HW 5 | 2(b) | Let $f(z)=e^{\overline{z}}$. Where does $f'(z)$ exist?

$$f(x+iy)=e^{\overline{x+iy}}=e^{x-iy}=e^x\left[\underbrace{\cos(-y)}_{\cos(y)}+i\underbrace{\sin(-y)}_{-\sin(y)}\right]=\underbrace{\left[e^x\cos(y)\right]}_{u(x,y)}+i\underbrace{\left[-e^x\sin(y)\right]}_{v(x,y)}$$

**CR-equations**

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x}=e^x\cos(y)$$
$$\frac{\partial v}{\partial y}=-e^x\cos(y)$$

$$\frac{\partial u}{\partial y}=-e^x\sin(y)$$
$$-\frac{\partial v}{\partial x}=e^x\sin(y)$$

Need:

$$e^x\cos(y)=-e^x\cos(y) \quad ①$$
$$-e^x\sin(y)=e^x\sin(y) \quad ②$$

$e^x\neq 0$ $\Rightarrow$

$$\cos(y)=-\cos(y) \quad ①$$
$$-\sin(y)=\sin(y) \quad ②$$

$\Downarrow$

$$2\cos(y)=0 \quad ①$$
$$2\sin(y)=0 \quad ②$$

$\Downarrow$

$$\cos(y)=0 \quad ①$$
$$\sin(y)=0 \quad ②$$

$\Rightarrow$

$\cos(y)=0$ iff $y=\frac{\pi}{2}+\pi k$ ①

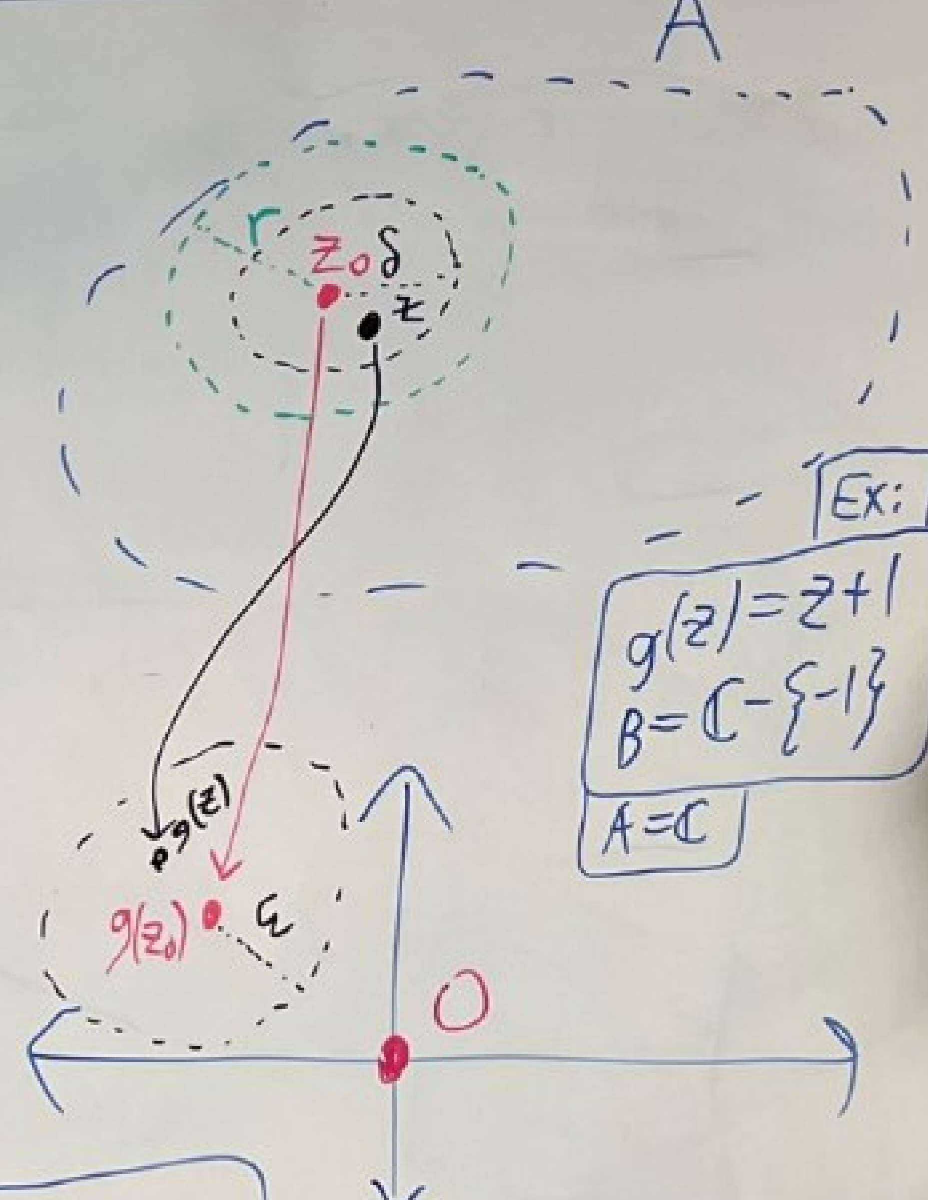
$\sin(y)=0$ iff $y=\pi k$ ②

No common solutions.

So, $f'(z)$ exists nowhere!

**HW 5** ⑤ Let $g$ be analytic on an open set $A$.
Let $B=\{z\in A \mid g(z)\neq 0\}$. Show (a) $B$ is open, and (b) $\frac{1}{g(z)}$ is analytic on $B$.

(a) Since $g$ is analytic on $A$, we know $g$ is continuous on $A$.
Let $z_0\in B$. Then $g(z_0)\neq 0$.
Since $g$ is continuous at $z_0$ we know $\lim_{z\to z_0} g(z)=g(z_0)$.
Since $z_0\in A$ and $A$ is open, there exists $r>0$ where $D(z_0;r)\subseteq A$.
Let $\varepsilon=\frac{|g(z_0)-0|}{2}=\frac{|g(z_0)|}{2}$.
Since $\lim_{z\to z_0} g(z)=g(z_0)$ there exists $\delta>0$ (make $\delta<r$)
where if $\underbrace{|z-z_0|<\delta}_{\text{same as: } z\in D(z_0;\delta)}$, then $\underbrace{|g(z)-g(z_0)|<\varepsilon}_{\text{same as: } g(z)\in D(g(z_0);\varepsilon)}$.




Ex:
$g(z)=z+1$
$B=\mathbb{C}-\{-1\}$
$A=\mathbb{C}$

Let $z\in D(z_0;\delta)$. Why is $g(z)\neq 0$. If $g(z)=0$, then
$|g(z_0)|=|g(z)-g(z_0)|<\varepsilon=\frac{|g(z_0)|}{2}$. Can't happen. Thus, $D(z_0;\delta)\subseteq B$
So, $z_0$ is an interior point and $B$ is open. $\boxed{(a)}$

## HW 6 ① (b)

$\int_\gamma (z^3+z)\,dz$ , $\gamma$ circle radius 3, centered at 0, counter-clockwise


$\gamma(t) = 0 + 3e^{it}, \; 0 \le t \le 2\pi$

$\gamma(t) = 3e^{it} = 3\cos(t) + i3\sin(t)$

$\gamma'(t) = -3\sin(t) + i3\cos(t) = i\left[3i\sin(t) + 3\cos(t)\right] = 3ie^{it}$

$$\int_0^{2\pi} \left[\underbrace{(3e^{it})^3}_{27e^{3it}} + \underbrace{(3e^{it})}_{+\,3e^{it}}\right] \cdot 3ie^{it}\,dt = \int_0^{2\pi} \left(81ie^{4it} + 9ie^{2it}\right)dt$$

$$= \left(\frac{81}{4i}\, i e^{4it} + \frac{9i}{2i} e^{2it}\right)\Bigg|_0^{2\pi} = \frac{81}{4}e^{8\pi i} + \frac{9}{2}e^{4\pi i} - \frac{81}{4}(1) - \frac{9}{2}(1) = \frac{81}{4}(1) + \frac{9}{2}(1) - \frac{81}{4}(1) - \frac{9}{2}(1) = 0$$

**HW 6**
- FTOC
- def of $\int_\gamma f$

- lines
- circles/half of circle

check:

$$\int_a^b e^{i\theta t}\,dt = \int_a^b \cos(\theta t) + i\sin(\theta t)\,dt$$

$$= \frac{1}{\theta}\sin(\theta t) - \frac{i}{\theta}\cos(\theta t)\Big|_{t=a}^{b}$$

$$= \frac{1}{i}\frac{1}{\theta}\left[i\sin(\theta t) + \cos(\theta t)\right]\Big|_{t=a}^{b}$$

$$= \frac{1}{i\theta}\left[e^{i\theta t}\right]_{t=a}^{b}$$

HW6

① (c) $\int_\gamma (x-y)\,dz$ $\gamma$ is line from $1$ to $1+i$

$$\gamma(t) = 1 + t[(1+i)-1] = \overset{x}{1} + i\,\overset{y}{t}, \quad 0 \le t \le 1$$

when plugged into f

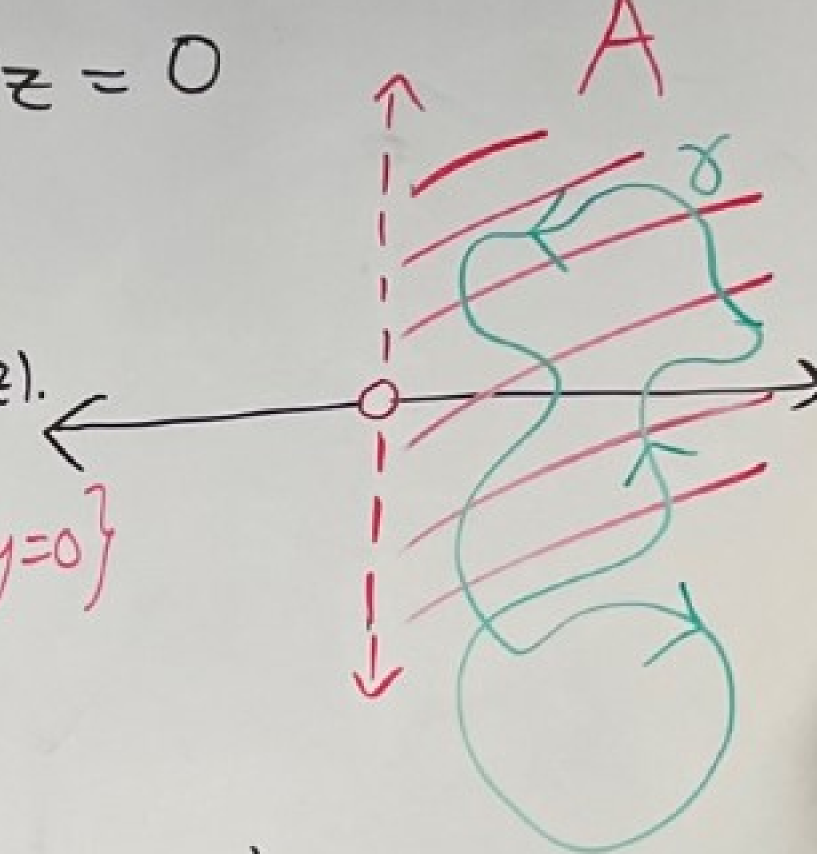
$$\gamma'(t) = 0 + i = i$$

$$\int_\gamma \underbrace{(x-y)}_{f(x+iy)=x-y}\,dz = \int_0^1 \underbrace{(1-t)}_{f(\gamma(t))} \cdot \underbrace{i\,dt}_{\gamma'(t)\,dt} = \cdots$$

$$f(\underset{x=1}{1} + i\,\underset{y=t}{t})$$

## Hw 6

5(a) Let $\gamma$ be a closed piecewise smooth curve lying entirely in $A = \mathbb{C} - \{\mathrm{Re}(z) \le 0\}$. Show that $\int_\gamma \frac{1}{z}\,dz = 0$

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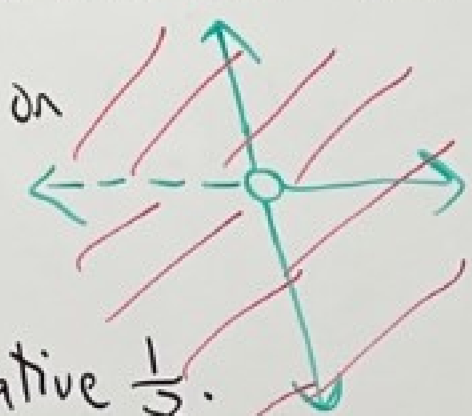
Consider $\log(z)$ where we use the principal branch of $\log(z)$.

This function is analytic on



$B = \mathbb{C} - \{x+iy \mid x \le 0 \text{ and } y = 0\}$

So, $\log(z)$ is also analytic on $A$ with derivative $\frac{1}{z}$.

Use FTOC, to get $\int_\gamma \frac{1}{z}\,dz = \log\left(\text{start point of } \gamma\right) - \log\left(\text{end point of } \gamma\right)$

$= 0$ (since $\gamma$ is closed)