On the Permanent of Certain $(0,1)$ Toeplitz Matrices

Bruno Codenotti* Valentino Crespi† Giovanni Resta‡]

Abstract

In this paper we obtain convenient expressions and/or efficient algorithms for the permanent of certain very sparse $(0,1)$ Toeplitz matrices. The classes of matrices considered here include some nontrivial examples of circulants to which none of the previous approaches could be successfully applied.

1 Introduction

The computation of the permanent of a matrix is a challenging task. The problem is computationally very hard, even for $(0,1)$ matrices. In fact, Valiant proved that computing the permanent of a $(0,1)$ matrix is $\#P$-complete (see [V179] and [V279]). The class $\#P$ contains those functions that can be computed in polynomial time by a counting (nondeterministic) Turing machine, and the $\#P$-complete problems represent the hardest problems within the class. The existence of a polynomial time algorithm for a $\#P$-complete problem would not only imply the existence of a polynomial time algorithm for NP-complete problems, but also the polynomial time computability of the number of solutions to NP-complete problems. Thus it is extremely unlikely that there is a polynomial time algorithm for computing the permanent. Actually, the best known algorithm for computing the permanent is due to Ryser [R63] and takes $O(n\,2^n)$ operations, where n is the matrix size.

More recently, several authors have found even stronger negative results, showing that it is also unlikely that

- the permanent can be efficiently (i.e., polynomial time) approximated [DLMV88];
- the permanent of random matrices (as opposed to the matrices used in Valiant reduction) can be efficiently computed even for a small fraction of the instances [FL92];
- the permanent of very sparse matrices can be efficiently computed [DLMV88].

The (above justified) little hope to get efficient algorithms for matrices without a very special structure, motivated a stream of research work oriented towards analyzing the permanent of restricted classes of matrices. Considerable attention has been devoted to matrices for which the permanent computation can be conveniently transformed into the computation

*Istituto di Matematica Computazionale del CNR, Via S. Maria 46, 56126-Pisa (Italy). e-mail: {codenotti,resta}@imc.pi.cnr.it.

†Dipartimento di Informatica, Università degli Studi di Milano, Milano (Italy). e-mail: crespi@dsi.unimi.it. Part of this work has been done while visiting IMC-CNR in Pisa.

‡[

of the determinant of a matrix of the same size (see [BS95] for updated references on this subject). Furthermore several authors have found either explicit expressions or recurrence formulas for the permanent of some special circulant matrices (see, e.g., [Mi85], [Mi87], [MSS69], [KP69]).

Note that the computation of the permanent of certain Toeplitz and in particular circulant matrices has applications to a number of combinatorial enumeration problems, e.g., the famous *Rencontres* and *Ménage* problems [Mi78, BR91].

This paper contains an analysis of the permanents of the most sparse Toeplitz matrices for which the problem is already non trivial. These matrices include some examples of circulants for which none of the previous approaches could be successfully employed.

For instance, we find efficient algorithms for the computation of the permanent of certain $(0,1)$-circulants with three nonzero entries per row, and, in some cases, we determine the explicit expression for their permanent. These results are obtained by taking advantage of the fact that these matrices have submatrices with very special properties.

The rest of this paper is organized as follows. In Section 2 we introduce the main notation used throughout the paper, we briefly review some of the work by Minc [Mi85, Mi87] on circulant matrices, and we present a simple analysis of the efficiency of his approach. In Section 3 we find the expression for the permanent of some $(0,1)$ Toeplitz matrices with at most three nonzero entries per row. These results will then be used in Section 4. Indeed they will be instrumental to the development of efficient strategies for computing the permanent of some $(0,1)$-circulant matrices with three nonzero elements per row (Section 4.1). For other kinds of $(0,1)$-circulant matrices with three nonzero elements per row, we take advantage of the circulant structure of certain submatrices in order to find explicitly the value of their permanent. This will then be the starting point to show that the value of the permanents of certain circulant matrices does not depend on all the information needed to describe them (Section 4.6). Supported by experimental evidences, in Section 5 we show that this last fact is just an example of the *symmetries* that come into play, and we present some conjectures and open questions related to structural and computational properties of the permanents of the matrices considered in the paper. Concluding remarks are in Section 6.

2 Preliminaries

Let Σ be the set of all permutations of the first n integers. The permanent of an $n \times n$ matrix $A = (a_{i,j})$ is defined as $\sum_{\sigma \in \Sigma} \prod_{i=1}^{n} a_{i,\sigma_i}$, where $\sigma = (\sigma_1, \ldots, \sigma_n)$. We will denote the permanent and the determinant of a square matrix A as $\mathrm{per}(A)$ and $\det(A)$, respectively. A $(0,1)$-matrix A is said to be *convertible* if there exists a $(-1,1)$-matrix X such that $\mathrm{per}(A) = \det(A \star X)$, and $\star$ denotes the elementwise product, i.e., the (i,j)-th entry of the matrix $A \star X$ is $a_{i,j}x_{i,j}$.

The permanent of a $(0,1)$ matrix has an interpretation in terms of both the digraph and the bipartite graph that can be associated with the matrix. More precisely, if A is an $n \times n$ $(0,1)$ matrix, we denote by $D(A)$ the digraph whose adjacency matrix is A and by $G[A]$ the $2n$-node bipartite graph associated with A in the natural way. Then the permanent of A is equal to the number of cycle covers of $D(A)$ as well as to the number of perfect matchings of $G[A]$. Recall that a cycle cover of $D(A)$ is a node disjoint covering of all the nodes of $D(A)$ in terms of its cycles, whereas a matching of $G[A]$ is a set of pairwise node disjoint edges, and a perfect matching (or 1-factor) of $G[A]$ is a matching such that each node of $G[A]$ is incident to exactly one of the edges forming the matching.

Let P be a permutation matrix. Sometimes we will denote the symmetric permutation PAP on the matrix A as $(\sigma_1, \ldots, \sigma_n)$, with the meaning that the σ_i-th row and column of A become the i-th row and column of PAP.

We will use the *Big-Oh* notation for orders of magnitudes, i.e., $O(m)$ will stand for "asymptotically not greater than cm, where c is a constant with respect to m".

Let P_n denote the $(0,1)$ $n \times n$ matrix with 1's only in positions $(i, i+1)$, $i = 1, 2, \ldots, n-1$, and $(n, 1)$. A matrix $P^{t_1} + P^{t_2} + \ldots + P^{t_k}$, where $0 \leq t_1 < t_2 < \ldots < t_k < n$, is called $(0,1)$-circulant matrix of type $(t_1, t_2, \ldots, t_k)$.

Metropolis, Stein and Stein in [MSS69] used a combinatorial argument to obtain linear recurrence formulas for the permanent of $(0,1)$-circulants of type $(0, 1, 2, \ldots, k-1)$. In [Mi85], Minc extended their results to obtain similar recurrences for a wider class of $(0,1)$-circulants.

More precisely, Minc's recurrence formulas consist of expressing $\text{per}(A_n)$, where A_n is an $n \times n$ $(0,1)$-circulant matrix of type $(t_1, t_2, \ldots, t_k)$, as a linear combination of $\text{per}(A_{n-i})$, for $i = 1, 2, \ldots, 2^{t_k - 1}$, under the assumption that $n \geq 2^{t_k} + 2t_k - 3$. So these recurrences can be applied only if $t_k = O(\log n)$. In addition, they provide an efficient way to compute the permanent only if t is very small compared with n. In particular it is easy to see that, if A is a $(0,1)$-circulant matrix of type $(t_1, t_2, \ldots, t_k)$, with $t_k = O(\log \log n)$, then $\text{per}(A)$ can be computed in $O(n \log^2 n)$ operations by solving a triangular linear system induced by Minc's recurrences.

On the other hand, these recurrences give no hints at how to compute the permanent when $2^{t_k} + 2t_k - 3 \geq n$.

The recurrence formulas of [Mi85] (see also [Mi87]) are obtained in terms of the *permanental compounds* of the matrix A_n. In some special cases, such as matrices of type $(0, 1, t)$, the permanental compounds turn out to be *convertible*. Taking advantage of this, it is possible to find asymptotic expressions for the n-th root of their permanents.

3 $(0,1)$ Toeplitz matrices with at most 3 nonzeros per row and column

Let us consider matrices of the form $T_n[i,j] = I_n + Q_n^i + (Q_n^T)^j$, where Q is the $n \times n$ upper triangular Toeplitz matrix whose first row is $[\ 0\ 1\ 0\ 0\ \ldots\ 0\ 0\ ]$.

For these matrices, which are easily seen to be convertible, we have the following result.

Theorem 1

$$\text{per}(I + Q^i + (Q^T)^j) = \begin{cases} \det(I + Q^i + (Q^T)^j) & \textit{if } \frac{i+j}{\gcd(i,j)} \textit{ is odd,} \\ \det(I - Q^i + (Q^T)^j) & \textit{otherwise.} \end{cases}$$

Proof. W.l.o.g. we assume that $i < j$. Let $A = Q^i + (Q^T)^j$, and consider $D(A)$. We can restrict ourselves to the case when there are cycles in $D(A)$, because the absence of cycles implies that the nonzero off diagonal entries do not contribute to the value of the permanent of $I + Q^i + (Q^T)^j$, which is thus equal to one.

We analyze the length of the cycles in $D(A)$. Let x be a node belonging to a cycle γ, and let $l = \text{length}(\gamma)$. Starting from x we reach the next node along γ either by adding i or by subtracting j as long as $x+i \leq n$ (resp. $x-j \geq 1$). Thus, γ can be represented by a sequence of integers $d_1, d_2, \ldots, d_l$, where d_k is equal to either i or $-j$, for $k = 1, 2, \ldots, l$ and the following properties hold:

1. $\sum_{k=1}^{l} d_k = 0$,

2. $\sum_{k=u}^{v} d_k \neq 0$, $\forall\ 1 \leq u < v \leq l$.

Let $m = \operatorname{lcm}(i,j)$ and h, k be the two positive integers such that $m = hi = kj$. Then $x + chi - ckj = x$, for any positive integer c, so that $\text{length}(\gamma) = c(h+k)$, for some $c \geq 1$, i.e., the lengths of the cycles must be multiples of $h+k$. Before proceeding we show that, for each cycle γ, $\text{length}(\gamma)$ must be exactly $h+k$.

We want to prove that any sequence of ch elements i and ck elements $-j$ satisfies condition 2 only if $c=1$. Let $s = h+k$ and $c > 1$, and suppose we are given a sequence $d_1, d_2, \ldots, d_{cs}$, satisfying condition 1, i.e., $\sum_{k=1}^{cs} d_k = 0$. This sequence contains exactly ch elements equal to i and ck elements equal to $-j$. Let $S_u = \sum_{k=u}^{u+s-1} d_k$. We introduce variables h_u and k_u, for $u = 1, 2, \ldots, cs - s + 1$, such that $S_u = h_u \cdot i - k_u \cdot j$. Let $\Delta^{(t)}h = h_t - h$, and $\Delta^{(t)}k = k_t - k$. We now consider the sequence $S_1, S_{s+1}, S_{2s+1}, \ldots, S_{(c-1)s+1}$. Since there are $c \cdot h$ elements equal to i, then $\sum_{t=0}^{c-1} \Delta^{(ts+1)}h = 0$. W.l.o.g. suppose that, for some t', $\Delta^{(t's+1)}h \neq 0$ and $\Delta^{(t's+1)}h < 0$. Then there must be another t'' such that $\Delta^{(t''s+1)}h > 0$. Suppose now that $t' < t''$, and consider the sequence $S_{t's+1}, S_{t's+2}, S_{t's+3}, \ldots, S_{t''s+1}$. Since for each u, $\Delta^{(u+1)}h$ is either $\Delta^{(u)}h$ or $\Delta^{(u)}h \pm 1$, then $\Delta^{(u)}h$ must be zero for some u, $t's+1 \leq u \leq t''s+1$, and thus condition 2 is violated.

If $h+k$ is odd, then all the cycle have odd length and the proof follows by a result of Bassett, Maybee and Quirk [BMQ68]. If $h+k$ is even, then both h and k must be odd by the property of lcm. Thus, after changing the sign to the entries of the main diagonal elements and to the entries of one of the two other diagonals, each cycle gets weight -1 (the weight of a path is obtained by multiplying altogether the weights of the edges belonging to it). Then we can apply Bassett, Maybee and Quirk result to guarantee that $\text{per}(A) = |\det(-I + Q^i - (Q^T)^j)|$. Finally, since the conversion matrix is unique up to multiplication of rows or columns by -1, we obtain $\text{per}(A) = \det(I - Q^i + (Q^T)^j)$ as claimed. □

In two special cases, there are explicit formulas for the permanent of $I + Q^i + (Q^T)^j$. These cases occur for

- $j = n - i + 1$, i.e., the matrix is circulant of type $(0, i)$;
- $j = ki$ (or $i = kj$), for a positive integer k, which include the cases when the matrix is symmetric ($i = j$) and when the matrix is lower (resp. upper) Hessenberg, i.e., $i = 1$ (resp. $j = 1$).

Indeed we have the following.

Theorem 2 *Let $F_j(n)$ be the n-th number of the sequence $F_j(k) = F_j(k-1) + F_j(k-j-1)$, with $F_j(h) = 1$, for any $h \leq j$. In particular, $F_1(n)$ is the $(n+1)$-th Fibonacci number. Then*

$$\begin{array}{lrcl}
(1) & \mathrm{per}(T_n[i,i]) & = & F_1\left(\lfloor \frac{n}{i} \rfloor\right)^i \cdot \left(\frac{F_1(\lceil \frac{n}{i} \rceil)}{F_1(\lfloor \frac{n}{i} \rfloor)}\right)^{n - \lfloor \frac{n}{i} \rfloor \cdot i}. \\
(2) & \mathrm{per}(T_n[i, n-i+1]) & = & 2^{\gcd(n,i)}. \\
(3) & \mathrm{per}(T_n[1,j]) & = & F_j(n) \quad (\textit{resp. } \mathrm{per}(T_n[i,1]) = F_i(n)). \\
(4) & \mathrm{per}(T_n[i,ki]) & = & \mathrm{per}(T_{\lceil \frac{n}{i} \rceil}[1,k])^{n - \lfloor \frac{n}{i} \rfloor i} \mathrm{per}(T_{\lfloor \frac{n}{i} \rfloor}[1,k])^{i - n + \lfloor \frac{n}{i} \rfloor i}.
\end{array}$$

Proof. Case 1. First of all note that, if $i = 1$, then the matrix is tridiagonal and it is easy to see that $\mathrm{per}(T_n[1,1]) = F_1(n)$.

If we let $n = q \cdot i + r$, with $0 \leq r < i$, then the equality to check can be restated as $\mathrm{per}(T_n[i,i]) = F_1(q)^{i-r} \cdot F_1(q+1)^r$, if $r \neq 0$, and $\mathrm{per}(T_n[i,i]) = F_1(q)^i$, if $r = 0$. Note that the digraph $D(T_n[i,i])$ consists of the disjoint union of $i - r$ chains of q nodes and r chains of $q+1$ nodes for a total amount of i chains. To show this, let x, $1 \leq x \leq i$, be one of the first i nodes of $D(T_n[i,i])$. Then x belongs to the chain

$$x \leftrightarrow x + i \leftrightarrow x + 2i \leftrightarrow \ldots \leftrightarrow x + k \cdot i,$$

where $x + k \cdot i \leq n$ and $x + (k+1) \cdot i > n$. The chain contains $k+1$ nodes. We can now estimate the value of k, which is the greatest integer such that $x + k \cdot i \leq n$. Thus $k = \lfloor \frac{n-x}{i} \rfloor$. Substituting $n = q \cdot i + r$, we get $k = \lfloor q + \frac{r}{i} - \frac{x}{i} \rfloor$, from which we finally obtain

$$k = \begin{cases} q & \text{for } x = 1, 2, \ldots, r, \\ q - 1 & \text{for } x = r+1, r+2, \ldots, i. \end{cases}$$

We can conclude that there are r disjoint chains of $k + 1 = q + 1$ nodes and $i - r$ chains of $k + 1 = q$ nodes. Furthermore, since these i chains involve $(i - r) \cdot q + r \cdot (q + 1) = n$ nodes overall and considering the degree of each node, then those chains are exactly the whole digraph $D(T_n[i,i])$. The statement then follows because the permanent of the chains is given by the appropriate Fibonacci number.

Case 2. The proof is easily seen by counting the number of matchings in the bipartite graph associated with the matrix.

Case 3. The proof follows by applying Laplace expansion.

Case 4. The proof is easily seen by reducing the matrix to block Hessenberg form, by means of a symmetric row and column permutation defined as

$$\left(1, i+1, 2i+1, \ldots, \left(\left\lfloor \frac{n}{i} \right\rfloor - 1\right) i + 1, 2, i+2, 2i+2, \ldots, \left(\left\lfloor \frac{n}{i} \right\rfloor - 2\right) i + 2, 3, \ldots\right).$$

□

Expression (4) of Theorem 2 for $\mathrm{per}(T_n[i,ki])$ can be generalized. Indeed, if we let $g = \gcd(i,j)$, then

$$\mathrm{per}(T_n[i,j]) = \mathrm{per}\left(T_{\lceil \frac{n}{g} \rceil}\left[\frac{i}{g}, \frac{j}{g}\right]\right)^{n - \lfloor \frac{n}{g} \rfloor g} \mathrm{per}\left(T_{\lfloor \frac{n}{g} \rfloor}\left[\frac{i}{g}, \frac{j}{g}\right]\right)^{g - n + \lfloor \frac{n}{g} \rfloor g}.$$

This equality is obtained by applying to the matrix $T_n[i,j]$ the row and column permutation defined as

$$(1, g+1, 2g+1, \ldots, \left(\left\lfloor \frac{n}{g} \right\rfloor - 1\right) g+1, 2, g+2, 2g+2, \ldots, \left(\left\lfloor \frac{n}{g} \right\rfloor - 2\right) g+2, 3, \ldots).$$

This leaves still open the problem of finding an explicit expression for the permanent of $T_n[i,j]$, when i and j are relatively prime.

A more general type of $(0,1)$ Toeplitz matrix with at most three nonzeros per row takes the form $Q_n^i + Q_n^j + (Q_n^T)^k$. For these matrices we can obtain results similar to those for matrices of the form $I_n + Q_n^h + (Q_n^T)^k$. In fact the matrix $A_n = Q_n^i + Q_n^j + (Q_n^T)^k$

1. is a submatrix of $I_{n+i} + Q_{n+i}^{j-i} + (Q_{n+i}^T)^{k+i}$,
2. contains as a submatrix $I_{n-i} + Q_{n-i}^{j-i} + (Q_{n-i}^T)^{k+i}$.

¿From (1) we can deduce that A_n is convertible, because the bipartite graph associated with $I_{n+i} + Q_{n+i}^{j-i} + (Q_{n+i}^T)^{k+i}$ is planar, which implies that the bipartite graph associated with A_n is also planar.
From (2), and by observing that the remaining submatrices of A_n have exactly a one per row or column, one can see that the actual conversion can be efficiently computed by adapting Brualdi and Shader's algorithm (see [BS95]) to this special case.

4 $(0,1)$ Circulant matrices with 3 nonzeros per row

Although circulant matrices have a very nice special structure, the evaluation of their permanent is far from trivial. Even for $(0,1)$-circulant matrices of type $(0, d_1, d_2)$ there is no *ad hoc* approach that can be efficiently adopted, for arbitrary values d_1 and d_2 (see Section 2). In this section we analyze several properties of the permanent of this kind of matrices and, in some cases, we find explicit expressions and/or efficient algorithms. Note that the bipartite graph associated with $(0,1)$-circulant matrices of type $(0, d_1, d_2)$ is 3-regular. The results of [DLMV88] show that, in general, counting the number of perfect matchings in 3-regular graphs is $\#P$-complete. This leads to the question of whether or not the circulant structure makes the problem significantly easier. The results of this section provide a partial positive answer to this question.

The main difference between our approach and that adopted in [Mi85, Mi87] is that we look for recurrence relations expressing the permanent of a circulant matrix in terms of smaller matrices not necessarily of circulant form, whereas Minc's recurrences insist on the circulant structure. This greater generality allows us to obtain more efficient computational schemes, in some cases.

4.1 An efficient algorithm

We show an algorithm for computing the permanent of $(0,1)$-circulants with three nonzero entries per row which takes advantage of the convertibility of some of their submatrices.

Before stating the result we need some simple Lemmas and Definitions.

Lemma 3 *Let A be a square $(0,1)$ matrix such that $G[A]$ is planar. Then the bipartite graph associated with any square submatrix of A is planar.*

Lemma 4 *Let A be a square $(0,1)$ matrix such that $a_{ij}=1$. Then*

$$\operatorname{per}(A)=\operatorname{per}(A-E_{ij})+\operatorname{per}(A(i|j))\,,$$

where E_{ij} denotes the matrix whose only nonzero entry is in position (i,j), and $A(i|j)$ denotes the matrix obtained by deleting the i-th row and j-th column of A.

Definition 1 *Let us denote with $\mathcal{P}_{k,n}$ the collection of all k-subsets of the n-set $\{1,2,\ldots,n\}$. Let A be a $(0,1)$ $n\times n$ matrix. Then, for $\alpha,\beta\in\mathcal{P}_{k,n}$, we denote with $A[\alpha,\beta]$ the $k\times k$ submatrix of A determined by rows $i\in\alpha$ and columns $j\in\beta$. Then $\operatorname{per}(A[\alpha,\beta])$ is called a* permanental k-minor *of A and we define $p_k(A)$ as the sum of all the permanental k-minors of A, i.e.,*

$$p_k(A)=\sum_{\alpha\in\mathcal{P}_{k,n}}\sum_{\beta\in\mathcal{P}_{k,n}}\operatorname{per}(A[\alpha,\beta])\,. \tag{1}$$

$p_k(A)$ counts the number of different selections of k ones in A, such that each row and column has at most a nonzero entry.

Lemma 5 *Let A be a $(0,1)$ $n\times n$ matrix, and let $a_{ij}=1$. Then $p_k(A)=p_k(A-E_{ij})+p_{k-1}(A(i|j))$, for $k\geq 2$, and $p_1(A)=p_1(A-E_{ij})+1$.*

Proof. Equality $p_1(A)=p_1(A-E_{ij})+1$ is clearly true. ¿From the definition of p_k, and separating the submatrices that contain the element a_{ij} from those that do not contain it, for $k\geq 2$, we have

$$p_k(A)=\sum_{\substack{\alpha\in\mathcal{P}_{k,n}\\ i\in\alpha}}\sum_{\substack{\beta\in\mathcal{P}_{k,n}\\ j\in\beta}}\operatorname{per}(A[\alpha,\beta])+\sum_{\substack{\alpha\in\mathcal{P}_{k,n}\\ i\notin\alpha\,\vee\, j\notin\beta}}\sum_{\beta\in\mathcal{P}_{k,n}}\operatorname{per}(A[\alpha,\beta])\,.$$

By Lemma 4, we can write

$$\begin{aligned}\operatorname{per}(A[\alpha,\beta]) &= \operatorname{per}(A[\alpha,\beta]-E_{ij})+\operatorname{per}(A[\alpha-\{i\},\beta-\{j\}])\\ &= \operatorname{per}((A-E_{ij})[\alpha,\beta])+\operatorname{per}(A[\alpha-\{i\},\beta-\{j\}]),\end{aligned}$$

if $i\in\alpha$, $j\in\beta$, while, if $i\notin\alpha\vee j\notin\beta$, we clearly have $\operatorname{per}(A[\alpha,\beta])=\operatorname{per}((A-E_{ij})[\alpha,\beta])$. Thus we obtain

$$\begin{aligned}p_k(A) &= \sum_{\substack{\alpha\in\mathcal{P}_{k,n}\\ i\in\alpha}}\sum_{\substack{\beta\in\mathcal{P}_{k,n}\\ j\in\beta}}\operatorname{per}((A-E_{ij})[\alpha,\beta])+\sum_{\substack{\alpha\in\mathcal{P}_{k,n}\\ i\in\alpha}}\sum_{\substack{\beta\in\mathcal{P}_{k,n}\\ j\in\beta}}\operatorname{per}(A[\alpha-\{i\},\beta-\{j\}])\\ &+\sum_{\substack{\alpha\in\mathcal{P}_{k,n}\\ i\notin\alpha\,\vee\, j\notin\beta}}\sum_{\beta\in\mathcal{P}_{k,n}}\operatorname{per}((A-E_{ij})[\alpha,\beta])\,.\end{aligned}$$

¿From the definition of p_k it follows that the sum of the first and the third terms of the last formula is indeed $p_k(A-E_{ij})$, while the second term corresponds to $p_{k-1}(A(i|j))$, and the thesis follows. □

Lemma 6 *Let $A = (a_{ij})$ be an $n \times n$ $(0,1)$ matrix, and let $z(A)$ denote the number of different $(0,1)$ matrices $M = (m_{ij})$ with at most one nonzero entry in each row and column, satisfying $M \leq A$, i.e., $m_{ij} \leq a_{ij}$, for all pairs (i,j). Then, for each nonzero entry a_{ij}, we have*

$$z(A) = \sum_{k=1}^{n} p_k(A) \tag{2}$$

$$z(A) = z(A - E_{ij}) + z(A(i|j)), \tag{3}$$

and, in general, if the matrix A has k nonzero entries, then $k+1 \leq z(A) \leq 2^k$.

Proof. Equality (2) easily follows from the definitions of $z(A)$ and $p_k(A)$, while (3) follows from (2) and from Lemma 5. □

We now prove a theorem that will be instrumental to the definition of an efficient algorithm for the computation of the permanent of circulants of type $(0, d_1, d_2)$.

We first need the following.

Lemma 7 *The permanent of an $n \times n$ convertible matrix A for which $G[A]$ is planar can be computed in $O(n^\gamma)$ time, $\gamma < 3$.*

Proof. The proof follows, e.g., from the results of [VV89], where it is shown that, if $G[A]$ is planar, then the overall running time for the computation of the permanent of A is dominated by the determinant computation. □

Theorem 8 *Let A, B, and C be $n \times n$ $(0,1)$ matrices such that $A = B + C$, and let $G[B]$ be planar. Then $\mathrm{per}(A)$ can be computed in $O(z(C)n^\gamma)$ time, $\gamma < 3$.*

Proof. The proof is by induction on the number k of ones in C.

- If $k = 0$, then $A = B$, $z(C) = 1$, and, since $G[A]$ is planar, then $\mathrm{per}(A)$ can be computed in time $O(n^\gamma)$, $\gamma < 3$, by Lemma 7.
- If $k > 0$, let us consider a one of C in position, say, (i,j), and let $C' = C - E_{ij}$. Then, by Lemma 4, we have

  $$\mathrm{per}(B+C) = \mathrm{per}(B+C-E_{ij}) + \mathrm{per}((B+C)(i|j)) = \mathrm{per}(B+C') + \mathrm{per}((B(i|j)+C(i|j))$$

  and by Lemma 6

  $$z(C) = z(C') + z(C(i|j)). \tag{4}$$

  The matrices $C' = C - E_{ij}$ and $C(i|j)$ are short of a nonzero entry with respect to C, while $G[B(i|j)]$ is planar by Lemma 3. Hence, by induction, we can claim that $\mathrm{per}(B + C')$ and $\mathrm{per}(B(i|j) + C(i|j))$ can be computed in $O(z(C')n^\gamma)$ and $O(z(C(i|j))(n-1)^\gamma)$ time, respectively. Summing up the two time bounds and using equality 4, we obtain

  $$O(z(C')n^\gamma) + O(z(C(i|j))(n-1)^\gamma) = O([z(C') + z(C(i|j))]n^\gamma) = O(z(C)n^\gamma),$$

  from which the thesis follows.

□

Lemma 9 *The bipartite graphs $G[I + Q^i + Q^j]$ and $G[I + Q^i + (Q^T)^j]$ are planar.*

Proof. Let $A = I_n + Q_n^i + (Q_n^T)^j$. We assume, w.l.o.g., that $\gcd(i, j) = 1$ (see the previous section). For simplicity, consider first the case $n = 2(i + j)$. The matrix A can be written as

$$A = \begin{bmatrix} U & B \\ C & V \end{bmatrix},$$

where $U = V = I_{\frac{n}{2}} + P_{\frac{n}{2}}^i$, $B = (Q_{\frac{n}{2}}^T)^j$ and $C = Q_{\frac{n}{2}}^i$. Since $\gcd(i, j) = 1$, then $\gcd(i, i + j) = 1$, and $\gcd(i, \frac{n}{2}) = 1$. This means that both $G[U]$ and $G[V]$ are cycles of n nodes, so that $G[A]$ is composed of two identical cycles connected by the $\frac{n}{2}$ edges in $G[B]$ and $G[C]$. To avoid edge intersections, we first draw the two cycles in a concentrical way, e.g., $G[U]$ inside $G[V]$. We show that the edges that connect them never cross each other. Let $h = \frac{n}{2}$. The nodes of $G[U]$ and $G[V]$ can be labeled consistently with the traversal direction of the cycles as

$$(1_r)^{in}, (1 + i)_c^{in}, (1 + i)_r^{in}, (1 + 2i)_c^{in}, (1 + 2i)_r^{in}, \ldots, (1 + (h - 1)i)_r^{in}, 1_c^{in},$$

and

$$(1_r)^{ou}, (1 + i)_c^{ou}, (1 + i)_r^{ou}, (1 + 2i)_c^{ou}, (1 + 2i)_r^{ou}, \ldots, (1 + (h - 1)i)_r^{ou}, 1_c^{ou},$$

respectively. (The symbols r, c, in, and ou are used to recall *row*, *column*, *inner cycle*, and *outer cycle*, respectively.)

Note that the edges in B and C can be drawn without crossovers starting from node $(1_r)^{ou}$ in the following way:

$$\begin{gathered} \{(1_r)^{ou}, (1 + i)_c^{in}\}, \\ \{(1 + i)_r^{in}, (1 + 2i)_c^{ou}\}, \\ \{(1 + 2i)_r^{ou}, (1 + 3i)_c^{in}\}, \\ \vdots \\ \{(1 + (h - 1)i)_r^{ou}, (1)_c^{in}\}. \end{gathered}$$

The above construction can be easily generalized to handle an arbitrary number of cycles, i.e., $n = k \cdot (i + j)$, k integer, by drawing them one inside the other in a concentrical way, and then applying the same strategy as above.

The planarity of $G[I_n + Q_n^i + Q_n^j]$ follows from the fact that it is a subgraph of $G[I_{n+i} + Q_{n+i}^{j-i} + (Q_{n+i}^T)^i]$. □

We are now ready to state our result.

Theorem 10 *Let $A = I_n + {P_n}^i + {P_n}^j$. Then* $\mathrm{per}(A)$ *can be computed in time $O(2^{i'+j'} n^{O(1)})$, where i' and j' are the two smallest numbers among $\{i, j, n - i, n - j\}$.*

Proof. The matrix A can be viewed as a Toeplitz matrix containing the identity and 4 diagonals of lengths $\{i, j, n - i, n - j\}$. It is thus possible to write $A = B + C$, where C contains the two shorter diagonals, of lengths i' and j', and B consists of the other three diagonals. By Lemma 9, $G[B]$ is planar and we can apply Theorem 8 to get the time bound $O(z(C)n^\gamma)$. The thesis follows since $z(C) \le 2^{i'+j'}$. □

From Theorem 10 we have that $\mathrm{per}(I_n + {P_n}^i + {P_n}^j)$ can be computed in polynomial time if i and j are either smaller than $O(\log n)$ or greater than $n - O(\log n)$.

In Section 4.4 we will use some properties of the bipartite graph $G[A]$ in order to strengthen the above result.

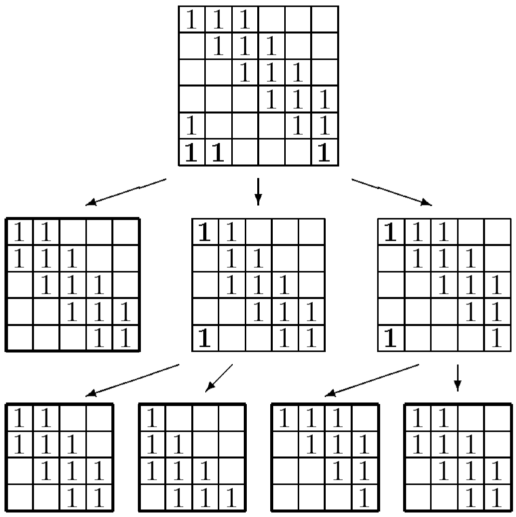

Figure 1: Laplace expansion for the permanent of $I + P + P^2$ for $n = 6$. The bold entries are those used in the elimination.

4.2 Permanent versus Determinant

The results of Section 4.1 can be used to describe some structural properties of the permanents of circulant matrices with three nonzeros per row, and, in particular, the relationship between these permanents and the determinants of Toeplitz matrices with at most three nonzeros per row.

Unlike the Toeplitz matrices of Section 3, $(0,1)$-circulant matrices of type $(0, d_1, d_2)$ are not convertible, except for very special cases, e.g., when $d_1 = 1$ and $d_2 = 2$ and the matrix has even size. Nevertheless their permanents bear some interesting connections with suitable determinants. These connections depend on the convertibility of the Toeplitz matrices of Section 3.

In fact, if one applies Laplace expansion to, say, a matrix of type $(0, 1, t)$, most of the submatrices induced after some steps, are the convertible Toeplitz matrices analyzed in Section 3.

The Laplace expansion for the permanent of $I+P+P^2$, shown in figure 1 for $n = 6$, outlines the close relationship of this matrix with the Fibonacci matrix $T_n[1, 1] = I + Q + Q^T$, whose permanent is $F(n+1)$. In particular one can see that $\mathrm{per}(I+P+P^2) = F(n)+2F(n-1)+2$. By comparing the expansion for the permanent and the determinant of $I + P + P^2$, and recalling that the matrix $T_n[1, 1]$ is convertible (and specifically that $\mathrm{per}(I+Q+Q^T) = \det(I-Q+Q^T)$) one can immediately see that, if n is even, then the matrix $I + P + P^2$ is convertible (see also [ST87]). In particular one obtains $\mathrm{per}(I + P + P^2) = \det(I + Q - (Q^T)^{n-1} - Q^2 + (Q^T)^{n-2})$.

Note that, if n is odd, $I + P + P^2$ is not convertible; nevertheless its permanent satisfies $\mathrm{per}(I + P + P^2) = 2 + \det(I + P - P^2)$, as one can readily check.

The same kind of analysis can be performed for the permanent of $I+P+P^3$. In this case, one of the possible ways to carry out Laplace expansion, leads to 14 submatrices, the sum of whose permanents is equal to $\mathrm{per}(I + P + P^3)$. In particular, one obtains two triangular matrices, and 12 convertible Toeplitz matrices, of sizes between $n-2$ and $n-6$, of the form $I+Q^2+Q^T$ and $Q+(Q^T)^2+Q^T$. If T_1 and T_2 denote the permanent of these (convertible) matrices, then

$$\begin{aligned}\text{per}(I+P+P^3) &= 2+T_1(n-2)+3T_1(n-3)+T_1(n-4) \\ &\quad +T_2(n-2)+2T_2(n-3)+T_2(n-4)+2T_2(n-5)+T_2(n-6)\end{aligned} \tag{5}$$

Summarizing, both $\text{per}(I+P+P^2)$ and $\text{per}(I+P+P^3)$ can be conveniently expressed in terms of a few determinants of Toeplitz matrices. These results can be generalized, although the corresponding formulas become more complicated.

4.3 Reduction

We show how certain permanent computations on circulant matrices of type $(0, d_1, d_2)$ can be reduced to the computation of either powers of permanents of smaller circulant matrices of the same type or permanents of circulants of type $(0, t_1, t_2)$, with $t_1 < d_1$ and $t_2 < d_2$.

Lemma 11 *Let A_n be an $n\times n$ $(0,1)$-circulant matrix of type $(0, da, db)$, i.e. $A_n = I_n + P_n^{da} + P_n^{db}$.*

1. *If n is a multiple of d, then*
$$\text{per}(A_n) = \text{per}(I_{\frac{n}{d}} + P_{\frac{n}{d}}^{a} + P_{\frac{n}{d}}^{b})^d.$$

2. *If $\gcd(d,n)=1$, then*
$$\text{per}(A_n) = \text{per}(I_n + P_n^a + P_n^b).$$

Proof.
(1) The proof follows from the fact that by simultaneous rows and columns permutations we can obtain from A_n a block matrix C_n of d square blocks of the type $B_{\frac{n}{d}} = I_{\frac{n}{d}} + P_{\frac{n}{d}}^{a} + P_{\frac{n}{d}}^{b}$.

(2) We show that $D(A_n)$ is isomorphic to $D(C_n)$, with $C_n = I_n + P_n^a + P_n^b$. Since $\gcd(d,n)=1$, d has a multiplicative inverse, denoted by d^{-1}, in the ring $\mathbf{Z}_n$ of the residue classes modulo n. Let $\psi : \mathbf{Z}_n \to \mathbf{Z}_n$ be the function defined as follows

$$\psi(x) = x \cdot d^{-1} \bmod n.$$

It turns out that this is an isomorphism between the two digraphs $D(A_n)$ and $D(C_n)$. In fact it is immediate to see that ψ is surjective and injective. In addition, we have to prove that ψ preserves the structure of the digraphs, i.e., for each pair of nodes $(x,y)\in\mathbf{Z}_n^2$, $(x,y)\in \mathrm{E}(D(A_n))$ if and only if $(\psi(x),\psi(y))\in\mathrm{E}(D(C_n))$. Indeed, if $(x,y)\in\mathbf{Z}_n^2$ then without loss of generality $y = x + d\cdot a \bmod n$. So

$$\begin{aligned}\psi(y) &= x\cdot d^{-1} + d\cdot a\cdot d^{-1} \bmod n \\ &= x\cdot d^{-1} + a \bmod n \\ &= \psi(x) + a \bmod n.\end{aligned}$$

Thus $(\psi(x),\psi(y))$ is an edge of $D(C_n)$. The converse can be proved in a similar way. □

As an example of application of Lemma 11, in Section 4.5 we find an explicit espression for the permanent of symmetric circulant matrices of the form $I+P^i+P^{n-i}$.

4.4 The bipartite graph $G[A_n]$

Let us consider $n \times n$ circulant matrices of type $(0, d_1, d_2)$, for n prime.

To analyze the permanent of the matrix $A_n = I + P^{d_1} + P^{d_2}$, we take into account the bipartite graph associated with the matrix $B_n = P^{d_1} + P^{d_2}$.

Definition 2 *Let $G = (X, Y; E)$ be a bipartite graph, where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. We say that a pair of nodes is symmetric if it is of the form $\{x_i, y_i\}$. Let M be a perfect matching of G. We denote by $X(M)$ and $Y(M)$ the set of nodes of X and Y on which the edges of M are incident, respectively. A symmetric matching of cardinality $m \leq n$ is a matching $M = \{e_1, e_2, \ldots, e_m\}$ of cardinality m such that for all $u \in X(M)$ there exists $v \in Y(M)$ such that u and v are symmetric.*

We have the following simple lemma, which shows that the problem of computing $\text{per}(A_n)$ can be reduced to the computation of the number of symmetric matchings in $G[B_n]$, and to $\text{per}(B_n) = 2^{\gcd(n,|d_1-d_2|)} = 2$, due to the primality of n.

Lemma 12 *Let A_n and B_n be as above and let*

$$m_k = \#\{\sigma \mid \sigma \text{ is a symmetric matching of cardinality } n-k \text{ in } G[B_n]\}. \quad (6)$$

Then

$$\text{per}(A_n) = 1 + \text{per}(B_n) + \sum_{k=1}^{n-1} m_k. \quad (7)$$

Proof. Equality (7) is based upon the observation that each perfect matching of $A_n = I + B_n$ is either a perfect matching of B_n or a perfect matching of A_n which contains k symmetric pairs of nodes for some k, $1 \leq k \leq n$. On the other hand a perfect matching of A_n containing exactly k pairs of symmetric nodes is a symmetric matching of B_n of cardinality k. □

We will show below how in some special cases it is possible to find a formula for $\text{per}(A_n)$ by evaluating the m_k's.

Note that a lemma similar to Lemma 12 could actually be stated in terms of $D(A_n)$ and $D(B_n)$. In this case we would get a special case of the known expansion

$$\text{per}(\lambda I_n + B_n) = \sum_{i=0}^{n} a_i \lambda^{n-i},$$

where the a_i's denote the number of cycle covers for subgraphs of $D(B_n)$ with i nodes (see, e.g., [CDS79] page 34).

Note that $G[B_n]$ is the disjoint union of two perfect matchings, since P^{d_1} and P^{d_2} represent two cyclic permutations of the nodes. As a result of such a union we get a finite set of $\gcd(n, |d_1 - d_2|)$ disjoint rings. So in our case, due to the primality of n, all the nodes of $G[B_n]$ belong to a unique ring and thus each pair of symmetric nodes are connected by a simple path of length $D = D(n, d_1, d_2)$.

The nodes of $G[B_n]$ can be drawn on a polygon. Let us choose one node of the polygon and label it with 1. Then we proceed clockwise and number the second node with 2, the third one with 3, and so on until we use the label $2n$ (see Figure 2).

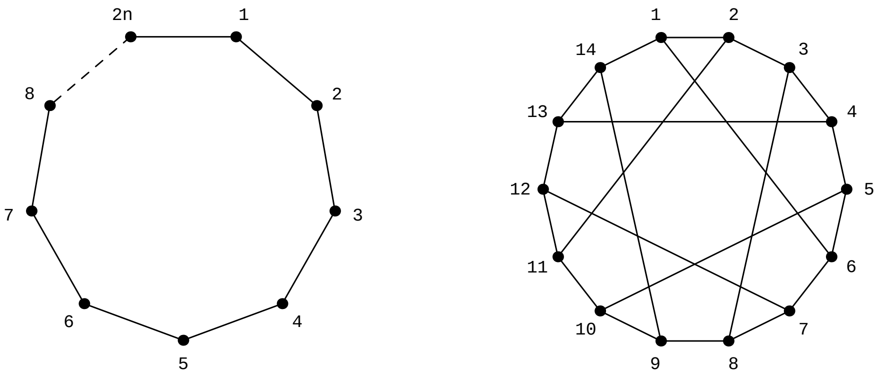

Figure 2: Clockwise numbering of $2n$ nodes and the graph for $n = 7$ and $D = 5$.

We now determine the value of D. Starting from, say, $u \in X$ we reach its symmetric node $v \in Y$ after an odd number of *moves* along a path on the ring. Note that moving on the ring from a node of X to a node of Y corresponds to adding d_1 modulo n whereas moving from a node of Y to a node of X corresponds to adding $n - d_2$ modulo n. In fact, to move from Y to X we need to consider the inverse of the cyclic permutation represented by P^{d_2}, i.e. the cyclic permutation represented by $(P^{d_2})^{-1} = P^{n-d_2}$. Thus, if x is the index of both u and v, we obtain the following equation

$$x + d_1 + (n - d_2) + d_1 + (n - d_2) + \ldots + d_1 \equiv x \pmod n$$

i.e.,

$$d_1 + h \cdot d_1 - h \cdot d_2 \equiv 0 \pmod n, \tag{8}$$

where $h = (D - 1)/2$. Equation (8) leads to the linear congruence

$$h \cdot (d_1 - d_2) \equiv -d_1 \pmod n \tag{9}$$

which has solutions if and only if

$$\gcd(n, d_1 - d_2) \text{divides} d_1. \tag{10}$$

In this case $\gcd(n, |d_1 - d_2|) = 1$ and certainly divides d_1. Furthermore, since n is prime, $\mathbf{Z}_n$ is a field and we can express h as

$$h = d_1 \cdot (d_2 - d_1)^{-1}, \tag{11}$$

so that

$$D = 2d_1 \cdot (d_2 - d_1)^{-1} + 1, \tag{12}$$

where both inversion and multiplication are computed in $\mathbf{Z}_n$. Note that we can express D as

$$D(n, i, j) = \min\{1 + 2h, 2n - 2h - 1\} \pmod{2n}$$

(In fact D and $2n - D$ play exactly the same role, when interpreted in the bipartite graph.)

To compute m_k, we can choose k pairs of symmetric nodes on $G[B_n]$, connect them two by two with k horizontal edges and count the number of perfect matchings of cardinality

$n-k$. Then m_k will be given by the sum of such matchings over all possible choices of k pairs. Pictorially, this translates into drawing k chords connecting two nodes of the polygon at distance D. Moreover, according to the new numbering, chords starting from odd nodes end up to even nodes moving clockwise by D edges, whereas starting from even nodes they end up to odd nodes moving counterclockwise by D edges (see Figure 2).

It is easy to see that, for each choice of k chords, there is either one or zero matchings. This amounts to saying that once we have removed from the polygon the nodes matched by the chords and their adjacent edges there is a matching if we are not left with simple paths of even length (odd number of nodes).

By analyzing the graph $G[B_n]$ according to Lemma 12, we have determined the following formula, which holds if $D(n,i,j) = n-2$, i.e., for $i=1$ and $j=3$, (and also if $D(n,i,j)=5$).

$$\mathrm{per}(A_n) = 3 + \sum_{k=1}^{n-1} m_k, \tag{13}$$

where

$$m_k = \begin{cases} \frac{n}{(k/2)!} \prod_{i=1}^{k/2-1} (n-k-i), & \text{if } k \text{ is even,} \\ \frac{n}{k!\cdot 2^{k-1}} \prod_{i=1}^{k-1} (n-k-2i), & \text{if } k \text{ is odd.} \end{cases} \tag{14}$$

The correctness of (14) can be shown by comparison with a formula that can be derived from a known recurrence [Mi85] for $I+P+P^3$, i.e., for $n \geq 12$,

$$\mathrm{per}(A_n) = \mathrm{per}(A_{n-1}) + \mathrm{per}(A_{n-2}) + \mathrm{per}(A_{n-3}) - \mathrm{per}(A_{n-4}) - \mathrm{per}(A_{n-5}) - \mathrm{per}(A_{n-6}) + 2.$$

Solving this linear recurrence, we obtain

$$\mathrm{per}(A_n) = 2 + \lfloor \frac{1}{2} + \alpha^n + \beta^n \rfloor, \tag{15}$$

where α and β are the real solutions of the equations x^3-x-1 and x^3-x^2-1, respectively[1], i.e.,

$$\alpha = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}},$$

and

$$\beta = \frac{1}{3} + \frac{1}{3}\sqrt[3]{\frac{29+3\sqrt{93}}{2}} + \frac{1}{3}\sqrt[3]{\frac{29-3\sqrt{93}}{2}}.$$

One can check that (15) coincides with (13).

Matrices of the form $I+P^i+P^j$, for n prime, and $D(n,i,j)$ small.

The permanent of the matrix $I+P^i+P^j$ depends on n and on $D(n,i,j)$. Note that, by varying i and j, $D(n,i,j)$ takes all possible odd values between 3 and n. In particular, from (12), we have that $D(n,1,j) = 1 + 2\cdot(j-1)^{-1}$. Since n is prime, it is easy to check that, for

[1] In particular, $\alpha = 1.3247\ldots$and $\beta = 1.4655\ldots$, whereas the other solutions of the correspondent equation $x^6-x^5-x^4-x^3+x^2+x+1=0$ are complex and their absolute values are less than 1.

$2 \leq j \leq n-1$, $D(n,1,j)$ goes through all possible values between 3 and n. This means that, if n is prime, the problem of computing $\mathrm{per}(I+P^i+P^j)$ can be reduced to the computation of $\mathrm{per}(I+P+P^k)$, for a suitable k.

Furthermore, note that the function $D_n(i,j) = D(n,i,j)$ is not injective, i.e., there are several pairs (i,j) with the same value of D. In addition, there are other symmetries, because the matrices $I+P^i+P^j$, $I+P^{j-i}+P^{n-i}$ and $I+P^{n-j+i}+P^{n-j}$ have all the same permanent whereas they may have different values of D (see Section 5).

We now outline a general approach to derive recurrence formulas for the permanent of matrices with a given value of $D(n,i,j)$.

We operate on the bipartite graph as follows:

- Being n prime, we can represent the graph $G[A_n]$ as a ring, corresponding to $G[B_n]$, with additional chords that connect two nodes of $G[B_n]$ at distance D.
- We consider $D-1$ adjacent nodes, starting from node 1.
- We enumerate all the different ways in which these $D-1$ nodes can be included in a perfect matching.
- The graph on the remaining $n-D+1$ nodes is an *open ring*, i.e. it does not contain a ring any more. This open ring can take only a few different shapes, and these depend on D. This makes it possible to enumerate the perfect matchings according to suitable recurrence formulas.
- Adding up the different terms, one gets a formula for the permanent depending on some simpler functions, each defined by a suitable recurrence.

In figure 3 we outline the case when $D=3$. The nodes considered are 1 and 9 and there are 5 possibilities overall to be analyzed.

In all of the 5 possibilities the ring is opened and we are left with a *necklace of trapezoidal elements*. If we denote by $S(n)$ the number of perfect matchings for a necklace of n trapezoidal elements, it is easy to prove that

$$\begin{cases} S(1) &= 2 \\ S(2) &= 3 \\ S(n) &= S(n-1)+S(n-2) \end{cases}$$

One gets the following formula for the permanent

$$P_{D=3}(n) = 2 + S(n-2) + 2S(n-3).$$

Note that, although the formula has been obtained under the assumption that n is prime, we experimentally found that it is valid for all values of n up to 25.

The above procedure can be repeated for $D=5$. In this case, to open the ring we have to consider $D-1=4$ nodes, and the number of possibilities that must be analyzed is 13. (In general, if we define $F(1)=F(2)=1$, and $F(n)=F(n-1)+F(n-2)$, then the number of possibilities to be analyzed is $F(D+2)$, e.g., if $D=7$, there are $F(9)=34$ cases.)

If $D=5$, the necklaces are of two types, as shown in figure 4.

For the number of perfect matchings of the two necklaces, we get the following formulas:

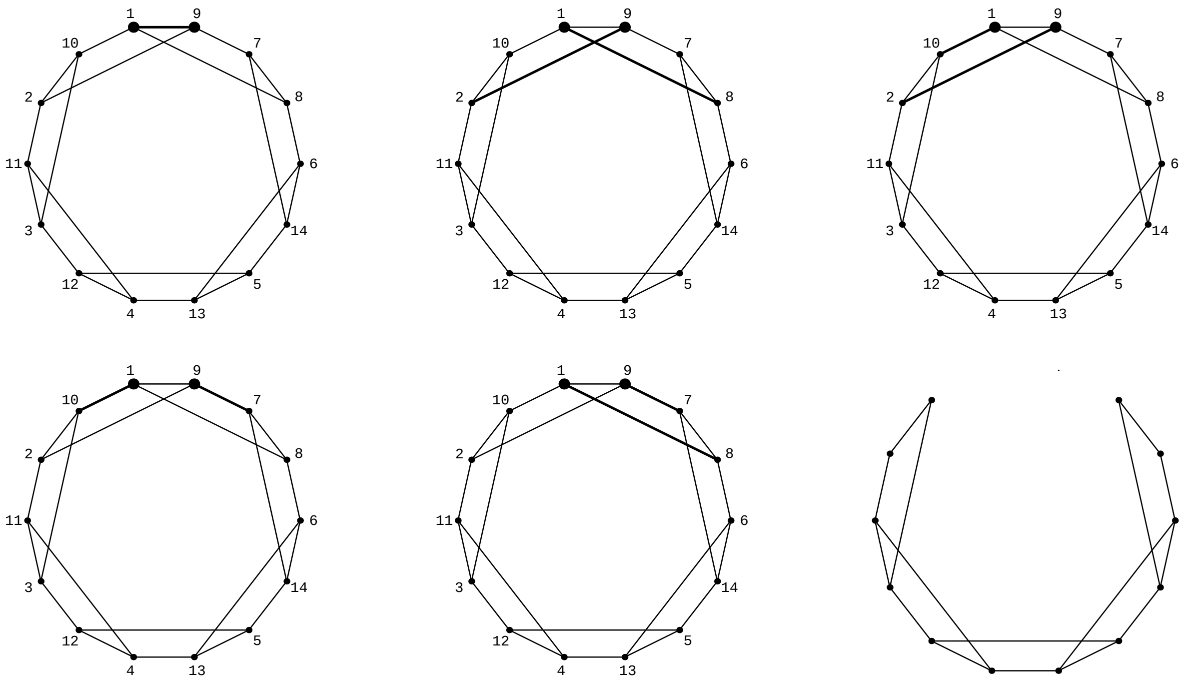

Figure 3: The 5 possibilities to include nodes 1 and 9 in a perfect matching, and the typical shape of an open ring.

$$\begin{cases} T_1(1) &= 2 \\ T_1(2) &= 3 \\ T_1(3) &= 4 \\ T_1(n) &= T_1(n-1) + T_1(n-3) \end{cases} \qquad \begin{cases} T_2(1) &= 1 \\ T_2(2) &= 2 \\ T_2(3) &= 2 \\ T_2(n) &= T_2(n-2) + T_2(n-3) \end{cases}$$

In this case, the recurrence for the permanent becomes

$$P_{D=5}(n) = 2 + T_1(n-4) + 3T_1(n-5) + T_1(n-6) + 2T_2(n-5) + 3T_2(n-6)\ .$$

(Note that the recurrences obtained here for $D = 3$ and $D = 5$ have some similarities with those obtained by other means in Section 4.2 for $I + P + P^2$ and $I + P + P^3$.)

The computational complexity of the above method can be analyzed by recalling Theorem 10. Indeed we have the following corollary, which implies that the permanent of $(0, 1)$-circulant matrices of type $(0, d_1, d_2)$ with $D = O(\log n)$ can be computed in polynomial time.

Corollary 13 *Let $A_n = I_n + {P_n}^i + {P_n}^j$, with n prime, and let $D = D(n, i, j)$. Then* $\mathrm{per}(A)$ *can be computed in time $O(2^{D/2}n^\gamma)$, $\gamma < 3$.*

Proof. With reference to Theorem 10, the thesis easily follows observing that the matrix $I_n + P_n + P_n^{\frac{D+1}{2}}$ has the same permanent as A_n, even if $D(n, 1, (D+1)/2) \neq D$. To see this property, we start drawing $G[A_n]$ in such a way that $G[P^i + P^j]$ is a ring, while I induces

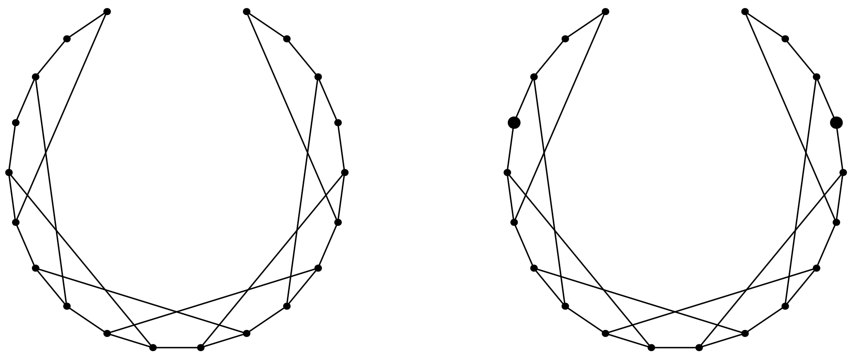

Figure 4: The two different necklaces for $D = 5$. Two nodes in the second one are already included in matchings.

chords connecting nodes of $G[P^i + P^j]$ at distance D in $G[P^i + P^j]$. Since the relabeling of the nodes of the bipartite graph does not affect the value of the permanent, we can sort the labels as shown in figure 5. According to this new ordering, it is easy to see that the ring now corresponds to $G[I + P]$, and the chords to $G[P^{\frac{D+1}{2}}]$. □

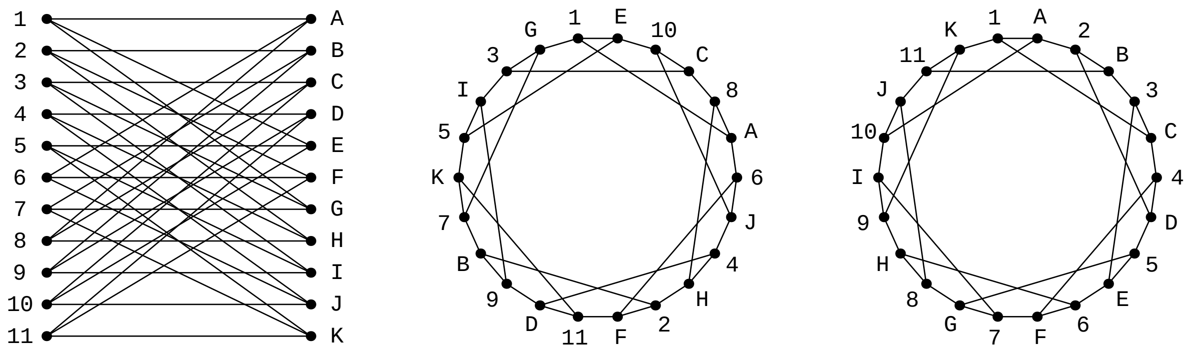

Figure 5: From left to right: $G[I + P^4 + P^6]$, for $n = 11$; another equivalent description of the same graph which shows that $D = 5$; and the graph with rearranged labels according to the scheme $1, A, 2, B, 3, C \ldots$, which corresponds to the matrix $I + P + P^3$.

4.5 Circulants of the form $I + P^i + P^{n-i}$

First of all, note that

$$\mathrm{per}(I + P^i + P^{n-i}) = \mathrm{per}(P^i(I + P^i + P^{n-i})) = \mathrm{per}(I + P^i + (P^i)^2).$$

We are now ready to state the following

Theorem 14 *Let $A_n = I_n + P_n^i + P_n^{n-i}$, and assume that $\frac{n}{\gcd(n,i)} \geq 5$. Then*

$$\mathrm{per}(A_n) = \left[\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{n}{\gcd(n,i)}} + \left(\frac{1-\sqrt{5}}{2}\right)^{\frac{n}{\gcd(n,i)}} + 2\right]^{\gcd(n,i)}.$$

Proof. Let $d = \gcd(i,n)$. Then, by the equivalence of types $(0,i,n-i)$ and $(0,i,2i)$ and from Lemma 11 we have

$$\begin{aligned}\mathrm{per}(I_n + P_n^i + P_n^{n-i}) &= \mathrm{per}(I_n + P_n^i + (P_n^i)^2) \\ &= \mathrm{per}(I_{\frac{n}{d}} + P_{\frac{n}{d}}^{\frac{i}{d}} + P_{\frac{n}{d}}^{2\cdot\frac{i}{d}})^d \\ &= \mathrm{per}(I_{\frac{n}{d}} + P_{\frac{n}{d}} + P_{\frac{n}{d}}^2)^d.\end{aligned}$$

By solving Minc's recurrences [Mi85], we can see that, for $k \geq 5$,

$$\mathrm{per}(I_k + P_k + P_k^2) = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k + 2.$$

□

4.6 Some further special structures

In this section, we first find the value of $\mathrm{per}(I+P+P^{\frac{n}{2}+1})$, for n even, and then, with similar techniques, we show that, if $\gcd(k,n) = 2$, then $\mathrm{per}(I_n + P_n^k + P_n^{\frac{n}{2}})$ does not depend on k, and, in particular

$$\mathrm{per}(I_n + P_n^k + P_n^{\frac{n}{2}}) = \mathrm{per}(I + P + P^{\frac{n}{2}+1}) = 2\sqrt{2^n} + 1.$$

Matrices of the form $I + P + P^{\frac{n}{2}+1}$.

Let n be even. We wish to evaluate the permanent of $I + P + P^{\frac{n}{2}+1}$. In this case, the matrix can be written as a 2×2 block matrix of the form

$$\begin{bmatrix} I_{n/2} + P_{n/2} & P_{n/2} \\ P_{n/2} & I_{n/2} + P_{n/2} \end{bmatrix},$$

i.e., the four blocks of size $\frac{n}{2} \times \frac{n}{2}$ are circulant matrices themselves. The proof of the following theorem takes advantage of the above property.

Theorem 15 *Let n be even. Then*

$$\mathrm{per}(I + P + P^{\frac{n}{2}+1}) = 2\sqrt{2^n} + 1.$$

Proof. We first transform $I + P + P^{\frac{n}{2}+1}$ into a matrix with a more convenient structure. Indeed, we make the following transformation, which does not affect the value of the permanent:

$$B = \begin{bmatrix} P_{n/2}^{n/2-1} & O \\ O & P_{n/2}^{n/2-1} \end{bmatrix} \cdot \begin{bmatrix} I_{n/2} + P_{n/2} & P_{n/2} \\ P_{n/2} & I_{n/2} + P_{n/2} \end{bmatrix} = \begin{bmatrix} I_{n/2} + P_{n/2}^{n/2-1} & I_{n/2} \\ I_{n/2} & I_{n/2} + P_{n/2}^{n/2-1} \end{bmatrix}.$$

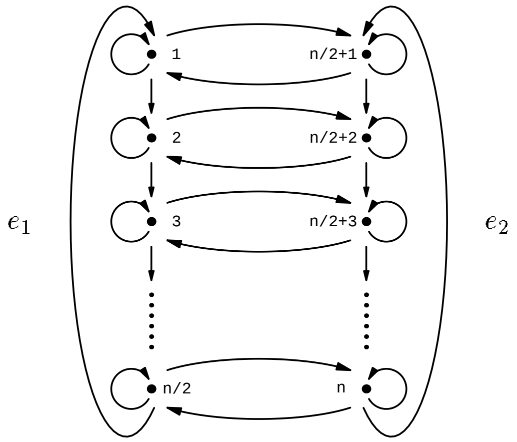

Figure 6: The digraph $D(B)$

We now compute the number of cycle covers of the digraph $D(B)$ (see figure 6). We consider the edges $e_1 = (\frac{n}{2}, 1)$ and $e_2 = (n, \frac{n}{2}+1)$, and define

$$\begin{aligned}
\mathcal{A} &= \{M \mid M \text{ is a cycle cover whose cycles do not contain neither } e_1 \text{ nor } e_2\},\\
\mathcal{B}_1 &= \{M \mid M \text{ is a cycle cover whose cycles contain } e_1 \text{ but not } e_2\},\\
\mathcal{B}_2 &= \{M \mid M \text{ is a cycle cover whose cycles contain } e_2 \text{ but not } e_1\},\\
\mathcal{C} &= \{M \mid M \text{ is a cycle cover whose cycles contain both } e_1 \text{ and } e_2\}.
\end{aligned}$$

We have that

$$\text{per}(B) = |\mathcal{A}| + |\mathcal{B}|_1 + |\mathcal{B}|_2 + |\mathcal{C}|,$$

where $|\mathcal{X}|$ denote the cardinality of the set $\mathcal{X}$.

It is immediate to see that $|\mathcal{C}| = 1$. For the other cardinalities, we have:

Cycle covers in $\mathcal{A}$.

Each cycle cover in $\mathcal{A}$ does not contain edges of type $(i, i+1)$, for $1 \leq i \leq \frac{n}{2} - 1$ and $\frac{n}{2} + 1 \leq i \leq n-1$. So the number of cycle covers is equal to that in the digraph $I_n + P_n^{\frac{n}{2}}$, i.e., $|\mathcal{A}| = 2^{\frac{n}{2}}$.

Cycle covers in $\mathcal{B}_1$ and $\mathcal{B}_2$.

For $\mathcal{B}_1$, each cycle cover corresponds to a simple path between node 1 and node $\frac{n}{2}$. In fact we can cover all the other nodes outside the path with their self-loops. The number of such paths is $2^{\frac{n}{2}-1}$.

Since $|\mathcal{B}|_1 = |\mathcal{B}|_2$ by symmetry, we finally obtain

$$\text{per}(B) = 2^{\frac{n}{2}} + 2 \cdot 2^{\frac{n}{2}-1} + 1 = 2^{\frac{n}{2}+1} + 1.$$

□

Matrices of the form $I + P^k + P^{\frac{n}{2}}$.

Let n be even. We prove that, under certain conditions, the value of k does not affect the permanent of $I + P^k + P^{\frac{n}{2}}$.

Theorem 16 *Let $n = 2d$, where d is an odd integer and $A_n = I_n + P_n^k + P_n^{\frac{n}{2}}$. If $\gcd(k,n) = 2$ then*

$$\mathrm{per}(A_n) = 2 \cdot \sqrt{2^n} + 1.$$

Proof. We prove that the digraph $D(A_n)$ is isomorphic to the digraph of the matrix

$$B_n = \begin{bmatrix} I_{n/2} + P_{n/2}^{\frac{n}{2}-1} & I_{n/2} \\ I_{n/2} & I_{n/2} + P_{n/2}^{\frac{n}{2}-1} \end{bmatrix}.$$

Note that, since d is odd and $\gcd(k,n) = 2$, then $D(P_n^k)$ consists of two disjoint cycles of d nodes of the form [2]:

$$(1, 1 + k \bmod n, 1 + 2k \bmod n, \ldots, 1 + (d-1)k \bmod n)$$

and

$$(\frac{n}{2} + 1, \frac{n}{2} + 1 + k \bmod n, \frac{n}{2} + 1 + 2k \bmod n, \ldots, \frac{n}{2} + 1 + (d-1)k \bmod n).$$

Thus we can relabel the nodes of $D(A_n)$ so that it consists of

- two cycles of length d of the form

$$(1, 1 + k \bmod n, 1 + 2k \bmod n, \ldots, 1 + (d-1)k \bmod n)$$

  and

$$(2, 2 + k \bmod n, 2 + 2k \bmod n, \ldots, 2 + (d-1)k \bmod n),$$

- the self-loops,
- d cycles of length 2 of the form $(x, x+1 \bmod n)$.

Consider $D(B_n)$. It consists of the self-loops, the two cycles of length d of the form $(1, 2, \ldots, d)$ and $(d+1, d+2, \ldots, n)$, and the d cycles of length 2 of the form $(x, x + \frac{n}{2} \bmod n)$. Again, we relabel the nodes of $D(B_n)$ as depicted in figure 7.
The isomorphism we are going to define maps the two relabeled cycles of $D(B_n)$, $(1, 3, 5, \ldots, n-1)$ and $(2, 4, 6, \ldots, n)$, onto the two relabeled cycles of $D(A_n)$, $(1, 1 + k \bmod n, \ldots, 1 + (d-1)k \bmod n)$ and $(2, 2 + k \bmod n, \ldots, 2 + (d-1)k \bmod n)$, respectively.

We define a function $\phi : \mathbf{Z}_n \to \mathbf{Z}_n$ as follows:

$$\phi(x) = \begin{cases} \frac{x-1}{2} \cdot k + 1 \bmod n & \text{if } x \text{ is odd,} \\ (\frac{x}{2} - 1) \cdot k + 2 \bmod n & \text{if } x \text{ is even.} \end{cases}$$

Now we prove that ϕ is an isomorphism between the two relabeled graphs $D(B_n)$ and $D(A_n)$. It is immediate to see that ϕ is surjective and injective. Let $(x, y) \in \mathrm{E}(D(B_n))$, with $x \neq y$. W.l.o.g. suppose that x is odd, so that $\phi(x) = \frac{x-1}{2} \cdot k + 1$. Then $x \in \{1, 3, 5, \ldots, n-1\}$, and y can be either $x + 1$ or $x + 2 \bmod n$. In the first case we have

$$\begin{aligned} \phi(x+1) &= (\frac{x+1}{2} - 1) \cdot k + 2 \\ &= (\frac{x-1}{2} \cdot k + 1) + 1 \\ &= \phi(x) + 1, \end{aligned}$$

[2] Here we use the cycle notation for permutations.

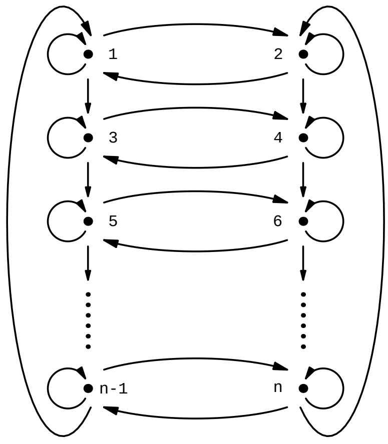

Figure 7: The relabeling of the nodes of $D(B)$.

which implies that $(\phi(x), \phi(x+1)) \in \mathrm{E}(D(A_n))$. In the second case we have

$$\begin{aligned}\phi(x+2 \bmod n) &= \frac{x+1}{2} \cdot k + 1 \bmod n \\ &= (\frac{x-1}{2} + 1) \cdot k + 1 \bmod n \\ &= (\frac{x-1}{2} \cdot k + 1) + k \bmod n \\ &= \phi(x) + k \bmod n,\end{aligned}$$

which still implies that $(\phi(x), \phi(x+2 \bmod n)) \in \mathrm{E}(D(A_n))$. The converse follows analogously, thus ϕ is a digraph isomorphism and then by Theorem 15 we obtain $\mathrm{per}(A_n) = \mathrm{per}(B_n) = 2 \cdot \sqrt{2^n} + 1$. □

Corollary 17 *Let $n = 2p$, where $p \neq 2$ is a prime number, and $A_n = I_n + P_n^k + P_n^{\frac{n}{2}}$. Then*

$$\mathrm{per}(A_n) = 2\sqrt{2^n} + 1.$$

Proof. If k is even, then $\gcd(k, 2p) = 2$, and the result follows from Theorem 16. If k is odd, then we have

$$\mathrm{per}(A_n) = \mathrm{per}(P_n^{\frac{n}{2}} A_n) = \mathrm{per}(I_n + P_n^{\frac{n}{2}} + P_n^{\frac{n}{2}+k}),$$

where $\frac{n}{2} + k$ is even. □

5 Conjectures and Open questions

As a support to our investigations on $(0,1)$-circulant matrices of type $(0, i, j)$, we have performed a number of experiments on matrices of size up to 31, from which we have derived several *structural* questions on their permanents.

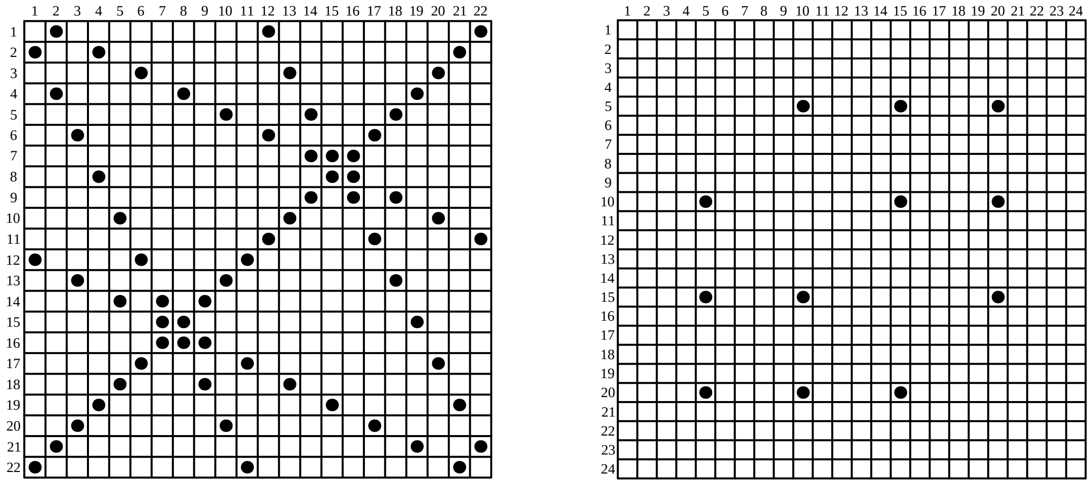

Figure 8: Pattern of maximal values of $\mathrm{per}(I + P^i + P^j)$, for $n = 23$ and $n = 25$. The entry (h, k) contains a filled circle if $\mathrm{per}(I + P^h + P^k) \geq \mathrm{per}(I + P^i + P^j)$, for all pairs (i, j).

Some of the unresolved questions led us to state the following conjecture and observation.

Conjecture. Because of symmetry, we assume, w.l.o.g., that $i < j$. The set of pairs of indices that maximize $\mathrm{per}(I_n + P_n^i + P_n^j)$ is given by

$$\begin{cases} \{\frac{n}{3}, \frac{2n}{3}\} & \text{if 3 divides } n \\ \{\frac{n}{4}, \frac{n}{2}\}, \{\frac{n}{4}, \frac{3n}{4}\}, \{\frac{n}{2}, \frac{3n}{4}\} & \text{if 4 divides } n \text{ and 3 does not divide } n \\ \{\frac{n}{p}i, \frac{n}{p}j\} \text{ for } i + j = n, i = 2j, i = 2j - n & \text{otherwise,} \end{cases}$$

where p is the smallest prime factor of n greater than or equal to 5 (see figure 8). As a consequence, we have that the cardinality of the set of pairs that maximize $\mathrm{per}(I + P^i + P^j)$ is equal to m, where

$$m = \begin{cases} 1 & \text{if 3 divides } n \\ 3 & \text{if 4 divides } n \text{ and 3 does not divide } n \\ \frac{3}{2}(p-1) & \text{otherwise.} \end{cases}$$

□

Observation. We have seen in Section 4.6 some examples of different matrices of the type $I + P^i + P^j$ with the same permanent. If n is prime, there are only a few different values for $\mathrm{per}(I + P^i + P^j)$, by varying i and j (see figure 9). By analyzing the bipartite graph $G[I + P^i + P^j]$, we have tried to understand this phenomenon. For instance, if $n = 31$, $D(n, i, j)$ can take all the 15 odd values $3, 5, 7, \ldots, 31$. By multiplying $I + P^i + P^j$ by P^{n-i} and P^{n-j} we have obtained the following "classes" of different values of D for which the permanent is the same:

$$\{11\}, \{3, 31\}, \{5, 21, 29\}, \{7, 15, 19\}, \{9, 13, 17\}, \{23, 25, 27\}.$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1		A	B	C	C	C	C	B	C	C	B	A	B	C	C	B	C	C	C	C	B	A
2	A		B	A	C	B	C	C	B	C	C	C	C	C	C	B	C	C	B	C	A	B
3	B	B		C	C	A	C	C	B	B	C	C	A	C	C	B	B	C	C	A	C	C
4	C	A	C		C	B	C	A	B	C	C	B	C	C	B	C	C	B	A	C	B	C
5	C	C	C	C		C	C	C	B	A	B	C	B	A	B	C	B	A	B	C	C	C
6	C	B	A	B	C		C	C	B	C	B	A	C	C	C	C	A	B	C	B	C	C
7	C	C	C	C	C	C		B	B	B	C	C	C	A	A	A	C	C	C	B	B	B
8	B	C	C	A	C	C	B		C	C	C	B	B	C	A	A	C	B	B	C	C	C
9	C	B	B	B	B	B	B	C		C	C	C	C	A	C	A	C	A	C	C	C	C
10	C	C	B	C	A	C	B	C	C		B	C	A	C	B	C	C	B	C	A	C	B
11	B	C	C	C	B	B	C	C	C	B		A	C	C	B	C	A	C	B	C	C	A
12	A	C	C	B	C	A	C	B	C	C	A		B	C	C	C	B	B	C	C	C	B
13	B	C	A	C	B	C	C	B	C	A	C	B		C	C	B	C	A	C	B	C	C
14	C	C	C	C	A	C	A	C	A	C	C	C	C		C	B	B	B	B	B	B	C
15	C	C	C	B	B	C	A	A	C	B	B	C	C	C		B	C	C	A	C	C	B
16	B	B	B	C	C	C	A	A	A	C	C	C	B	B	B		C	C	C	C	C	C
17	C	C	B	C	B	A	C	C	C	C	A	B	C	B	C	C		C	B	A	B	C
18	C	C	C	B	A	B	C	B	A	B	C	B	A	B	C	C	C		C	C	C	C
19	C	B	C	A	B	C	C	B	C	C	B	C	C	B	A	C	B	C		C	A	C
20	C	C	A	C	C	B	B	C	C	A	C	C	B	B	C	C	A	C	C		B	B
21	B	A	C	B	C	C	B	C	C	C	C	C	C	B	C	C	B	C	A	B		A
22	A	B	C	C	C	C	B	C	C	B	A	B	C	C	B	C	C	C	C	B	A	

Figure 9: Permanents of $I + P^i + P^j$, when $n = 23$. The entry (h, k) contains the value of the permanent of $I + P^h + P^k$. These permanents take only 3 different values, namely $A = 64081$, $B = 7225$ and $C = 4097$.

Experimentally we have checked that the actual *partition* is:

$$\{11\}, \{3,31\}, \{5,21,29\}, \{7,15,19,23,25,27\}, \{9,13,17\},$$

which shows that the problem hides further symmetries.

The knowledge of all these symmetries can be computationally very useful. In fact one could try to reduce the computation of $\mathrm{per}(I_n + P_n^i + P_n^j)$ to that of the permanent of a matrix of type $(0, i', j')$, where i' and j' satisfy the requirements for which Theorem 10 provides, e.g., a polynomial time algorithm (see Table 1).

n	#	$i'+j'$	n	#	$i'+j'$	n	#	$i'+j'$
5	1	2	31	6	6	67	12	9
7	2	3	37	7	7	71	12	9
11	2	3	41	7	7	73	13	9
13	3	4	43	8	7	79	14	10
17	3	4	47	8	7	83	14	9
19	4	5	53	9	7	89	15	10
23	4	5	59	10	8	97	17	11
29	5	5	61	9	11	101	17	11

Table 1: For n prime, $3 \leq n \leq 101$, we report, in the second column, the number of different values taken by $\mathrm{per}(I_n + P_n^i + P_n^j)$. In the third column we indicate the maximum value of $i'+j'$ (with reference to the notation of Theorem 10) that one has to consider in order to compute any of the possible different values taken by the permanent. Note that the values reported in both columns are actually upper bounds, since they are not outcomes of experimental results, but are obtained from known relationships between the values of $\mathrm{per}(I + P^i + P^j)$, as i and j vary.

□

Summarizing, we would like to raise the following questions:

- Which are the properties of the pairs (i, j) that maximize the value of the permanent of a $(0, 1)$-circulant matrix of type $(0, d_1, d_2)$? What is the number of such pairs?
- How many different values can the permanents of $(0, 1)$-circulant matrices of type $(0, d_1, d_2)$ take?
- Is there an algorithm for the computation of the permanent of a $(0, 1)$-circulant matrix of type $(0, 1, t)$ that runs in polynomial time for all the values of t? Is this the case at least if n is prime?

6 Conclusions

In this paper we have provided a contribution to the investigation on the permanent of some sparse and structured $(0, 1)$ matrices. We found that the permanents of the matrices analyzed here present several strong properties which sometimes make their computation tractable.

In particular, by building upon a few basic and simple facts, i.e., that

- the permanent of $(0,1)$ symmetric Toeplitz matrices with three diagonals grows as a generalized Fibonacci sequence with the matrix size,
- the permanent of $(0,1)$-circulant matrices with 2 nonzeros per row grows exponentially with the gcd between the matrix size and the index that identifies its nonzero off-diagonal,
- the permanent of certain circulant matrices and of symmetric Toeplitz matrices is a power of the permanent of a smaller matrix of the same type,
- the permanent of certain non convertible circulant matrices can be expressed as a sum of a few determinants of Toeplitz matrices,

we have been able to devise more general formulas and efficient algorithms for the computation of permanents of $(0,1)$ nonsymmetric Toeplitz matrices with three diagonals and $(0,1)$-circulant matrices with three nonzeros per row.

We believe that the partial results obtained in this paper deserve further investigation. The goal is to achieve a deeper understanding of the structural and computational properties of the permanents of certain Toeplitz matrices.

References

[BMQ68] L. Bassett, J. Maybee, and J. Quirk. Qualitative Economics and the scope of the correspondence principle. *Econometrica 36:544-563 (1968).*

[BR91] R.A. Brualdi, and H.J. Ryser. Combinatorial matrix theory. *Cambridge University Press (1991).*

[BS95] R.A. Brualdi, and B.L. Shader. Matrices of sign-solvable linear systems. *Cambridge University Press (1995).*

[CDS79] D.M. Cvetkovic, M. Doob, and H. Sachs. Spectra of Graphs. *Academic Press (1979).*

[DLMV88] P. Dagum, M. Luby, M. Mihail, and U. Vazirani. Polytopes, Permanents, and Graphs with Large Factors. *Proc. 27th IEEE Symp. on Found. of Comput. Sc. (1988).*

[FL92] U. Feige, and C. Lund. On the Hardness of Computing the Permanent of Random Matrices. *Proc. 24th ACM Symp. on the Theory of Comput. 643-654 (1992).*

[KP69] B.W. King, and F.D. Parker. A Fibonacci Matrix and the permanent function. *Fibonacci Quart. 7:539-544 (1969).*

[MSS69] N. Metropolis, M.L. Stein, and P.R. Stein. Permanents of cyclic $(0,1)$ matrices. *J.Combinatorial Theory B 7:291-321 (1969).*

[Mi78] H. Minc. Permanents. *Encyclopedia of Mathematics and its Appl. Vol. 6 (1978).*

[Mi85] H. Minc. Recurrence Formulas for Permanents of $(0,1)$ Circulants. *Linear Algebra and its Appl. 71:241-265 (1985).*

[Mi87] H. Minc. Permanental Compounds and Permanents of (0, 1) Circulants. *Linear Algebra and its Appl. 86:11-42 (1987).*

[R63] H.J. Ryser. Combinatorial Mathematics. *Carus Mathematical Monograph No. 14 (1963).*

[ST87] P. Seymour and C. Thomassen. Characterization of Even Directed Graphs. *J. Combinatorial Theory B, 42:36-45 (1987).*

[V179] L.G. Valiant. The complexity of computing the permanent. *Theoretical Computer Science 8:189-201 (1979).*

[V279] L.G. Valiant. Completeness classes in algebra. *ACM Symp. on the Theory of Comput. 249-261 (1979).*

[VV89] V.V. Vazirani. NC Algorithms for Computing the Number of Perfect Matchings in $K_{3,3}$-Free Graphs and Related Problems. *Information and Comput. 80 152-164 (1989).*