

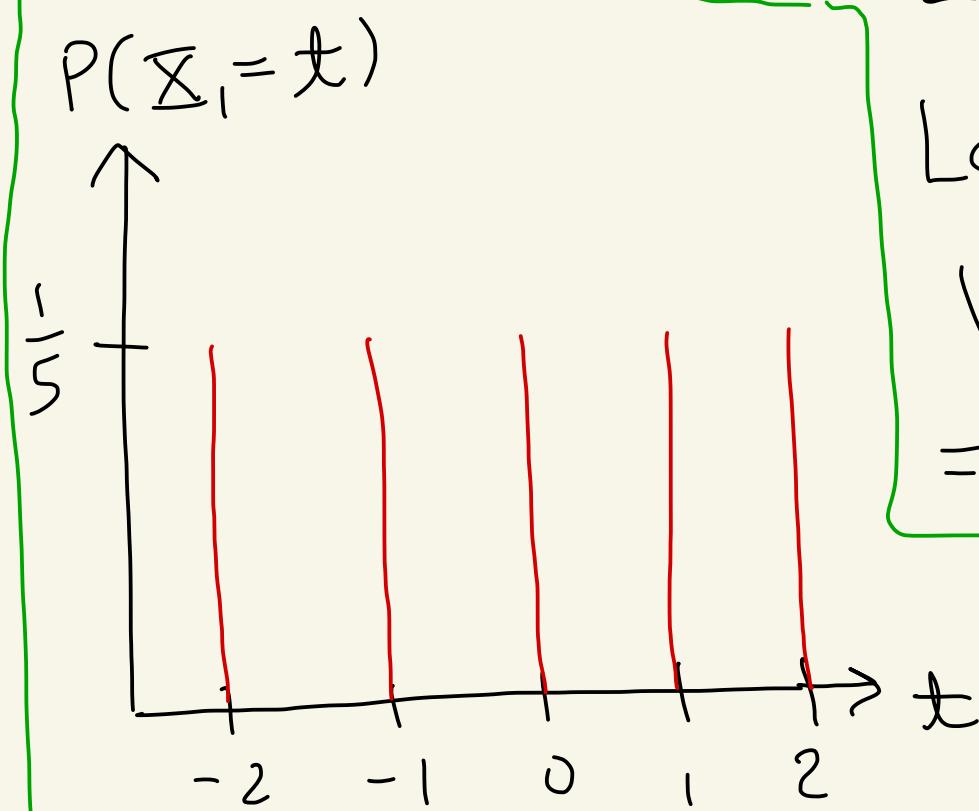
Math 4740

11/27/23



Previously we had two examples with the same expected value of 0 but the data was spread out differently. Let's called the variance / standard deviation of those examples.

Ex:



We calculated $E[X_1] = 0$.

Let's calculate

$$\begin{aligned} \text{Var}(X_1) &= E[X_1^2] - (E[X_1])^2 \\ &= E[X_1^2] - 0^2 \end{aligned}$$

$$= E[X_1^2]$$

We have

$$E[\bar{X}_1^2] = (-2)^2 \cdot \underbrace{\left(\frac{1}{5}\right)}_{P(\bar{X}_1 = -2)} + (-1)^2 \cdot \underbrace{\left(\frac{1}{5}\right)}_{P(\bar{X}_1 = -1)}$$

$$+ (0)^2 \left(\frac{1}{5}\right) + (1)^2 \left(\frac{1}{5}\right) + (2)^2 \left(\frac{1}{5}\right)$$

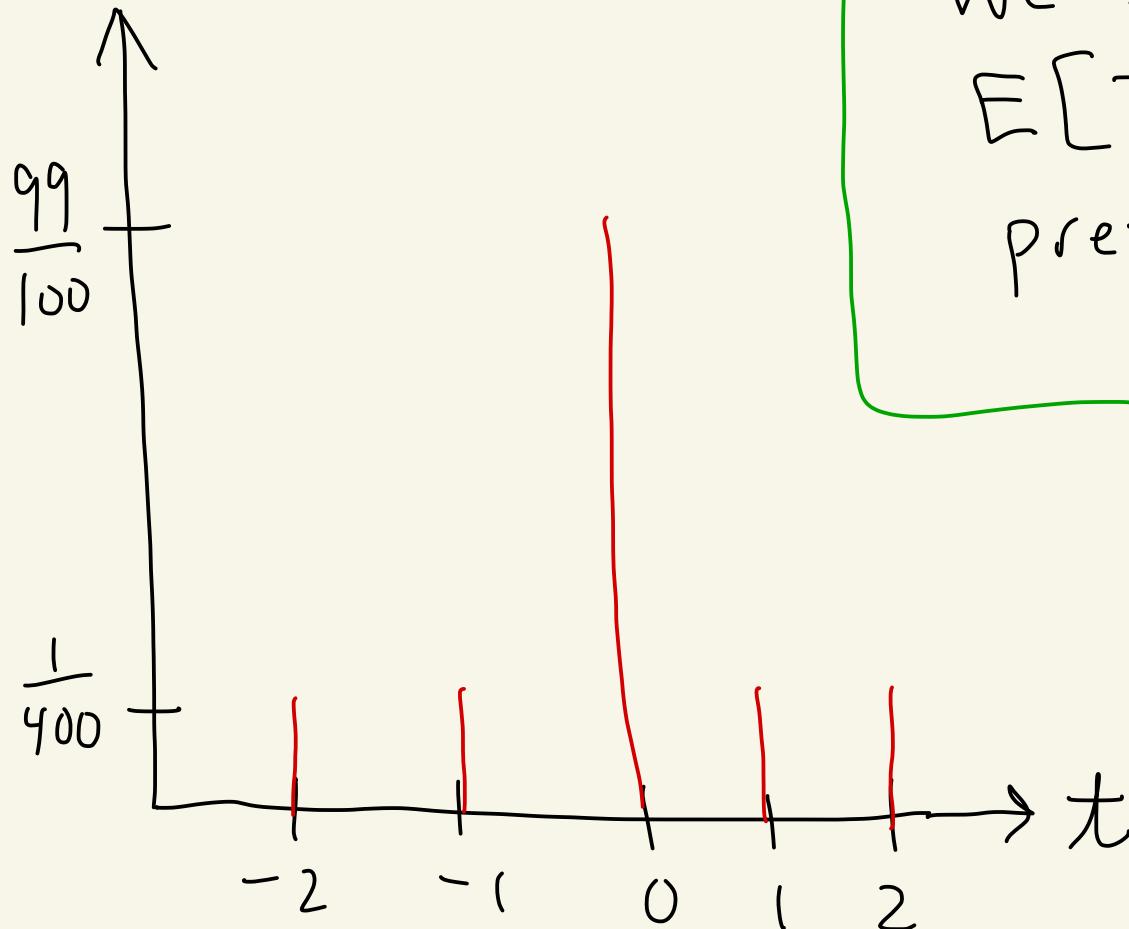
$$= (4 + 1 + 0 + 1 + 4) \cdot \frac{1}{5} = 2$$

So, $\boxed{\text{Var}(\bar{X}_1^2) = 2}$

Then, $\boxed{\sigma_{\bar{X}_1} = \sqrt{\text{Var}(\bar{X}_1^2)} = \sqrt{2} \approx 1.414}$

We also had the following example:

$$P(\bar{X}_2 = t)$$



We saw that
 $E[\bar{X}_2] = 0$
 previously.

Thus,

$$\begin{aligned} \text{Var}(\bar{X}_2) &= E[\bar{X}_2^2] \\ &= E[\bar{X}_2]^2 - \underbrace{\left(E[\bar{X}_2]\right)^2}_{0} \\ &= E[\bar{X}_2^2] \end{aligned}$$

And,

$$\begin{aligned} E[\bar{X}_2^2] &= (-2)^2 \cdot \underbrace{\left(\frac{1}{400}\right)}_{P(\bar{X}_2 = -2)} + (-1)^2 \left(\frac{1}{400}\right) \\ &\quad + (0)^2 \left(\frac{99}{100}\right) + (1)^2 \left(\frac{1}{400}\right) + (2)^2 \left(\frac{1}{400}\right) \\ &= \frac{10}{400} = \frac{1}{40} \end{aligned}$$

Thus,

$$\text{Var}(\bar{X}_2) = \frac{1}{40}$$

$$\sigma_{\bar{X}_2} = \sqrt{\text{Var}(\bar{X}_2)} = \sqrt{\frac{1}{40}} \approx 0.158$$

Theorem: Let \bar{X} be a binomial random variable with parameters n and p . Then,

$$\text{Var}(\bar{X}) = np(1-p)$$

$$\sigma_{\bar{X}} = \sqrt{np(1-p)}$$

Proof: Recall that $E[\bar{X}] = np$.

We have that

$$\begin{aligned}
 E[\bar{X}^2] &= \sum_{i=0}^n i^2 \binom{n}{i} p^i (1-p)^{n-i} \\
 &= \sum_{i=1}^n i^2 \frac{n!}{i!(n-i)!} \cdot p^i (1-p)^{n-i} \\
 &= np \sum_{i=1}^n i \frac{n!}{(i-1)!(n-i)!} \cdot p^{i-1} (1-p)^{n-i} \\
 &\quad \text{(circled: } k = i-1 \text{)} \\
 &= np \sum_{k=0}^{n-1} (k+1) \frac{(n-1)!}{k!((n-1)-k)!} \cdot p^k (1-p)^{(n-1)-k}
 \end{aligned}$$

$$= np \sum_{k=0}^{n-1} k \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$$

$E[\bar{X}]$ where \bar{X} is a binomial random variable w/ parameters $n-1$ & p .

$$+ np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$$

use $(a+b)^l = \sum_{j=0}^l \binom{l}{j} a^j b^{l-j}$

binomial thm:

$$= (np) \cdot (n-1) \cdot p + (np) \cdot (p + (1-p))^{n-1}$$

$$= n^2 p^2 - np^2 + np$$

Thus,
 $\text{Var}(\bar{X}) = E[\bar{X}^2] - (E[\bar{X}])^2$

$$= n^2 p^2 - np^2 + np - (np)^2$$

$$= np - np^2 = np(1-p).$$



Ex: Suppose we flip a coin 100 times. Let Σ be the number of heads that occur.

Then, Σ is a binomial random variable with $n = 100$ and $p = \frac{1}{2}$

Probability of heads
on a single flip

Then,

$$E[\Sigma] = np = 100\left(\frac{1}{2}\right) = 50$$

$$\text{Var}(\Sigma) = np(1-p) = 100\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right) = 25$$

$$\sigma_{\Sigma} = \sqrt{25} = 5$$

Showed in topic 5

Theorem (Markov's Inequality)

Let \bar{X} be a non-negative discrete random variable.

non-negative means:

$\bar{X}(w) \geq 0$ for all w in the sample space

Let $\mu = E[\bar{X}]$.

Then for any real number $t > 0$ we have that

$$P(\bar{X} \geq t) \leq \frac{\mu}{t}$$

(since μ is fixed)

Note: $\frac{\mu}{t} \rightarrow 0$ as $t \rightarrow \infty$

$$P(X=k)$$



add all these to get $P(\bar{X} \geq t)$

Proof:

Let A be the range of the function \bar{X} .

Let

$$B = \{x \mid x \in A \text{ and } x \geq t\}.$$

Then,

$$E[\bar{X}] = \sum_{x \in A} x \cdot P(\bar{X} = x)$$

$$\geq \sum_{x \in B} x \cdot P(\bar{X} = x)$$

since
 $x \geq t$
if $x \in B$

$$\geq \sum_{x \in B} t \cdot P(\bar{X} = x)$$

since
 $B \subseteq A$
and
 \bar{X} is
non-negative

$$= t \sum_{x \in B} P(\bar{X} = x) = t P(\bar{X} \geq t)$$

Thus, $P(\bar{X} \geq t) \leq \frac{E[\bar{X}]}{t}$. \square

Theorem: (Chebychev's Inequality)

Let \bar{X} be a discrete random variable. Let $\mu = E[\bar{X}]$.

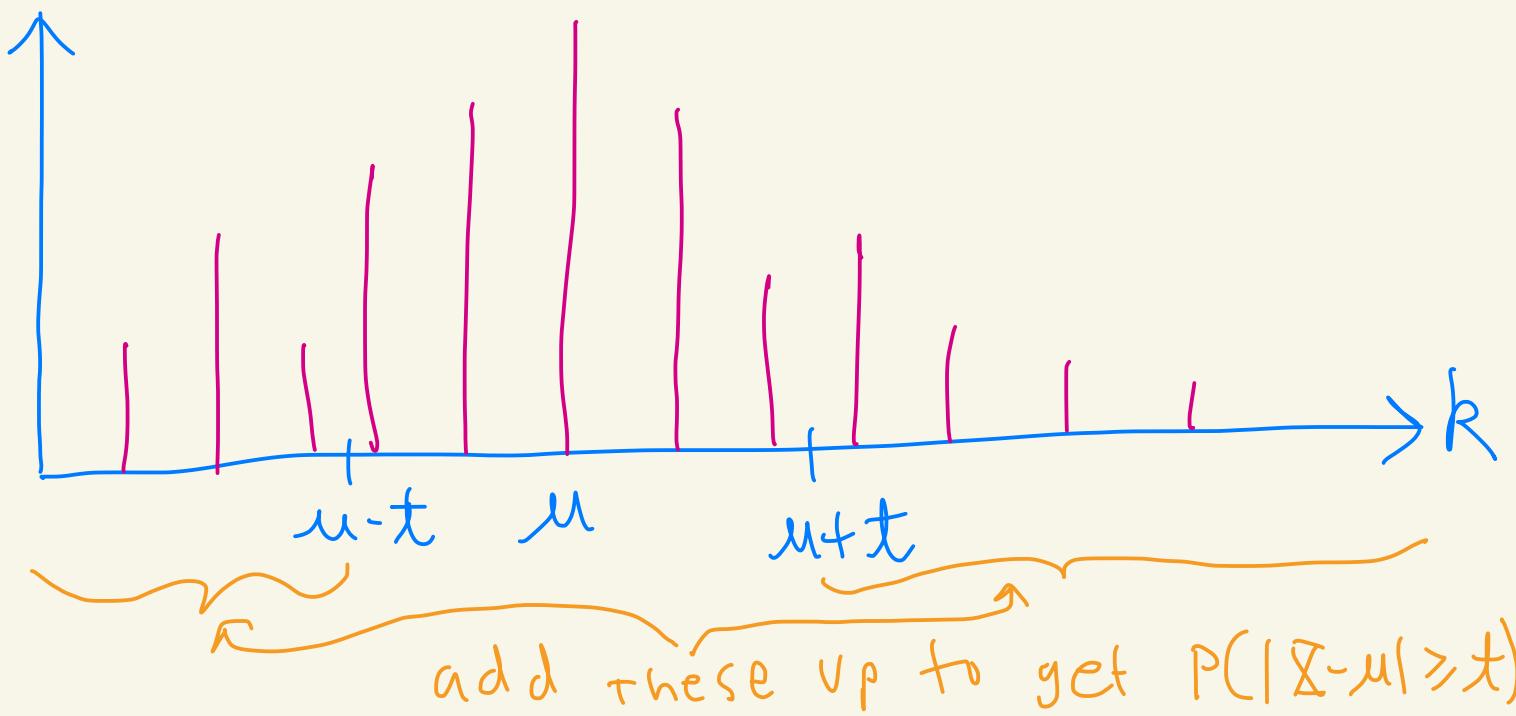
Let $\sigma = \sqrt{\text{Var}(\bar{X})}$.

Then for any $t > 0$, we have

$$P(|\bar{X} - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

means: $P(\{\omega \mid \omega \in S \text{ with } |\bar{X}(\omega) - \mu| \geq t\})$

$$P(\bar{X} = k)$$



Proof: The random variable $(\bar{X} - \mu)^2$ is non-negative.

So, Markov's inequality gives:

$$P((\bar{X} - \mu)^2 \geq t^2) \leq \frac{E[(\bar{X} - \mu)^2]}{t^2}$$

same as

$$P(|\bar{X} - \mu| \geq t)$$

$$= \frac{\text{Var}(\bar{X})}{t^2}$$

$$= \frac{\sigma^2}{t^2}$$



Ex: (HW 6 #5(b))

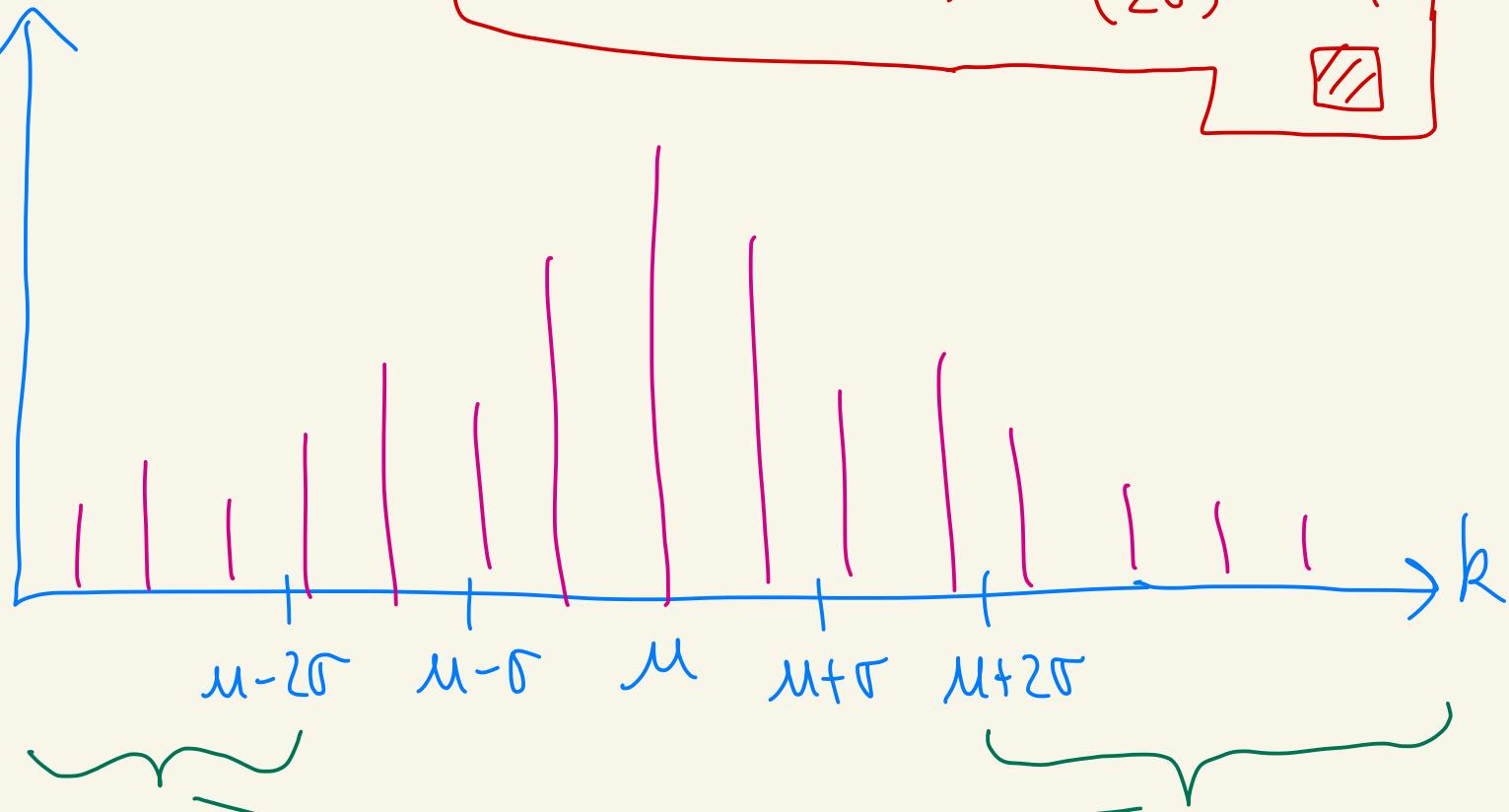
Let \bar{X} be a discrete random variable with $\mu = E(\bar{X})$ and $\sigma = \sqrt{\text{Var}(\bar{X})}$.

Show that $P(|\bar{X} - \mu| \geq 2\sigma) \leq \frac{1}{4}$

$$P(\bar{X} = k)$$

Pf: By Chebyshev:

$$P(|\bar{X} - \mu| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4}$$



add up to get $P(|\bar{X} - \mu| \geq 2\sigma)$