Math 2550-01
4/25/24

Recall from Tuesday last week

A is an $n \times n$ matrix

$A\vec{x} = \lambda \vec{x}$ where $\vec{x} \neq 0$

$\lambda \leftarrow$ eigenvalue

$\vec{x} \leftarrow$ eigenvector

You can find the eigenvalues of A by solving

$$\det(A - \lambda I_n) = 0$$

$$E_\lambda(A) = \{ \vec{x} \mid A\vec{x} = \lambda \vec{x} \}$$

eigenspace for λ

Facts / Defs

Let A be an $n \times n$ matrix with eigenvalue λ.

1. The eigenspace $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.
2. The dimension of $E_\lambda(A)$ is called the <u>geometric multiplicity</u> of λ
3. The <u>algebraic multiplicity</u> of λ is the multiplicity of λ as a root of the characteristic polynomial of A.
4. $$\begin{pmatrix} \text{geometric} \\ \text{multiplicity} \\ \text{of } \lambda \end{pmatrix} \leq \begin{pmatrix} \text{algebraic} \\ \text{multiplicity} \\ \text{of } \lambda \end{pmatrix}$$

Example we started last time

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\det(A - \lambda I_3) = -(\lambda-2)^{2}(\lambda-1)^{1}$$

Eigenvalue	algebraic multiplicity
$\lambda = 1$	1
$\lambda = 2$	2

A basis for eigenspace $E_1(A)$ for $\lambda=1$ is $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$. So, $\dim(E_1(A)) = 1$.

Thus, the geometric multiplicity of $\lambda=1$ is 1.

Let's now get a basis for $E_2(A)$.

We need to solve $A\vec{x} = 2\vec{x}$.

So, need to solve

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} -2c \\ a + 2b + c \\ a + 3c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix}$$

$$\begin{pmatrix} -2a - 2c \\ a + c \\ a + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Need to solve

$$\boxed{\begin{array}{rcl} -2a - 2c &=& 0 \\ a + c &=& 0 \\ a + c &=& 0 \end{array}}$$

Let's solve it!

$$\left(\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[-R_1+R_3 \to R_3]{2R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} a + c &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a

free: b, c

Solution:

$b = t$

$c = u$

$a = -c = -u$

Thus, if $\vec{x}$ solves $A\vec{x} = 2\vec{x}$ then

$$\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -u \\ t \\ u \end{pmatrix}$$

$$= \begin{pmatrix} -u \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$$

$$= u\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ can make any $\vec{x}$ that solves $A\vec{x} = 2\vec{x}$.

So, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span the eigenspace $E_2(A)$

You can verify that $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

So, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is a basis for $E_2(A)$.

So, $\dim(E_2(A)) = 2$.

Thus, the geometric multiplicity of $\lambda = 2$ is 2.

Summary table

eigenvalue λ	alg. mult. of λ	basis for $E_\lambda(A)$	geometric mult. λ
$\lambda = 1$	1	$\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$	1
$\lambda = 2$	2	$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	2

$\boxed{\text{Ex:}}$ (HW 8 #1(b))

Let $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

Find eigenvalues, basis for each eigenspace, alg. & geometric mult. of each eigenvalue.

Eigenvalue time!

characteristic polynomial

$\det(A - \lambda I_2) =$

$= \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

$= \det\left(\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}\right)$

$= \det\begin{pmatrix} 10-\lambda & -9 \\ 4 & -2-\lambda \end{pmatrix}$

$$= (10-\lambda)(-2-\lambda) - (-9)(4)$$

$$= -20 - 10\lambda + 2\lambda + \lambda^2 + 36$$

$$= \lambda^2 - 8\lambda + 16$$

$$= (\lambda - 4)(\lambda - 4)$$

$$= (\lambda - 4)^{2}$$

The only eigenvalue is $\lambda = 4$.

The algebraic multiplicity of $\lambda = 4$ is 2.

Let's find a basis for the eigenspace $E_4(A) = \{\vec{x} \mid A\vec{x} = 4\vec{x}\}$

Let's solve!

$$\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 4\begin{pmatrix} a \\ b \end{pmatrix} \quad \leftarrow A\vec{x} = 4\vec{x}$$

$$\begin{pmatrix} 10a - 9b \\ 4a - 2b \end{pmatrix} = \begin{pmatrix} 4a \\ 4b \end{pmatrix}$$

$$\begin{pmatrix} 6a - 9b \\ 4a - 6b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Need to solve:

$$\boxed{\begin{aligned} 6a - 9b &= 0 \\ 4a - 6b &= 0 \end{aligned}}$$

Solving we get:

$$\left(\begin{array}{cc|c} 6 & -9 & 0 \\ 4 & -6 & 0 \end{array}\right) \xrightarrow{\frac{1}{6}R_1 \to R_1} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 4 & -6 & 0 \end{array}\right)$$

$$\xrightarrow{-4R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

This gives:

$$\boxed{\begin{aligned} a - \tfrac{3}{2}b &= 0 \\ 0 &= 0 \end{aligned}}$$

leading: a

free: b

Thus,

$$\boxed{\begin{aligned} b &= t \\ a &= \tfrac{3}{2}b = \tfrac{3}{2}t \end{aligned}}$$

Ergo, any solution $\vec{x}$ to $A\vec{x} = 4\vec{x}$ is of the form

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{3}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$$

Thus, a basis for $E_4(A)$ is $\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$. So the geometric multiplicity of $\lambda = 4$ is $\dim(E_4(A)) = 1$

Summary table for $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$

Eigenvalue λ	alg. mult. of λ	basis for $E_\lambda(A)$	geometric mult.
$\lambda = 4$	2	$\begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$	1