Proposition from last time

We showed (13) $|z+w| \geq \big||z|-|w|\big|$

Let's show (14)

$$|z-w| = |z+(-w)| \overset{(13)}{\geq} \big||z|-|-w|\big| = \big||z|-|w|\big|.$$

So, $|z-w| \geq \big||z|-|w|\big|$

$\blacksquare$

Two theorems (proofs in HW 1 - optional, ie proofs not on test)

<u>De Moivre's Theorem</u> Let $z = r[\cos(\theta) + i\sin(\theta)]$.
If n is a positive integer, then

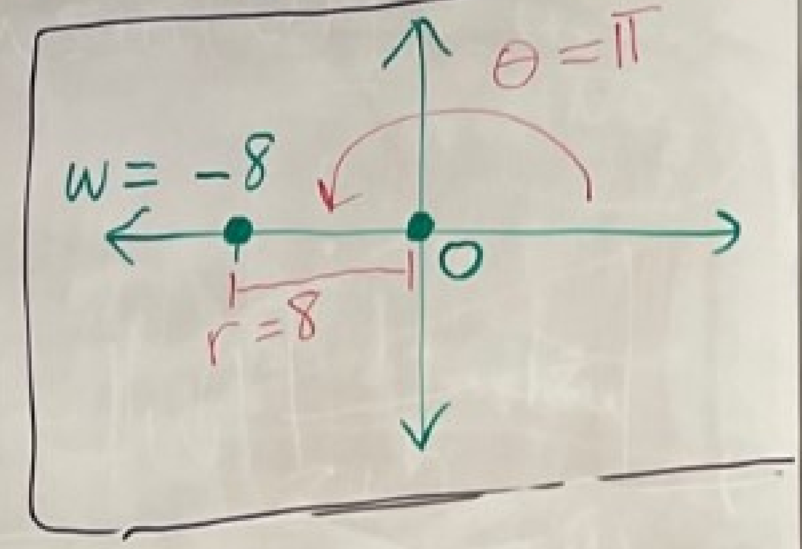
$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

<u>Theorem:</u> Let $w = r[\cos(\theta) + i\sin(\theta)]$ and let n be a positive integer, where $w \neq 0$. Then the solutions to $\boxed{z^n = w}$ are given by

$$\boxed{z_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]}$$ where $k = 0, 1, 2, \ldots, n-1$

Ex: Find the solutions to $z^3 = \underbrace{-8}$.

$w = -8 = 8\left[\cos(\pi) + i\sin(\pi)\right]$



$(n=3)$ Want the cube roots of -8.

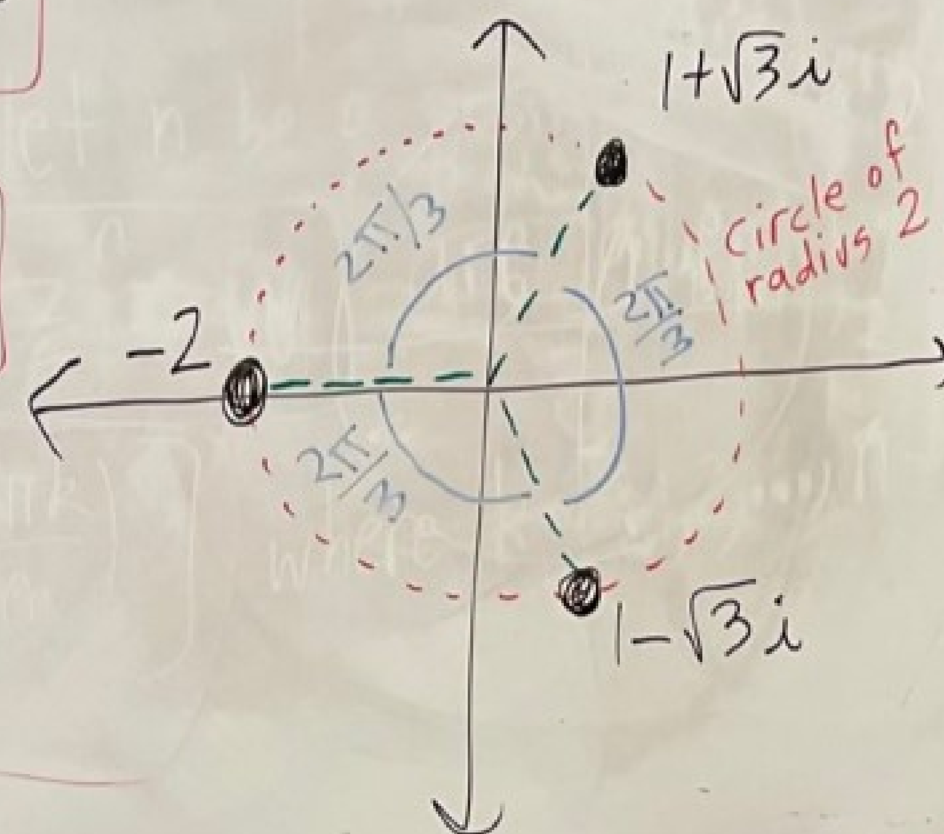
$$\boxed{z_k = \underbrace{8^{1/3}}_{2}\left[\cos\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right)\right],\ k = 0, 1, 2.}$$

$$z_0 \overset{k=0}{=} 2\cdot\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right] = 2\left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] = \boxed{1 + \sqrt{3}\,i}$$

$$z_1 \overset{k=1}{=} 2\left[\cos\left(\underbrace{\frac{\pi}{3} + \frac{2\pi}{3}}_{\pi}\right) + i\sin\left(\underbrace{\frac{\pi}{3} + \frac{2\pi}{3}}_{\pi}\right)\right] = 2\left[-1 + i\cdot 0\right] = \boxed{-2}$$

$$z_2 \overset{k=2}{=} 2\left[\cos\left(\underbrace{\frac{\pi}{3} + \frac{4\pi}{3}}_{5\pi/3}\right) + i\sin\left(\underbrace{\frac{\pi}{3} + \frac{4\pi}{3}}_{5\pi/3}\right)\right] = 2\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$$

$$= \boxed{1 - \sqrt{3}\,i}$$



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HW 2 - Elementary functions

We want to make e^z where it agrees with e^x for $z=x$ real.

Def: Let $z=x+iy$ be a complex number.

Define $e^z = e^x\left[\cos(y)+i\sin(y)\right]$ where e^x is the usual real-number e^x

Ex: $e^{5+\pi i} = e^5\left[\cos(\pi)+i\sin(\pi)\right] = e^5\left[-1+0i\right] = -e^5$

$e^{\ln(2)+\frac{\pi}{4}i} = e^{\ln(2)}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right] = 2\left[\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right] = \sqrt{2}+i\sqrt{2}$

$e^{\pi i/3} = e^{0+\frac{\pi}{3}i} = e^0\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right] = 1\cdot\left[\frac{1}{2}+\frac{\sqrt{3}}{2}i\right] = \frac{1}{2}+\frac{\sqrt{3}}{2}i$

$$e^{\sqrt{2}-3i} = e^{\sqrt{2}}\left[\cos(-3)+i\sin(-3)\right]$$

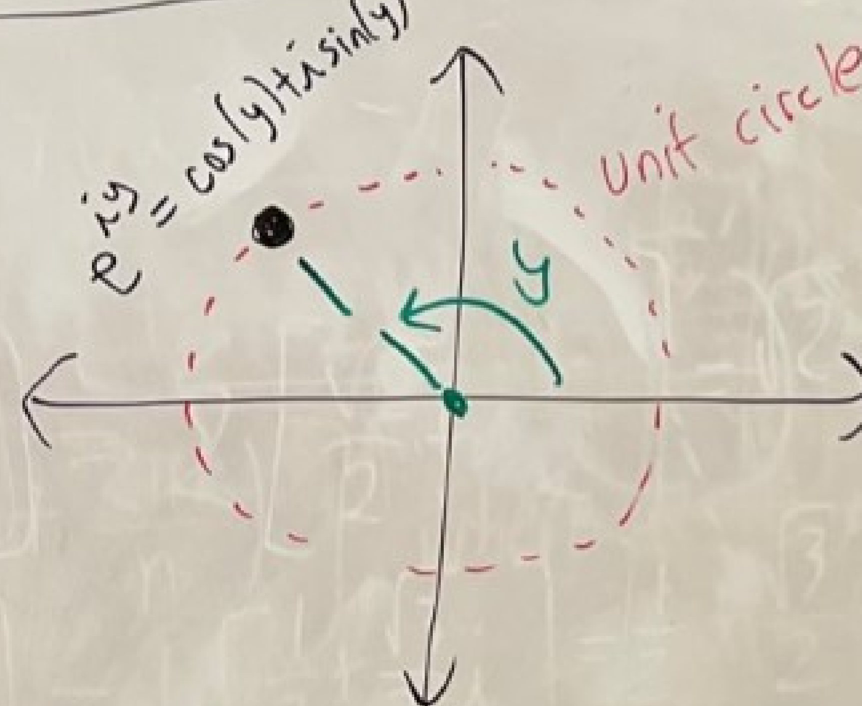
Note: If $z = x + i0$ is a real number, then

$$e^z = e^x\left[\cos(0)+i\sin(0)\right] = e^x\left[1+i0\right] = e^x.$$

Our new function e^z agrees with the real-valued e^x when $z=x$ is real

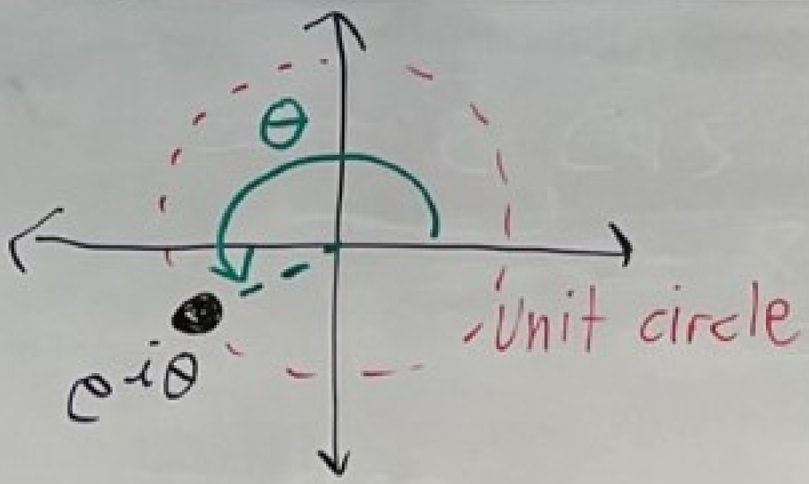
Note: If $z = 0+iy$ then

$$e^z = e^{0+iy} = e^0\left[\cos(y)+i\sin(y)\right] = \underbrace{\cos(y)+i\sin(y)}_{\text{On the unit circle}}$$



Note: If θ is a real number,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$



can use this in $z^n = w$ formula

$$z_k = r^{1/n} e^{\frac{\theta}{n} + \frac{2\pi k}{n}}, \quad k = 0, 1, \ldots, n-1$$

$$e^{x+iy} = \underbrace{e^{x}}_{r}\left[\cos(\underset{\uparrow\atop\theta}{y}) + i\sin(\underset{\uparrow\atop\theta}{y})\right]$$

polar form already