Math 4740

10/2/23

Calendar

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M TEST 1	W Topic 3/4
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Study guide is on the website

Ex: (HW 3 #3 modified)

Suppose you roll two 8-sided dice. You can't see the outcome, but your friend can. They tell you that the sum of the dice is divisible by 5. What is the probability that both dice have landed on 5?

$S = \{(a,b) \mid a,b = 1,2,\dots,8\}$

$|S| = 8^2 = 64$

$F = \{(a,b) \mid a+b \text{ is divisible by } 5\}$

$\boxed{E = \{(5,5)\}}$

Want: $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$

We have

$$\boxed{F = \{(1,4), (2,3), (2,8), (3,2), (3,7), (4,1), (4,6), (5,5), (6,4), (7,3), (7,8), (8,2), (8,7)\}}$$

$$\boxed{E \cap F = \{(5,5)\}}$$

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{(1/64)}{(13/64)}$$

$$= 1/13$$

$$\approx 0.7692\ldots$$

$$\approx 7.7\%$$

Theorem: Let (S, Ω, P) be a probability space.

① Let A and B be events and $P(A) > 0$. Then

$$P(A \cap B) = P(A) \cdot P(B|A)$$

② Let $A_1, A_2, \ldots, A_n$ be events with $P(A_1 \cap A_2 \cap \cdots \cap A_n) > 0$.

Then,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot P(A_4|A_1 \cap A_2 \cap A_3) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

③ (Law of total probability)

Suppose $S = E_1 \cup E_2 \cup \cdots \cup E_n$
where each $E_i \neq \phi$, and
$E_i \cap E_j = \phi$ if $i \neq j$,

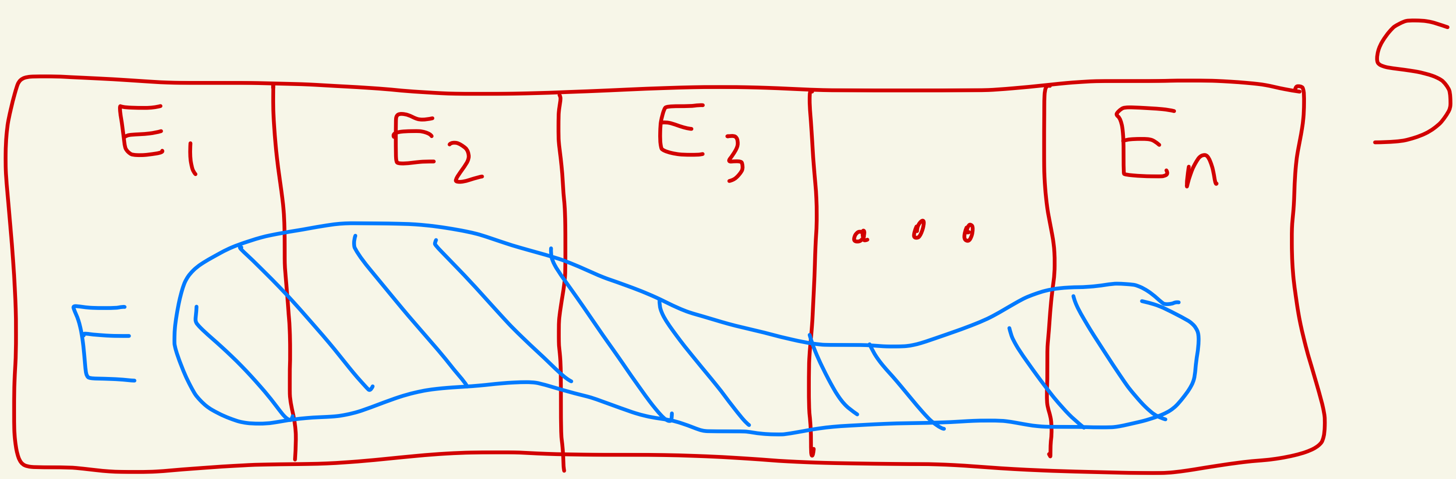
S is broken into n disjoint events

and $P(E_i) \neq 0$ for each i.

Then for event E we have

$$
\begin{aligned}
P(E) = &\ P(E|E_1)\cdot P(E_1) \\
&+ P(E|E_2)\cdot P(E_2) \\
&+ \cdots \\
&+ P(E|E_n)\cdot P(E_n)
\end{aligned}
$$

$$
\begin{aligned}
&P(E \cap E_1) \\
&+ \\
&P(E \cap E_2) \\
&+ \\
&\vdots \\
&+ \\
&P(E \cap E_n)
\end{aligned}
$$



Ex: Suppose there are three boxes. In box 1, there are two 4-sided dice. In box 2, there are two 6-sided dice. In box 3, there are two 8-sided dice.

Suppose you randomly pick a box (each box is equally likely to be chosen), then you take the dice out of that box and roll them.

What is the probability that the sum of the dice is 8?

Solution:

Picked box 1	$\rightarrow$	sum of dice is 8: $\{(4,4)\}$	$\rightarrow$	probability of sum 8 when box 1 is picked: $1/16$

picked box 2 $\rightarrow$

$$\frac{\text{sum of dice is } 8}{\{(2,6),(3,5),(4,4),(5,3),(6,2)\}}$$

$\rightarrow$

$$\frac{\text{probability of sum 8 when box 2 is picked}}{5/36}$$

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picked box 3 $\rightarrow$

$$\frac{\text{sum of dice is } 8}{\{(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1)\}}$$

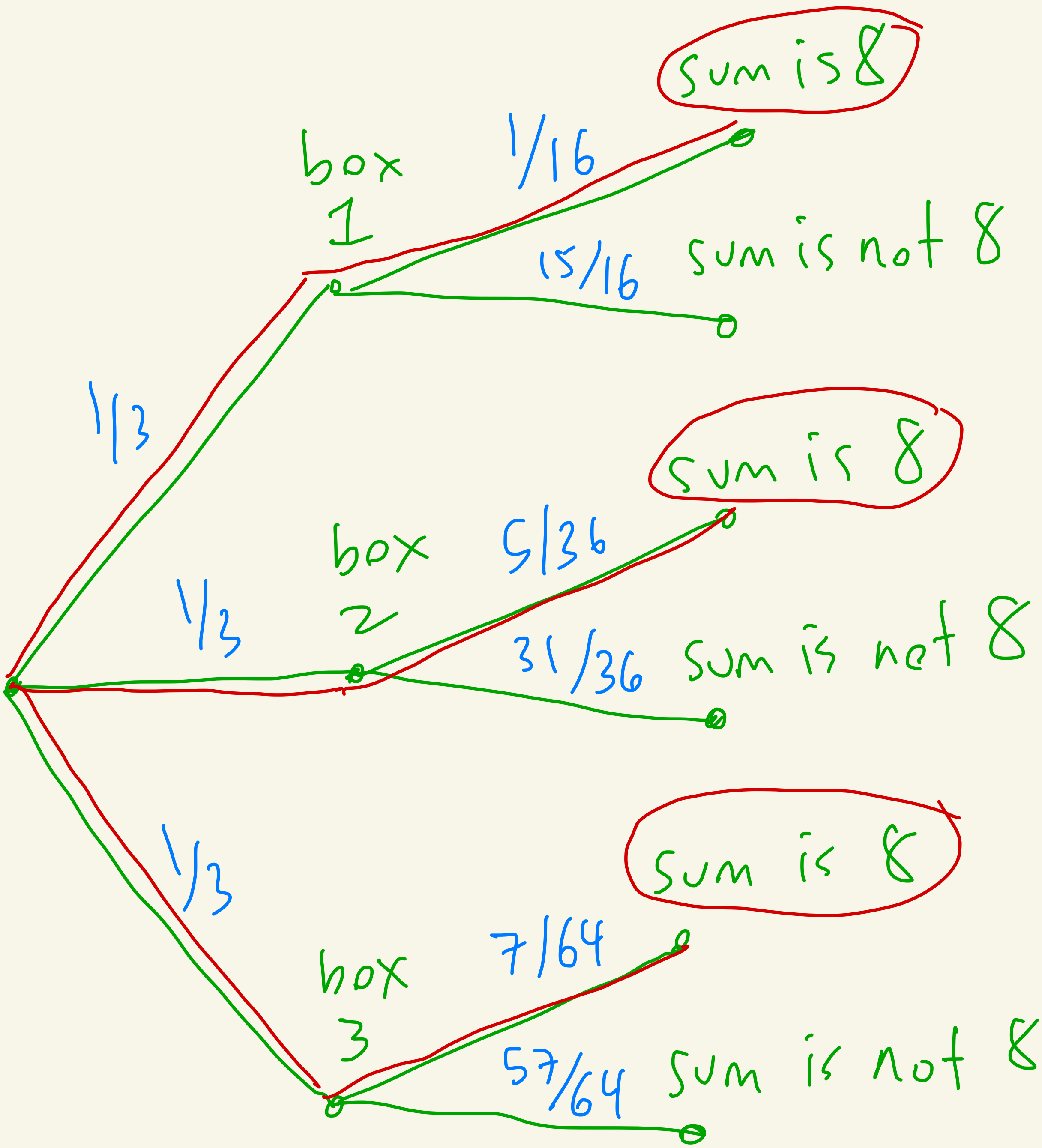
$\rightarrow$

$$\frac{\text{probability of sum 8 when box 3 is picked}}{7/64}$$

$$
\begin{aligned}
&P(\text{sum of dice is } 8) \\
&= P\left(\begin{matrix}\text{sum of}\\ \text{dice is } 8\end{matrix} \,\middle|\, \text{box 1 picked}\right) \cdot P\left(\begin{matrix}\text{box 1}\\ \text{picked}\end{matrix}\right) \\
&+ P\left(\begin{matrix}\text{sum of}\\ \text{dice is } 8\end{matrix} \,\middle|\, \text{box 2 picked}\right) \cdot P\left(\begin{matrix}\text{box 2}\\ \text{picked}\end{matrix}\right) \\
&+ P\left(\begin{matrix}\text{sum of}\\ \text{dice is } 8\end{matrix} \,\middle|\, \text{box 3 picked}\right) \cdot P\left(\begin{matrix}\text{box 3}\\ \text{picked}\end{matrix}\right)
\end{aligned}
$$

$$= \left(\tfrac{1}{16}\right)\left(\frac{1}{3}\right) + \left(\tfrac{5}{36}\right)\left(\frac{1}{3}\right) + \left(\tfrac{7}{64}\right)\left(\frac{1}{3}\right)$$

$$= \frac{11,456}{110,592} \approx 0.1036\ldots \approx \boxed{10.36\%}$$



Ex: (Monty Hall)

Let's redo the probability of the switch strategy for Montey Hall (start with door 1 and switch after Monty reveals another door).

$$P(\text{win car})$$

$$= P\left(\begin{matrix}\text{win}\\ \text{car}\end{matrix} \middle| \begin{matrix}\text{car behind}\\ \text{door 1}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{car behind}\\ \text{door 1}\end{matrix}\right)$$

$$+ P\left(\begin{matrix}\text{win}\\ \text{car}\end{matrix} \middle| \begin{matrix}\text{car behind}\\ \text{door 2}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{car behind}\\ \text{door 2}\end{matrix}\right)$$

$$+ P\left(\begin{matrix}\text{win}\\ \text{car}\end{matrix} \middle| \begin{matrix}\text{car behind}\\ \text{door 3}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{car behind}\\ \text{door 3}\end{matrix}\right)$$

$$= (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right)$$

$$= 2/3$$

Sometimes $P(E|F)$ is not equal to $P(E)$ and sometimes it is.

Suppose $P(E|F) = P(E)$.

Then, $$\frac{P(E \cap F)}{P(F)} = P(E).$$

So, $P(E \cap F) = P(E) \cdot P(F)$

<u>Def:</u> We say that two events E and F are <u>independent</u> if

$$P(E \cap F) = P(E) \cdot P(F)$$

otherwise we say they are <u>dependent</u>.

Note: Suppose $P(E) > 0$ and $P(F) > 0$

E and F are independent

is equivalent to

$$P(E \cap F) = P(E) \cdot P(F)$$

is equivalent to

$$\frac{P(E \cap F)}{P(E)} = P(F) \text{ and } \frac{P(E \cap F)}{P(F)} = P(E)$$

is equivalent to

$$P(F \mid E) = P(F) \text{ and } P(E \mid F) = P(E)$$