Ex: Suppose you flip a coin 40 times. Let \bar{X} be the number of heads.

Approximate $P(\bar{X} = 20)$.

We have:

$$\begin{array}{l} n=40 \\ P=\frac{1}{2} \end{array} \quad \left. \begin{array}{l} np=20 \\ \sqrt{np(1-p)}=\sqrt{10} \end{array} \right\}$$

$$P(\bar{X}=20) = P(19.5 \leq \bar{X} \leq 20.5)$$

\bar{X} can
only be
Whole #
values

$$= P\left(\frac{19.5-20}{\sqrt{10}} \leq \frac{\bar{X}-20}{\sqrt{10}} \leq \frac{20.5-20}{\sqrt{10}}\right)$$

$$= P\left(\frac{\bar{X}-np}{\sqrt{np(1-p)}}\right)$$

$$\approx P(-0.16 \leq \frac{\bar{X}-20}{\sqrt{10}} \leq 0.16)$$

$$\approx \Phi(0.16) - \Phi(-0.16)$$

DeMoivre
Laplace
even though
 $n=40$
is small

$$= \Phi(0.16) - [1 - \Phi(0.16)]$$

$\Phi(-x) = 1 - \Phi(x)$

$$= 2\Phi(0.16) - 1$$

$$\approx 2[0.5636] - 1$$

$$\approx 0.1272 \approx 12.72\%$$

Is this accurate? Yes!

$$P(X=20) = \binom{40}{20} \cdot \left(\frac{1}{2}\right)^{20} \left(1-\frac{1}{2}\right)^{40-20}$$

$$= \frac{137,846,528,820}{1,099,511,627,776}$$

$$\approx 0.125371 \approx 12.54\%$$