Math 4740

10/11/23

Ex: Suppose you roll two 6-sided die, one green and one red.

Let E be the event that the green die is 1.

Let F be the event that the red die is 3.

Are these events independent?

$$S = \left\{ (g,r) \;\middle|\; \begin{array}{l} g = 1,2,3,4,5,6 \\ r = 1,2,3,4,5,6 \end{array} \right\} \leftarrow |S| = 36$$

$$E = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)\}$$

$$F = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$$

$$E \cap F = \{(1,3)\}$$

$$P(E \cap F) = \frac{1}{36}$$

$$P(E) \cdot P(F) = \frac{6}{36} \cdot \frac{6}{36} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

So, $P(E \cap F) = P(E) \cdot P(F)$

Thus, E and F are independent.

Ex: Suppose you roll two 6-sided die, one green and one red.

Let E be the event that the sum of the dice is 6.

Let F be the event that the red die equals 4.

$$S = \{(g,r) \mid g,r = 1,2,3,4,5,6\} \leftarrow \boxed{|S| = 36}$$

$$E = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$$

$$F = \{(1,4),(2,4),(3,4),(4,4),(5,4),(6,4)\}$$

$$E \cap F = \{(2,4)\}$$

$P(E \cap F) = 1/36 \approx 0.0278\ldots$

$P(E) \cdot P(F) = (5/36)(6/36) = \frac{5}{216} \approx 0.0231\ldots$

Thus, $P(E \cap F) \neq P(E) \cdot P(F)$.

So, E and F are not independent.

<u>Def:</u> (General def of independence)

In a probability space (S, Ω, P) the events $E_1, E_2, \ldots, E_n$ are said to be <u>independent</u> if for every $2 \leq k \leq n$ we have that

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \cdot P(E_{i_2}) \cdots P(E_{i_k})$$

whenever $1 \leq i_1 < i_2 < \cdots < i_k \leq n$

Ex: E_1, E_2, E_3 are independent if all of the following are true:

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

$k=2$, $i_1=1$, $i_2=2$

$$P(E_1 \cap E_3) = P(E_1) \cdot P(E_3)$$

$k=2$, $i_1=1$, $i_2=3$

$$P(E_2 \cap E_3) = P(E_2) \cdot P(E_3)$$

$k=2$, $i_1=2$, $i_2=3$

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2) \cdot P(E_3)$$

$k=3$, $i_1=1$, $i_2=2$, $i_3=3$

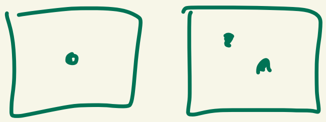
<u>Theorem</u>: Let S be a sample space of a repeatable experiment. Let A and B be events where $A \cap B = \phi$ [they don't overlap. This is called disjoint events] Suppose further that each time we repeat the experiment S, the experiment is independent of the previous times we did experiment S. Suppose we keep repeating S until either A or B occurs and then we stop. Then the probability that A occurs before B is given by

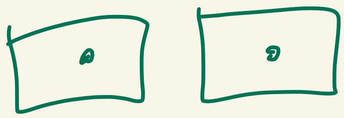
$$\frac{P(A)}{P(A)+P(B)}$$

Proof: See Spring 2022 notes $\boxed{///}$

Ex: Suppose we roll two 6-sided die over and over. Let A be the event that the sum of the dice is 5. Let B be the event that the sum of the dice is 7. We keep rolling the dice until either A or B happens and then we stop. Whats the probability that A occurs before B, ie that we roll sum of 5 before we roll sum of 7?

Ex:

roll 1 –  $\leftarrow$ sum = 3

roll 2 –  $\leftarrow$ sum = 2

roll 3 –  $\leftarrow$ sum = 5

Sum is 5 occured before sum is 7

$$S = \{(a,b) \mid a,b = 1,2,3,4,5,6\} \leftarrow \boxed{|S| = 36}$$

$$A = \{(1,4),(2,3),(3,2),(4,1)\}$$

$$B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$P(A) = 4/36$$

$$P(B) = 6/36$$

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probability sum is 5 occurs before

sum is 7 is [ie A before B]

$$\frac{P(A)}{P(A)+P(B)} = \frac{4/36}{4/36+6/36} = \frac{4}{10} = \boxed{\frac{2}{5}} = \boxed{40\%}$$

probability sum is 7 occurs before sum is 5 occurs is [ie B before A]

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36+4/36} = \boxed{\frac{6}{10}} = \boxed{60\%}$$

Topic 4 – Random Variables, Expected Value, Games

<u>Def:</u> Let (S, Ω, P) be a probability space. A <u>random variable</u> is a function

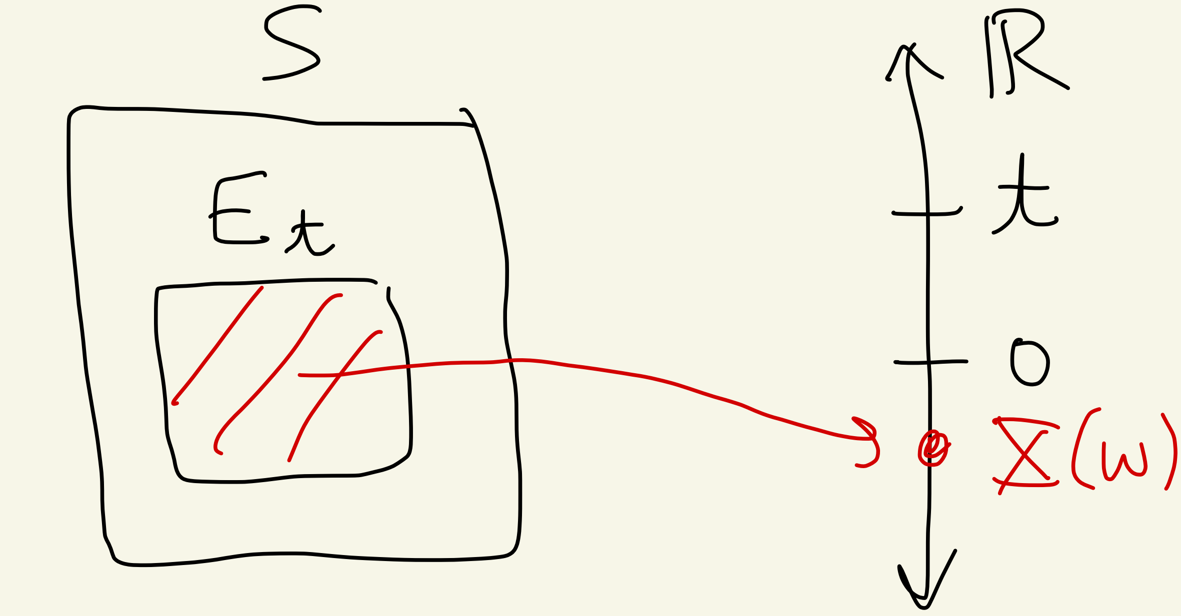
$X : S \to \mathbb{R}$ such that for all real numbers t

X is a function
input $= S$
output $=$ real #s $= \mathbb{R}$

we have that

$$E_t = \{ w \mid w \in S \text{ and } X(w) \leq t \}$$

is an event in Ω.



<u>Note:</u> The condition on E_t means we can calculate $P(E_t)$. In our class when S is finite and Ω is all subsets of S this condition will always occur so a random variable is just a function $X : S \to \mathbb{R}$

Def: Let X be a random variable on a probability space (S, Ω, P). We say that X is *discrete* if the range of X can be enumerated as a list of values $x_1, x_2, x_3, \ldots$

$\boxed{\text{For Math 3450: Ie, the range of } X \text{ is finite or countably infinite}}$