Math 2550-03

4/25/24

Recall from Tuesday last week

A is an $n \times n$ matrix.

$$A\vec{x} = \lambda \vec{x}, \quad \vec{x} \neq \vec{0}$$

$\lambda \leftarrow$ eigenvalue

$\vec{x} \leftarrow$ eigenvector

You can find the eigenvalues of A by solving the characteristic polynomial

$$\det(A - \lambda I_n) = 0$$

$$E_\lambda(A) = \{ \vec{x} \mid A\vec{x} = \lambda \vec{x} \}$$

Eigenspace for λ

Facts/Defs

Let A be an $n \times n$ matrix.

Let λ be eigenvalue of A.

① The eigenspace $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

② The dimension of $E_\lambda(A)$ is called the <u>geometric multiplicity of λ</u>.

③ The <u>algebraic multiplicity of λ</u> is the multiplicity of λ as a root of the characteristic polynomial of A.

④ $\begin{pmatrix} \text{geometric multiplicity} \\ \text{of } \lambda \end{pmatrix} \leq \begin{pmatrix} \text{algebraic} \\ \text{multiplicity} \\ \text{of } \lambda \end{pmatrix}$

Example we started last time

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

characteristic polynomial:

$$\det(A - \lambda I_3) = -(\lambda - 2)^{2}(\lambda - 1)^{1}$$

eigenvalue λ	algebraic multiplicity of λ
$\lambda = 1$	1
$\lambda = 2$	2

We also found a basis for $E_1(A) = \{ \vec{x} \mid A\vec{x} = 1 \cdot \vec{x} \}$ it was $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$. Thus, the geometric multiplicity of $\lambda = 1$ is $\dim(E_1(A)) = \underbrace{1}_{\text{1 vector in basis}}$

Let's now find a basis for $E_2(A) = \{ \vec{x} \mid A\vec{x} = 2\vec{x} \}$

Want to solve $A\vec{x} = 2\vec{x}$.

So need to solve

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} \leftarrow A\vec{x} = 2\vec{x}$$

$$\begin{pmatrix} -2c \\ a + 2b + c \\ a + 3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$$\begin{pmatrix} -2a - 2c \\ a + c \\ a + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} -2a - 2c &= 0 \\ a + c &= 0 \\ a + c &= 0 \end{aligned}}$$

Let's solve:

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1 + R_3 \to R_3]{2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives:

$$\boxed{\begin{array}{rl} a \qquad + c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{array}}$$

leading: a

free: c, b

Solution:

$$\boxed{\begin{array}{l} b = t \\ c = u \\ a = -c = -u \end{array}}$$

Thus, if $\vec{x}$ solves $A\vec{x} = 2\vec{x}$ then

$$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -u \\ t \\ u \end{pmatrix}$$

$$= \begin{pmatrix} -u \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

$$= u \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So all solutions of $A\vec{x} = 2\vec{x}$ are linear combinations of $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Thus, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span the eigenspace $E_2(A)$.

You can verify that $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

Thus, $\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ is a basis for $E_2(A)$. So, $\lambda = 2$ has geometric multiplicity $\dim(E_2(A)) = 2$.

Summary table for A:

eigenvalue λ	alg. mult. of λ	basis for $E_\lambda(A)$	geometric mult.
$\lambda = 1$	1	$\begin{pmatrix}-2\\1\\1\end{pmatrix}$	1
$\lambda = 2$	2	$\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$	2

$\boxed{\text{Ex:}}$ (HW 8 #1(b))

Let $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

Find the eigenvalues, basis for each eigenspace, alg. & geom. mult. of each eigenvalue.

Eigenvalue time!

characteristic poly.

$$\det(A - \lambda I_2) =$$

$$= \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}\right)$$

$$= \det \begin{pmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{pmatrix}$$

$$= (10-\lambda)(-2-\lambda) - (-9)(4)$$

$$= -20 - 10\lambda + 2\lambda + \lambda^2 + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$$= (\lambda - 4)(\lambda - 4)$$

$$= (\lambda - 4)^{2}$$

eigenvalue λ	alg. mult.
$\lambda = 4$	2

Let's get a basis for

$$E_4(A) = \{ \vec{x} \mid A\vec{x} = 4\vec{x} \}$$

Need to solve $A\vec{x} = 4\vec{x}$.

Let's solve!

$$\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix} \quad \leftarrow A\vec{x} = 4\vec{x}$$

$$\begin{pmatrix} 10a - 9b \\ 4a - 2b \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \end{pmatrix}$$

$$\begin{pmatrix} 6a - 9b \\ 4a - 6b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives:

$$\boxed{\begin{aligned} 6a - 9b &= 0 \\ 4a - 6b &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right) \xrightarrow{\frac{1}{6}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So we get:

$$\boxed{\begin{aligned} a - \tfrac{3}{2}b &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a

free: b

Solutions:

$$\boxed{\begin{aligned} b &= t \\ a &= \tfrac{3}{2}b = \tfrac{3}{2}t \end{aligned}}$$

Thus if $\vec{x}$ solves $A\vec{x} = 4\vec{x}$

then $\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3/2\, t \\ t \end{pmatrix} = t \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

Thus, a basis for $E_4(A)$ is $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$. Thus, $\lambda = 4$ has geometric multiplicity $\dim(E_4(A)) = 1$.

Summary table for A

eigenvalue λ	alg. mult. of λ	basis for $E_\lambda(A)$	Geometric mult. of λ
$\lambda = 4$	2	$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$	1