Math 4740

11/1/23

Topic 5 - Binomial Random Variables

A Bernoulli trial is an experiment with two possible outcomes: success or failure.

Suppose success occurs with probability p and failure with probability $1-p$.

Ex: Experiment = flipping a coin

success = heads $\leftarrow p = 1/2$
failure = tails $\leftarrow 1-p = 1/2$

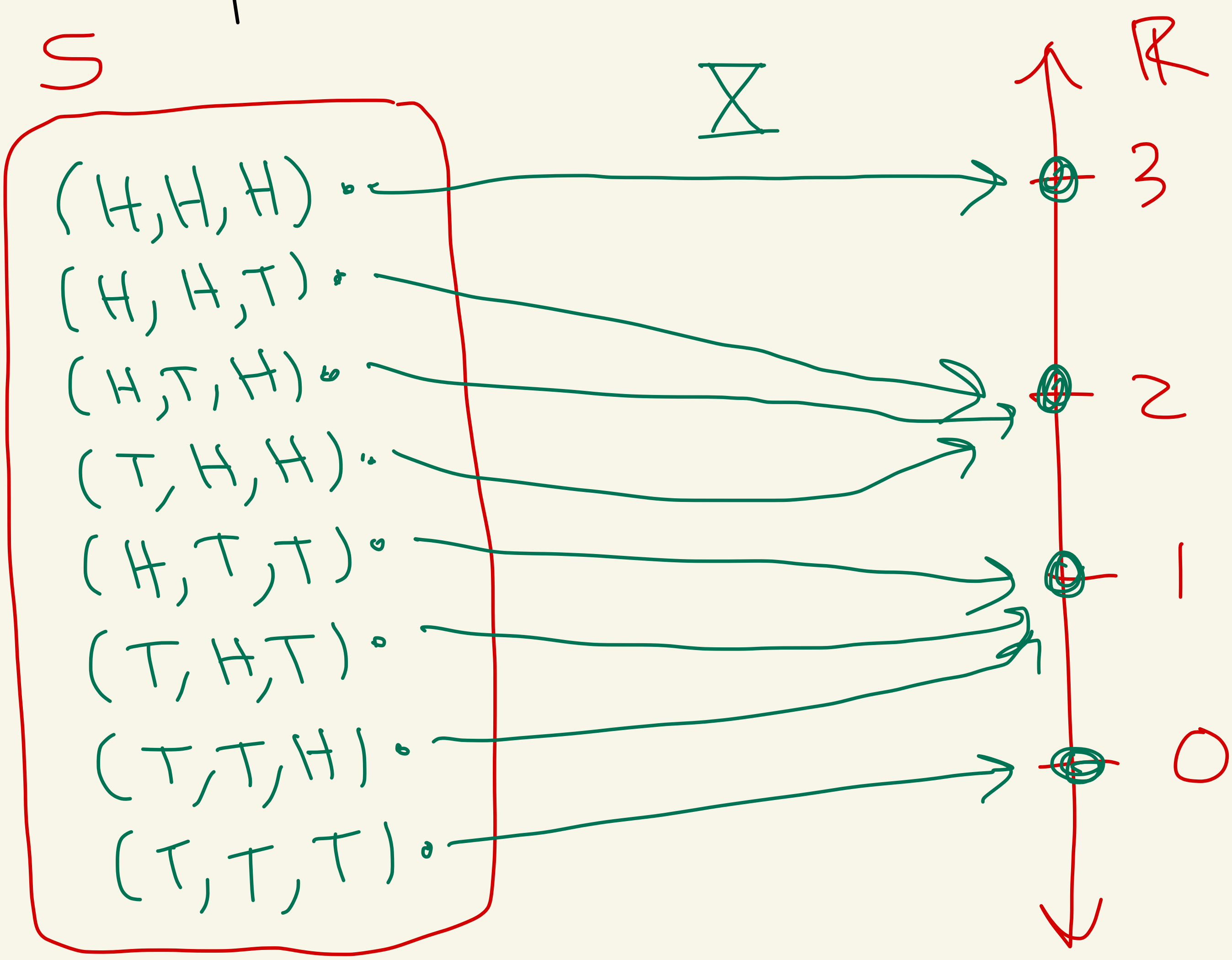
Ex: Experiment = rolling two 6-sided die

success = sum of dice is 7 ← $p = 6/36$

failure = sum of dice is not 7 ← $1-p = 30/36$

Now suppose that n Bernoulli trials, each with success p, are performed independently. Let X be the number of successes. Then X is called a <u>binomial random variable with parameters n and p</u>.

Ex: Suppose the Bernoulli trial is flipping a coin and success is heads with probability $p = 1/2$. Let's repeat this experiment $n=3$ times and let X be the number of heads that occur. Then X is a binomial random variable with parameters $n=3$, $p = 1/2$.



<u>Theorem</u>: Let X be a binomial random variable with parameters n and p. Then,

$$P(X=k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{if } k=0,1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

<u>proof</u>: Let k be $0,1,2,\ldots,$ or n.

Let's calculate $P(X=k)$.

How many ways can you get exactly k successes in n trials?

<u>Ex</u>: $n=4$, $k=2$

$\underline{s}$ $\underline{s}$ $\underline{f}$ $\underline{f}$ $\leftarrow \binom{4}{2}=6$

pick 2 spots where the successes go

ssff sffs fsfs
sfsf fssf ffss

For general n and k, there are $\binom{n}{k}$ ways you can get exactly k successes in n trials.

Each of these sequences has probability $\underbrace{p^k}_{k \text{ successes}}\underbrace{(1-p)^{n-k}}_{n-k \text{ failures}}$

because of independence.

Ex:

$$\underline{s} \quad \underline{s} \quad \underline{f} \quad \underline{f}$$

$$p \cdot p \cdot (1-p) \cdot (1-p) = p^2 (1-p)^{4-2}$$

Thus, $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$.

If $k \neq 0, 1, \ldots, n$, then X can never equal k, so

$P(X = k) = 0$. $\blacksquare$

Ex: Suppose we flip a coin 3 times. What is the probability that exactly 2 heads occur?

Let $n = 3$, success = head,

$p = 1/2$, X = # of heads in 3 flips.

flips

$$P(X=2) = \binom{n}{2} p^2 (1-p)^{3-2}$$

successes

$$= \binom{3}{2} \cdot \underbrace{\left(\tfrac{1}{2}\right)^2}_{\text{2 successes}} \underbrace{\left(1-\tfrac{1}{2}\right)^1}_{\text{1 failure}}$$

or heads or tails

$$= 3 \cdot \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right) = \frac{3}{8}$$

<u>Ex:</u> What if flip a coin 100 times and you want the probability of exactly 48 heads occuring?

$n = 100$ ← # flips

$k = 48$ ← # successes/heads

$p = 1/2$ ← probability of one success

$1 - p = 1 - 1/2 = 1/2$ ← probability of one failure

X = # successes/heads

$$P(X=48) = \binom{100}{48}\cdot\left(\frac{1}{2}\right)^{48}\left(1-\frac{1}{2}\right)^{100-48}$$

$$= \binom{100}{48}\cdot\left(\frac{1}{2}\right)^{48}\cdot\left(\frac{1}{2}\right)^{52}$$

$$= \binom{100}{48}\cdot\frac{1}{2^{100}}$$

$$= \frac{93{,}206{,}558{,}875{,}049{,}876{,}949{,}581{,}681{,}100}{1{,}267{,}650{,}600{,}228{,}229{,}401{,}496{,}703{,}205{,}376}$$

$$\approx 0.073527 \approx \boxed{7.35\%}$$

Ex: Suppose we flip a coin 20 times. What is the probability of getting between 10 and 12 heads? (ie 10, 11, or 12 heads).

$n = 20 \leftarrow$ # flips

$p = 1/2 \leftarrow$ probability of success/heads

$1-p = 1/2 \leftarrow$ probability of failure/tails

X = # of successes/heads

$$P(10 \leq X \leq 12)$$

$$= P(X=10) + P(X=11) + P(X=12)$$

$$= \binom{20}{10} \cdot \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{10}}_{\text{failure}} + \binom{20}{11} \underbrace{\left(\frac{1}{2}\right)^{11}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{9}}_{\text{failure}}$$

$$+ \binom{20}{12} \underbrace{\left(\frac{1}{2}\right)^{12}}_{\text{success}} \underbrace{\left(\frac{1}{2}\right)^{8}}_{\text{failure}}$$

$$= \frac{\binom{20}{10} + \binom{20}{11} + \binom{20}{12}}{2^{20}}$$

$$= \frac{184{,}756 + 167{,}960 + 125{,}970}{1{,}048{,}576}$$

$$\approx 0.456511\ldots \approx \boxed{45.65\%}$$

<u>Ex:</u> Suppose we roll two 6-sided dice 20 times. Suppose we say that a sum of 7 or 11 on the die is a success, and any other sum is a failure.

Let X = # of successes.

What is $P(X=12)$?

$$n = 20$$

$$p = \underbrace{\frac{6}{36}}_{\text{sum is } 7} + \underbrace{\frac{2}{36}}_{\text{sum is } 11} = \frac{8}{36}$$ ← probability of success

$$1-p = 1 - \frac{8}{36} = \frac{28}{36}$$ ← probability of failure

$$P(X=12) = \binom{20}{12} \cdot \underbrace{\left(\frac{8}{36}\right)^{12}}_{12 \text{ successes}} \cdot \underbrace{\left(\frac{28}{36}\right)^{8}}_{8 \text{ failures}}$$

$$= \binom{20}{12} \cdot \left(\frac{2}{9}\right)^{12} \cdot \left(\frac{7}{9}\right)^{8}$$

$$= \frac{\binom{20}{12} \cdot 2^{12} \cdot 7^{8}}{9^{20}}$$

$$= \frac{(125{,}970)(4096)(5{,}764{,}801)}{12{,}157{,}665{,}459{,}056{,}928{,}801}$$

$$\approx 0.000244659 \approx \boxed{0.024\%}$$

<u>Thm:</u> Let X be a binomial random variable with parameters n and p, then $E[X] = np$.

<u>proof:</u> See Spring notes. $\blacksquare$

Intuition: Say we tossed a coin 100 times then we would expect the average # of heads to be $n \cdot p = 100 \cdot \frac{1}{2} = 50$