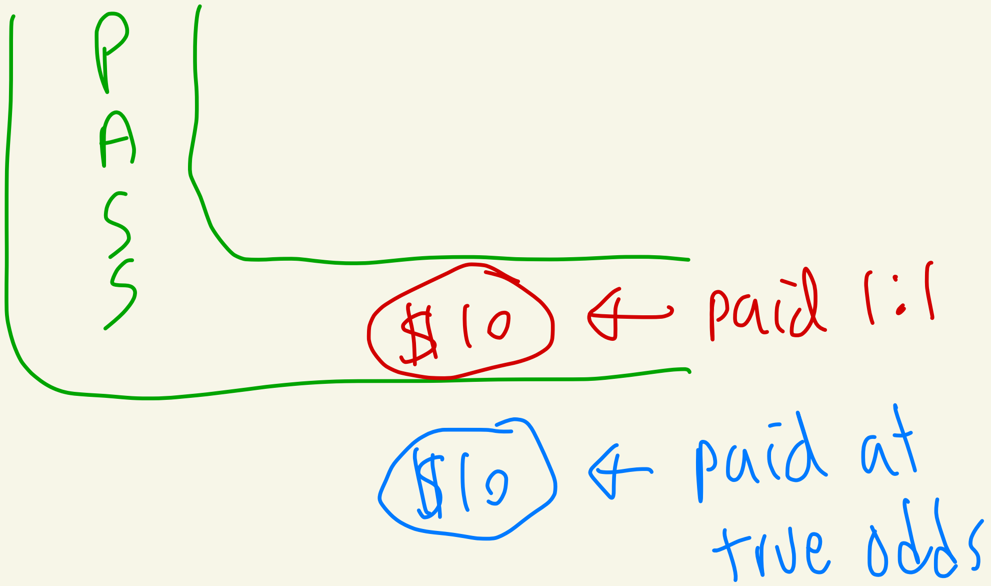
Math 4740
10/30/23

Let's look at an expected value for free odds betting.

Suppose you bet \$10 on the pass line and if a point is made then you bet an additional \$10 as a free odds bet. Let X be the amount won or lost.

Let's calculate $E[X]$.



$$E[X] = \underbrace{(\$10)\left(\frac{8}{36}\right)}_{\text{7 or 11 on come out roll}} + \underbrace{(-\$10)\left(\frac{4}{36}\right)}_{\text{2, 3, or 12 on come out roll}}$$

$$+ \underbrace{\overset{\boxed{\text{4 or 10}}\downarrow}{2}\cdot(\$30)\left(\frac{3}{36}\right)\left(\frac{3}{9}\right)}_{\text{4 or 10 as point and we won}} + \underbrace{\overset{\boxed{\text{4 or 10}}\downarrow}{2}\cdot(-\$20)\left(\frac{3}{36}\right)\left(\frac{6}{9}\right)}_{\text{4 or 10 as point and we lost}}$$

$10 ← pass line 1:1

$20 ← free odds 2:1

$30

-$10 ← pass line

-$10 ← free odds

-$20

$$+ \underbrace{\overset{\boxed{\text{5 or 9}}\downarrow}{2}\cdot(\$25)\left(\frac{4}{36}\right)\left(\frac{4}{10}\right)}_{\text{5 or 9 as Point and we won}} + \underbrace{\overset{\boxed{\text{5 or 9}}\downarrow}{2}\cdot(-\$20)\left(\frac{4}{36}\right)\left(\frac{6}{10}\right)}_{\text{5 or 9 as point and we lost}}$$

$10 ← pass line 1:1

-$10 ← Pass line

$15 ← free odds 3:2
$25

−$10 ← free odds
−$20

$$+\overset{\boxed{\text{6 or 8}}}{\downarrow}\!\!\underbrace{2\cdot(\$22)\left(\frac{5}{36}\right)\left(\frac{5}{11}\right)}_{\text{6 or 8 as point and we won}} + \overset{\boxed{\text{6 or 8}}}{\downarrow}\!\!\underbrace{2\cdot(-\$20)\left(\frac{5}{36}\right)\left(\frac{6}{11}\right)}_{\text{6 or 8 as point and we lost}}$$

$10 ← pass line 1:1
$12 ← free odds 6:5
$22

−$10 ← pass line
−$10 ← free odds
−$20

$$=\boxed{-\$\frac{14}{99}} \approx \boxed{-\$0.1414}$$

So if you follow this betting strategy then you lose about 14¢ per game in the long run.

Let's now put the above in "per \$1 bet" terms to compare with our non-free odds betting expected value. Let's see what the average amount bet is with our \$10 pass line / \$10 free odds betting scheme is.

$$\begin{pmatrix}\text{average}\\ \text{amount}\\ \text{bet}\end{pmatrix} = \underbrace{\overbrace{(\$10)}^{\text{amount bet}}\overbrace{\left(\frac{12}{36}\right)}^{\text{probability we bet it}}}_{\text{come out roll is } 7, 11, 2, 3, 12} + \underbrace{\overbrace{(\$20)}^{\text{amount bet}}\overbrace{\left(\frac{24}{36}\right)}^{\text{probability we bet it}}}_{\text{come out roll } 4, 5, 6, 8, 9, 10}$$

$$= (\$10)\left(\frac{1}{3}\right) + (\$20)\left(\frac{2}{3}\right)$$

$$= \left(\$\frac{50}{3}\right) \approx \boxed{\$16.67}$$

Expected value per dollar wagered is

$$\frac{-\$0.1414}{\$16.67} \approx \boxed{-\$0.0085}$$

Recall that with just \$1 bet on the pass line and no free odds bet the expected value was $\boxed{-\$0.01414}$

St. Petersburg Paradox

Goes back to the 1700's.

A casino offers a game to a single player. A fair coin is tossed at each stage.

The pot (amount won) starts at \$2 and doubles every time a head is flipped. The first time a tail is flipped the game ends and the player wins whats in the pot.

How much would you pay to play this game? You don't get back what you paid, just what you win.

Ex: pay $5 to play

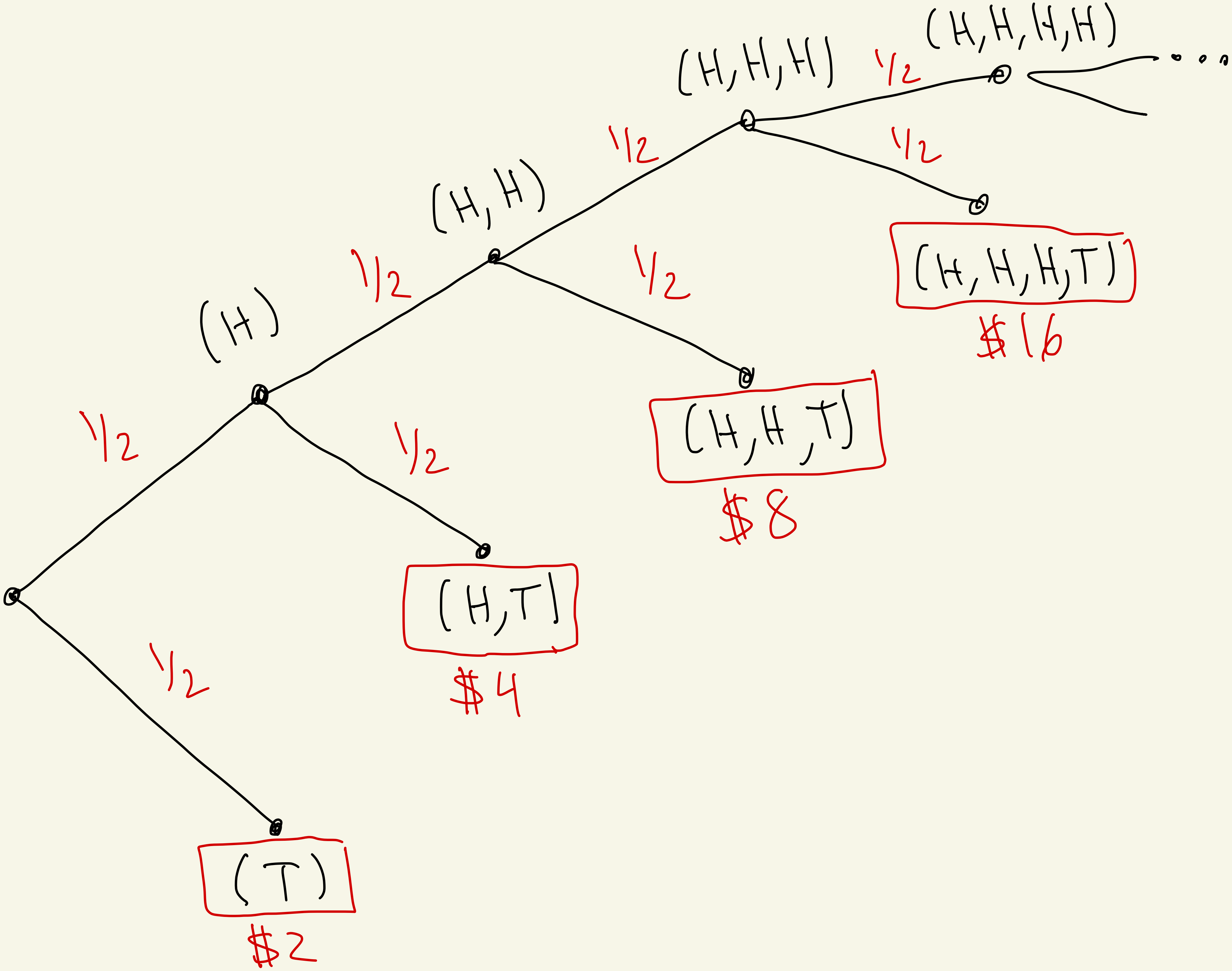
Pot (amount won)	flip
$2	H
$4	H
$8	H
$16	T

game stops

winnings = $16

amount paid = − $5

net amount won = $11


(H,H,H,H)
...
(H,H,H)
1/2
1/2
(H,H)
1/2
1/2
(H,H,H,T)
$16
(H)
1/2
1/2
(H,H,T)
$8
1/2
1/2
(H,T)
$4
1/2
(T)
$2

Let X be the net amount won.

$$E[X] = (-\$\text{ amount paid})$$

$$+ (\$2)\underbrace{\left(\frac{1}{2}\right)}_{\text{Probability won \$2}} + (\$4)\underbrace{\overbrace{\left(\frac{1}{4}\right)}^{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}}_{\text{Probability won \$4}}$$

$$+ (\$8)\underbrace{\left(\frac{1}{8}\right)}_{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} + (\$16)\left(\frac{1}{16}\right) + \dots$$

$$= (-\$\text{ amount paid})$$

$$+ \$1 + \$1 + \$1 + \dots$$

$$= \infty$$

This game has infinite expected value

But winning \$$2^n$ has probability $\frac{1}{2^n}$.

To win at least $\$2^{20} = \$1{,}048{,}576$ has probability equal to

$$\underbrace{\frac{1}{2^{20}}}_{\text{probability to win } \$2^{20}} + \underbrace{\frac{1}{2^{21}}}_{\text{probability to win } \$2^{21}} + \underbrace{\frac{1}{2^{22}}}_{\text{probability to win } \$2^{22}} + \cdots$$

$$= \frac{1}{2^{20}}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots\right]$$

$$= \frac{1}{2^{20}}\left[\frac{1}{1 - 1/2}\right] = \frac{1}{2^{20}}\left[2\right] = \frac{1}{2^{19}} \approx \Bigg($$

$$\uparrow$$

$$1 + x + x^2 + \cdots = \frac{1}{1-x} \quad \text{if } -1 < x < 1$$

$x = 1/2$ here geometric sum }

$$\rightarrow \approx \boxed{0{,}000001907\ldots}$$