Math 4680

11/16/22

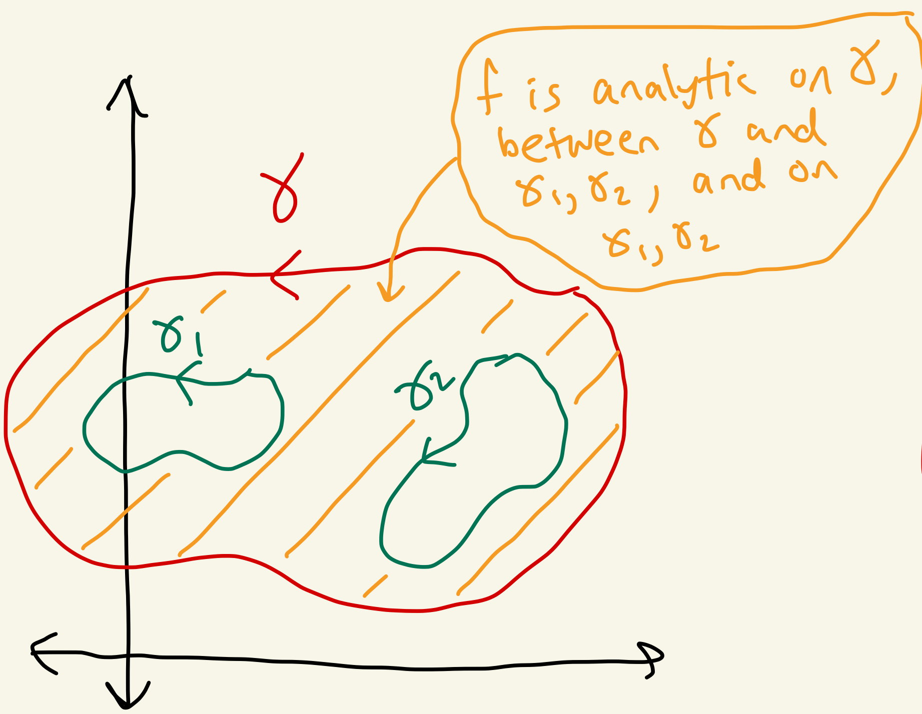
Theorem: Suppose that γ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are simple, closed, piecewise smooth curves such that

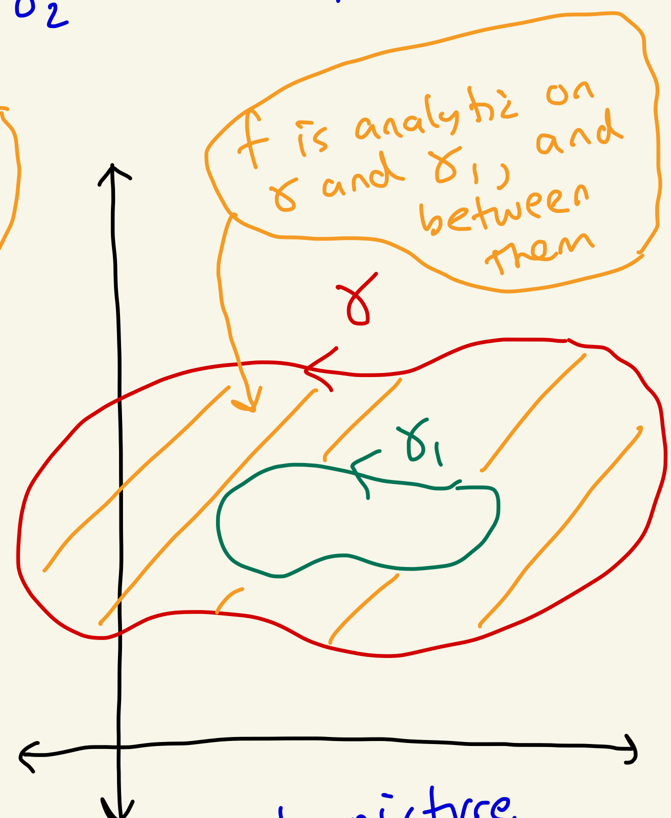
(a) γ is oriented counterclockwise

(b) $\gamma_1, \gamma_2, \ldots, \gamma_n$ are all oriented counterclockwise, are all interior to γ, and the interiors of $\gamma_1, \gamma_2, \ldots, \gamma_n$ have no points in common.

If f is analytic throughout the closed set consisting of all points within and on γ except for points interior to any γ_k, then

$$\int_{\gamma} f = \sum_{k=1}^{n} \int_{\gamma_k} f = \int_{\gamma_1} f + \int_{\gamma_2} f + \cdots + \int_{\gamma_k} f$$





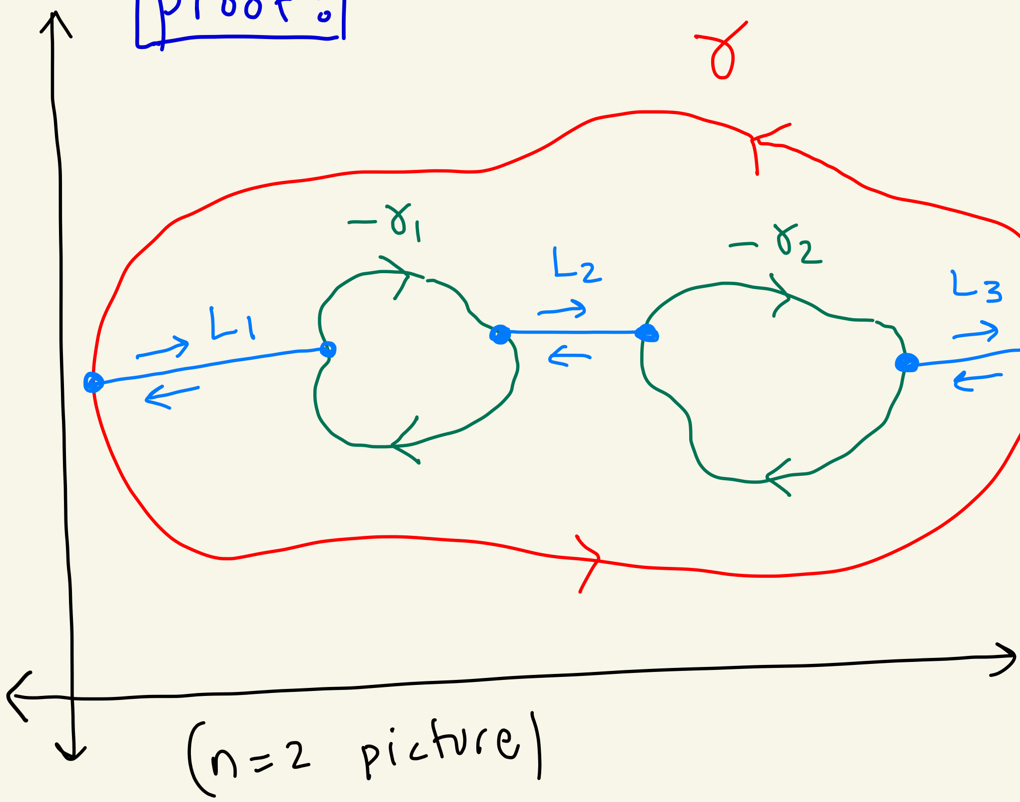
$n = 2$ picture

$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

$n = 1$ picture

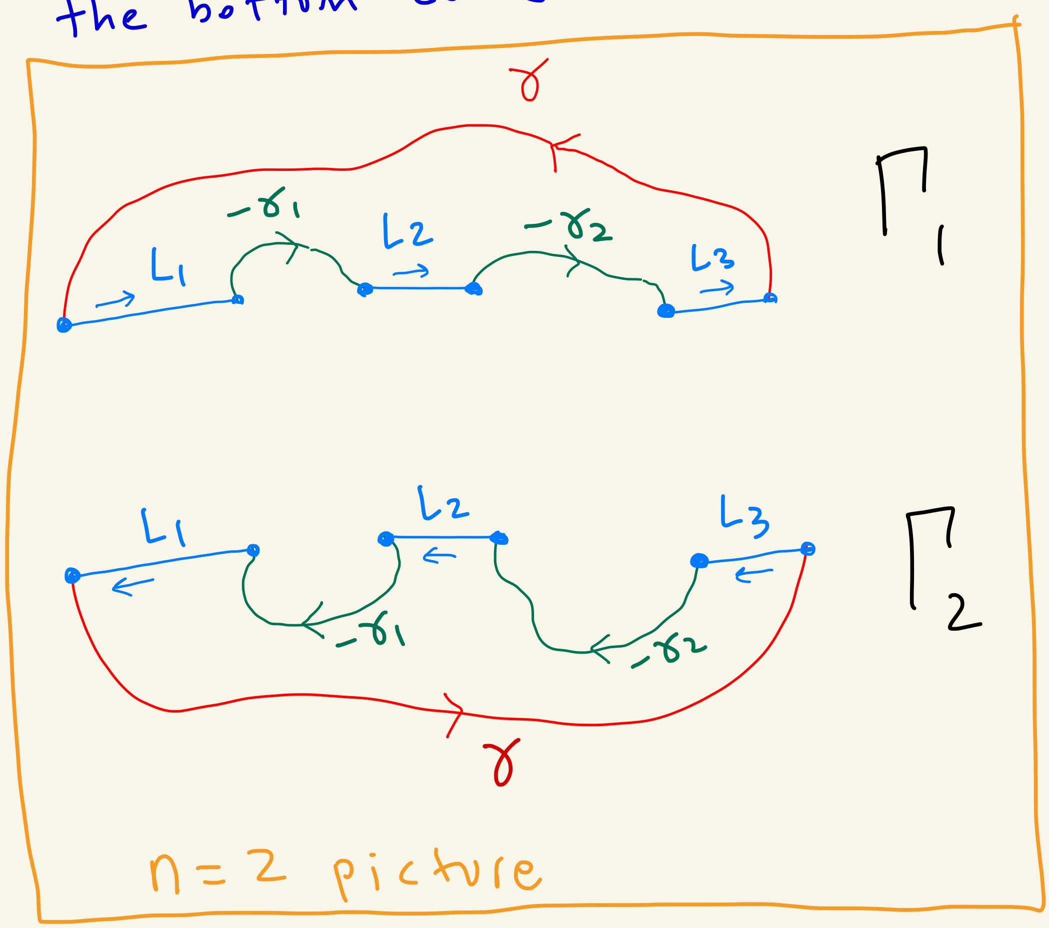
$$\int_{\gamma} f = \int_{\gamma_1} f$$

|proof:|



Make the lines $L_1, L_2, \ldots, L_{n+1}$ as in the picture on the left. And consider $-\gamma_k$ as shown.

Let Γ_1 be the top curve and Γ_2 be the bottom curve



By Cauchy's theorem

$$\int_{\Gamma_1} f = 0$$

and

$$\int_{\Gamma_2} f = 0$$

Thus,

$$0 = \int_{\Gamma_1} f + \int_{\Gamma_2} f = \int_{\Gamma_1 + \Gamma_2} f$$

$$= \int_{\gamma} f + \sum_{k=1}^{n} \int_{-\gamma_k} f$$

Lines L_k cancel each other

$$= \int_{\gamma} f - \sum_{k=1}^{n} \int_{\gamma_k} f$$

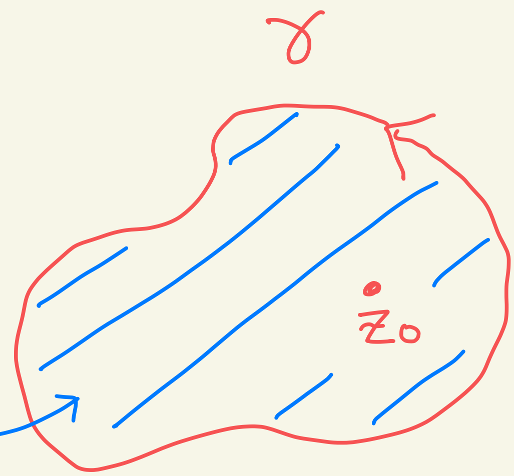
So, $0 = \int_{\gamma} f - \sum_{k=1}^{n} \int_{\gamma_k} f.$

Thus, $\int_{\gamma} f = \sum_{k=1}^{n} \int_{\gamma_k} f.$ $\blacksquare$

Theorem: (Cauchy Integral Formula)

Let f be analytic everywhere within and on a simple, closed, piecewise smooth curve γ, where γ is oriented in the counterclockwise direction. If z_0 is any point interior to γ then

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z - z_0}\, dz$$



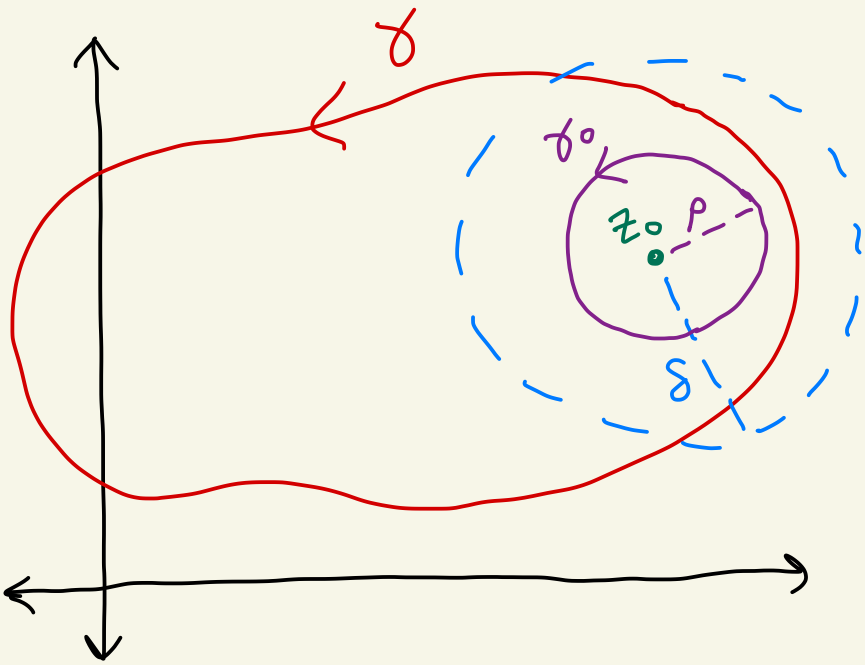
f is analytic in and on γ

proof:

Let z_0 be a point interior to γ.
Let $\varepsilon > 0$ be fixed.
Since f is continuous at z_0, there exists $\delta > 0$ where if $|z - z_0| < \delta$ and z is in the domain of f, then $|f(z) - f(z_0)| < \varepsilon$.

$$\lim_{z \to z_0} f(z) = f(z_0)$$

By the Jordan-curve theorem, the interior of γ is open. Thus there exists $\rho > 0$ where the circle $|z - z_0| = \rho$ is interior to γ. Choose ρ such that $\rho < \delta$.



Let γ_0 denote the circle $|z - z_0| = \rho$ oriented counterclockwise.

Since $\dfrac{f(z)}{z - z_0}$ is analytic on γ, in between γ and γ_0, and on γ_0, by the previous theorem we have

$$\int_{\gamma} \frac{f(z)}{z - z_0}\, dz = \int_{\gamma_0} \frac{f(z)}{z - z_0}\, dz$$

Thus,

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz - 2\pi i f(z_0)$$

$$= \underbrace{\int_{\gamma_0} \frac{f(z)}{z-z_0}\,dz}_{\text{from above}} - f(z_0)\underbrace{\int_{\gamma_0} \frac{dz}{z-z_0}}_{\substack{\text{earlier result} \\ \text{this is } 2\pi i}}$$

$$= \int_{\gamma_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

The arclength of γ_0 is $2\pi\rho$.

Thus, $$\left|\int_{\gamma_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| < \frac{\varepsilon}{\underbrace{\rho}} \cdot \underbrace{2\pi\rho}_{\boxed{\text{arclength of } \gamma_0}}$$

$$\boxed{\begin{array}{l}\text{If } z \text{ is on } \gamma_0, \text{ then} \\ |f(z)-f(z_0)| < \varepsilon \\ \text{and } |z-z_0| = \rho. \text{ So if } z \text{ is} \\ \text{on } \gamma_0 \text{ then } \left|\dfrac{f(z)-f(z_0)}{z-z_0}\right| < \dfrac{\varepsilon}{\rho}\end{array}}$$

Thus,

$$\left| \int_{\gamma} \frac{f(z)}{z - z_0} dz - 2\pi i f(z_0) \right|$$

$$= \left| \int_{\gamma_0} \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\varepsilon}{\rho} \cdot 2\pi\rho = 2\pi\varepsilon.$$

Thus, $\left| \int_{\gamma} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right|$ is smaller than any positive number since ε can be any positive number.

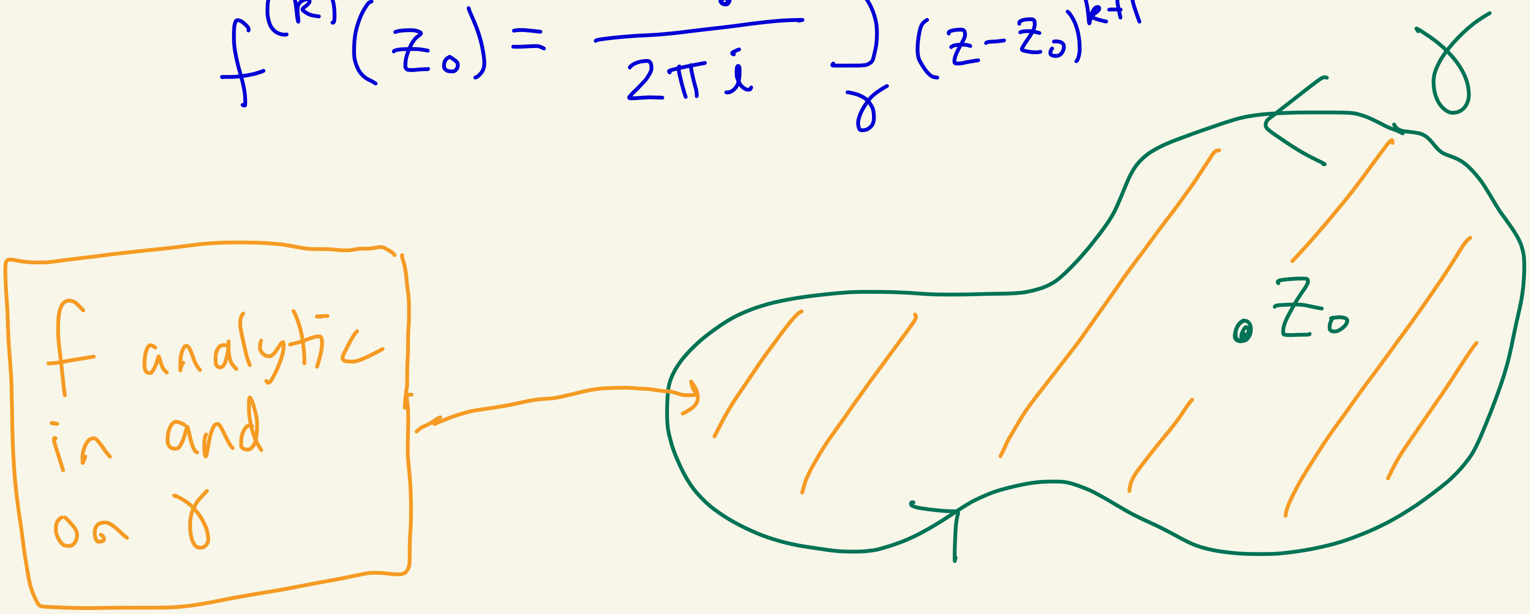
Thus, $\left| \int_{\gamma} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) \right| = 0.$

So, $\int_{\gamma} \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$ $\blacksquare$

Theorem: (Generalized Cauchy Integral Theorem)

Let f be analytic everywhere within a simple, closed, piecewise smooth curve γ.
Let γ be oriented counterclockwise.
If z_0 is any point interior to γ, then f is infinitely differentiable at z_0 and

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\, dz$$



Proof: See notes or Hoffman/Marsden book. $\blacksquare$