Math 4680

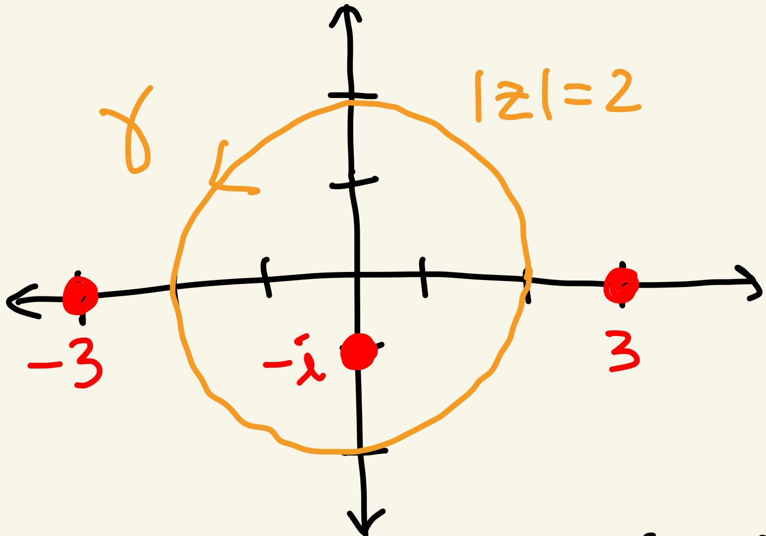
11/28/22

Plan	M	W
	11/28 - Finish final stuff	11/30 - Not on final
	12/5 - Not on final	12/7 - Review day
	12/12 - FINAL 12 - 2	

Ex: Let γ be the circle $|z|=2$ oriented in the counter-clockwise direction.

Evalute

$$\int_\gamma \frac{z}{(9-z^2)(z+i)}\,dz$$



$\dfrac{z}{(9-z^2)(z+i)}$ is analytic except when $(9-z^2)(z+i)=0$ which is at $z=\pm 3, -i$

$f(z) = z/(9-z^2)$

f is analytic on and inside γ

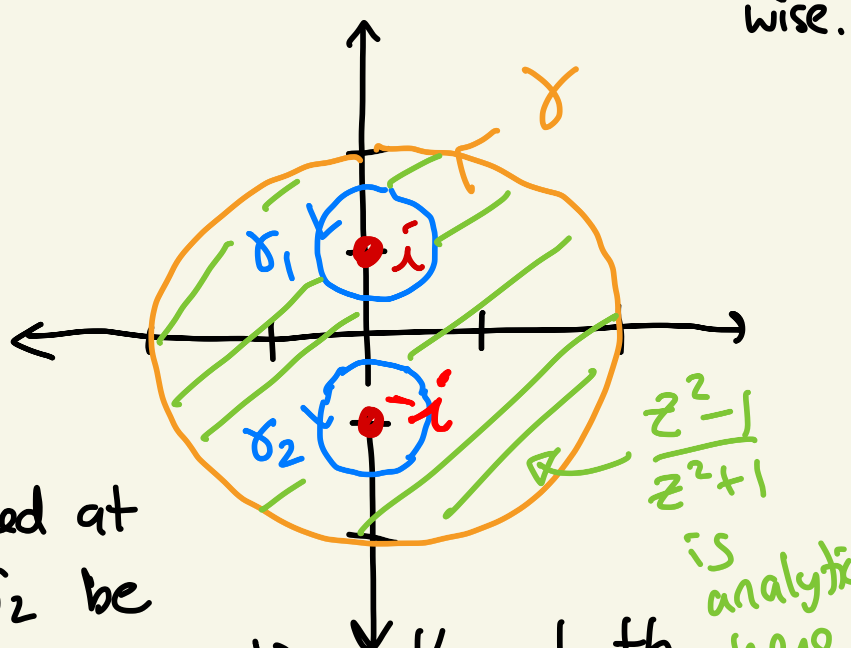
So,

$$\int_\gamma \frac{z}{(9-z^2)(z+i)}\,dz = \int_\gamma \frac{\boxed{\left(z/(9-z^2)\right)}}{(z-(-i))}\,dz = 2\pi i\, f(-i)$$

Cauchy Integral thm

$$= 2\pi i\left(\frac{-i}{9-(-i)^2}\right) = \frac{-2\pi i^2}{10} = \boxed{\frac{\pi}{5}}$$

Ex: Calculate $\displaystyle\int_\gamma \frac{z^2-1}{z^2+1}\,dz$ where γ is the circle of radius 2 centered at 0, oriented counter-clock wise.



Note: $\frac{z^2-1}{z^2+1}$ is analytic everywhere except when $z^2+1=0$ which is when $z=\pm i$.

Let γ_1 be the circle centered at i with radius $1/2$ and γ_2 be the circle centered at $-i$ with radius $1/2$, both oriented counter-clockwise.

Then $\frac{z^2-1}{z^2+1}$ is analytic on $\gamma, \gamma_1, \gamma_2$ and between γ and γ_1, γ_2.

So,
$$\int_\gamma \frac{z^2-1}{\underbrace{z^2+1}}\,dz = \int_{\gamma_1} \frac{z^2-1}{z^2+1}\,dz + \int_{\gamma_2} \frac{z^2-1}{z^2+1}\,dz$$

$$\boxed{\frac{z^2-1}{(z+i)(z-i)}} = \int_{\gamma_1} \frac{(z^2-1)/(z+i)}{(z-i)}\,dz + \int_{\gamma_2} \frac{(z^2-1)/(z-i)}{(z-(-i))}\,dz$$

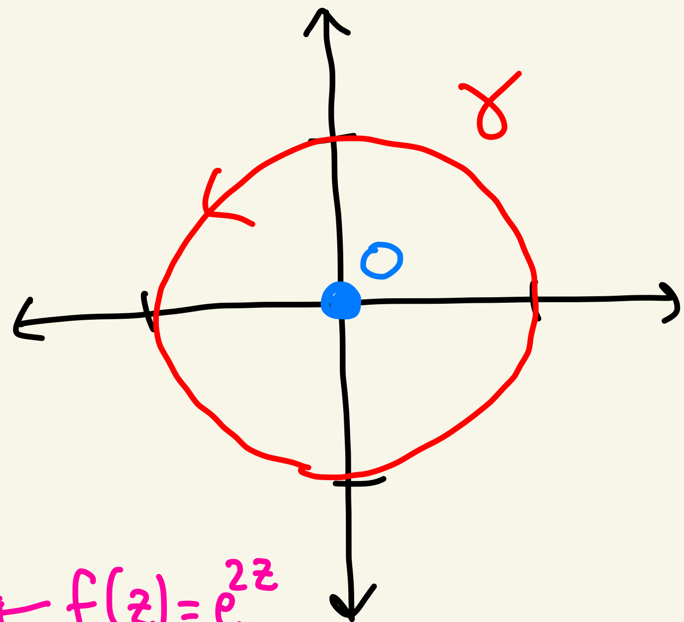
$$= 2\pi i\left[\frac{(i)^2-1}{(i+i)}\right] + 2\pi i\left[\frac{(-i)^2-1}{(-i-i)}\right]$$

Cauchy integral formula magic

$$= 2\pi i\left[\frac{-2}{2i}\right] + 2\pi i\left[\frac{-2}{-2i}\right]$$

$$= -2\pi + 2\pi = 0$$

Ex: Let γ be the unit circle, oriented counterclockwise.



Evaluate $\displaystyle\int_\gamma \frac{e^{2z}}{z^4}\,dz$.

$\dfrac{e^{2z}}{z^4}$ is analytic everywhere except at $z=0$.

$f(z) = e^{2z}$

So, $\displaystyle\int_\gamma \frac{e^{2z}}{z^4}\,dz = \int_\gamma \frac{e^{2z}}{(z-0)^4}\,dz = \frac{2\pi i}{3!} f^{(3)}(0)$

f is analytic on γ and inside γ

Use cauchy-integral formula

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$$= \frac{2\pi i}{6}\, 8e^{2(0)} = \boxed{\frac{8}{3}\pi i}$$

Topic 11 – More cool theorems

NOT ON FINAL

<u>Theorem:</u> Let $A \subseteq \mathbb{C}$ be an open set and $f: A \to \mathbb{C}$ is analytic on A. Then, $f^{(k)}$ exists and is also analytic on A for all $k \geq 1$.

[$f^{(k)}$ means the k-th derivative]

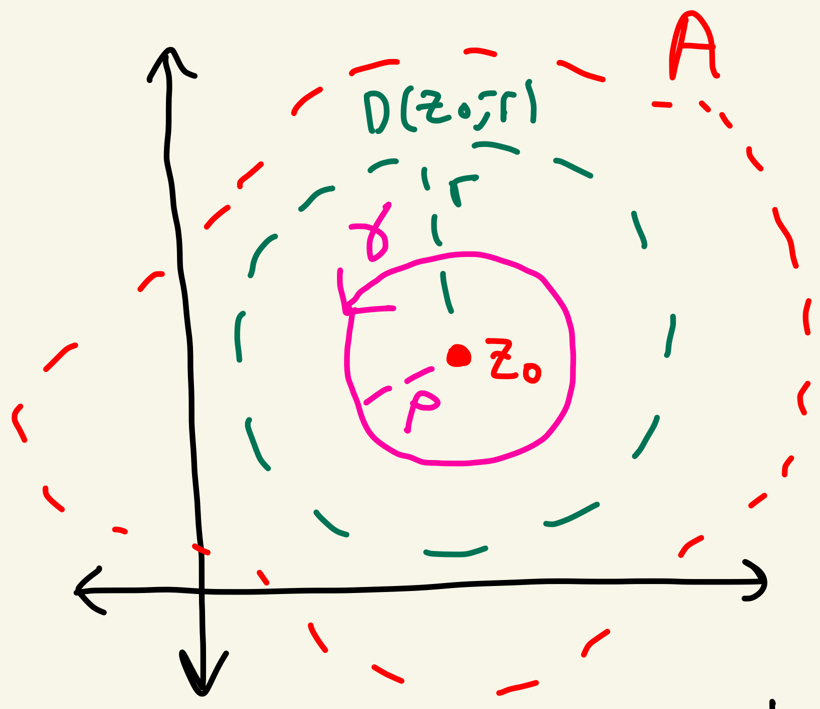
<u>proof:</u> We will show first that $f^{(k)}$ exists at all points in A.

Let $z_0 \in A$.

Since A is open, there exists $r > 0$ where $D(z_0; r) \subseteq A$.

Pick $0 < \rho < r$.

[For example, $\rho = \frac{r}{2}$ works]



Let γ be a circle of radius ρ centered at z_0

Since γ is inside $D(z_0; r)$ we know γ is interior to A.

Then, f is analytic inside and on γ.

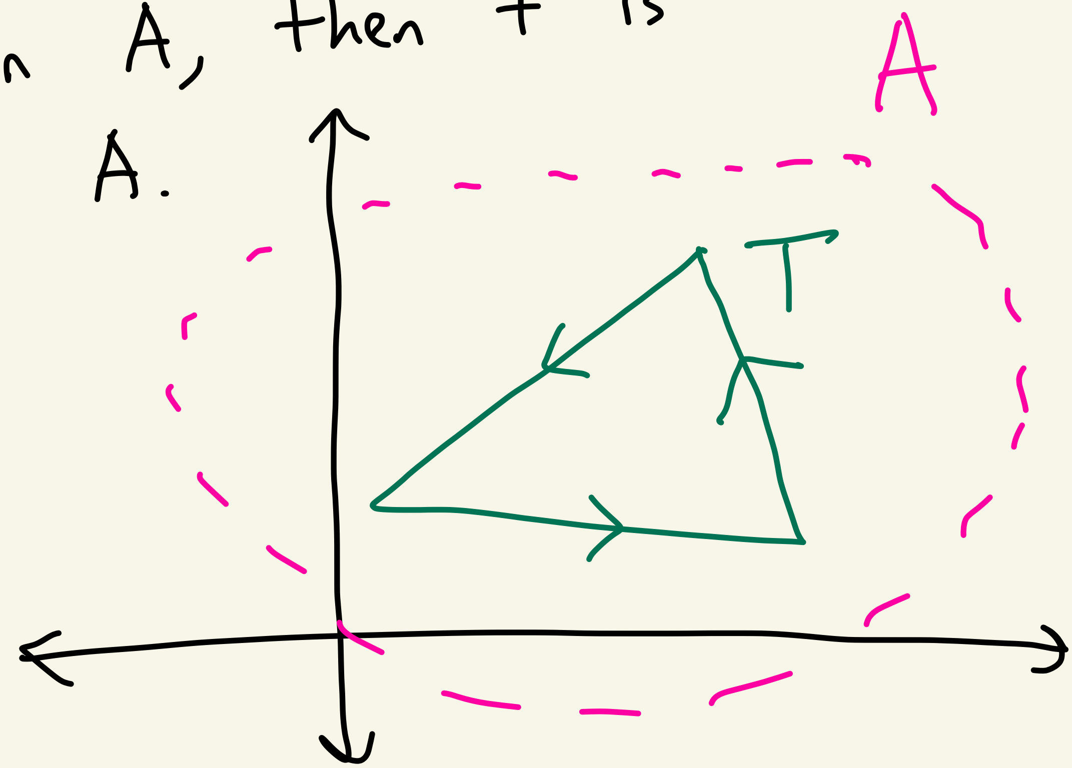
By the Cauchy-integral formula,

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$

So, $f^{(k)}$ exists at all points in A for $k \geq 1$.

And $f^{(k)}$ is analytic on A because $f^{(k+1)}$ exists on A. $\blacksquare$

Morera's Theorem Let $A \subseteq \mathbb{C}$ be a region (open and path connected) and $f: A \to \mathbb{C}$ be continuous on A. If $\int_T f = 0$ for every triangular path T in A, then f is analytic in A.



Proof:

See online notes $\blacksquare$