On the Lovász Number of Certain Circulant Graphs

Valentin E. Brimkov[1], Bruno Codenotti[2], Valentino Crespi[1], and Mauro Leoncini[2,3]

[1] Department of Mathematics, Eastern Mediterranean University, Famagusta, TRNC
{brimkov, crespi}.as@mozart.emu.edu.tr

[2] Istituto di Matematica Computazionale del CNR, Via S. Maria 46, 56126-Pisa, Italy
codenotti@imc.pi.cnr.it

[3] Facoltà di Economia, Università di Foggia, Via IV Novembre 1, 71100 Foggia, Italy
leoncini@imc.pi.cnr.it

Abstract. The theta function of a graph, also known as the Lovász number, has the remarkable property of being computable in polynomial time, despite being "sandwiched" between two hard to compute integers, i.e., clique and chromatic number. Very little is known about the explicit value of the theta function for special classes of graphs. In this paper we provide the explicit formula for the Lovász number of the union of two cycles, in two special cases, and a practically efficient algorithm, for the general case.

1 Introduction

The notion of capacity of a graph was introduced by Shannon in [14], and after that labeled as *Shannon capacity*. This concept arises in connection with a graph representation for the problem of communicating messages in a zero-error channel. One considers a graph G, whose vertices are letters from a given alphabet, and where adjacency indicates that two letters can be confused. In this setting, the maximum number of one-letter messages that can be sent without danger of confusion is given by the independence number of G, here denoted by $\alpha(G)$. If $\alpha(G^k)$ denotes the maximum number of k-letter messages that can be safely communicated, we see that $\alpha(G^k) \geq \alpha(G)^k$. Moreover one can readily show that equality does not hold in general (see, e.g., [11]). The Shannon capacity of G is the number $\Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}$, which, by the previous observations, satisfies $\Theta(G) \geq \alpha(G)$, where equality does not need to occur.

It was very early recognized that the determination of the Shannon capacity is a very difficult problem, even for small and simple graphs (see [8, 13]). In a famous paper of 1979, Lovàsz introduced the theta function $\vartheta(G)$, with the explicit goal of estimating $\Theta(G)$ [11].

Shannon capacity and Lovász theta function attracted a lot of interest in the scientific community, because of the applications to communication issues, but also due to the connections with some central combinatorial and computational questions in graph theory, like computing the largest clique and finding the chromatic number of a graph (see [2–4, 6] for a sample of the wealth of different results and applications of $\vartheta(G)$ and $\Theta(G)$). Despite a lot of work in the field, finding the explicit value of the theta function for interesting special classes of graphs is still an open problem.

In this paper we present some results on the theta function of circulant graphs, i.e., graphs which admit a circulant adjacency matrix. We recall that a circulant matrix is fully determined by its first row, each other row being a cyclic shift of the previous one. Such graphs span a wide spectrum, whose extremes are the single cycle and the complete graph. We either give a formula or an algorithm for computing the Lovász number of circulant graphs given by the union of two cycles. The algorithm is based on the computation of the intersection of halfplanes and (although its running time is $O(n \log n)$ in the worst case, as compared with the linear time achievable through linear programming) is very efficient in practice, since it exploits the particular geometric structure of the intersection.

2 Preliminaries

There are several equivalent definitions for the Lovász theta function (see, e.g., the survey by Knuth [10]). We give here the one that comes out of Theorem 6 in [11], because it requires only little technical machinery.

Definition 1. *Let G be a graph and let $\mathbf{A}$ be the family of matrices $A = (a_{ij})$ such that $a_{ij} = 0$ if i and j are adjacent in G. Also, let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ denote the eigenvalues of A. Then*

$$\vartheta(G) = \max_{A \in \mathbf{A}} \left\{ 1 - \frac{\lambda_1(A)}{\lambda_n(A)} \right\} .$$

Combining the fact that $\Theta(G) \leq \vartheta(G)$ with the easy lower bound $\Theta(C_5) \geq \sqrt{5}$, Lovász has been able to determine exactly the capacity of C_5, the pentagon, which turns out to be $\sqrt{5}$.

For several families of simple graphs, the value of $\vartheta(G)$ is given by explicit formulas. For instance, in the case of odd cycles of length n we have

$$\vartheta(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)} .$$

We now sketch the proof of correctness of the above formula (see [10] for more details), because it will be instrumental to the more general results obtained in this paper.

With reference to the definition of the Lovász number which resorts to the minimum of the largest eigenvalue over all feasible matrices (Section 6 in [10]), in the case of n-cycles, we have that a feasible matrix has ones everywhere, except on the superdiagonal, subdiagonal and the upper-right and lower-left corners, i.e. it can be written as $C = J + xP + xP^{-1}$, where J is a matrix whose entries are all equal to one, and P is the permutation matrix taking j into $(j+1) \mod n$. It is well known and easy to see that the eigenvalues of C are $n+2x$, and $x(\omega^j + \omega^{-j})$, for $j = 1, \ldots, n-1$, where $\omega = e^{2\pi i/n}$. The minimum over x of the maximum of these values is obtained when $n + 2x = -2x \cos \pi/n$, which immediately leads to the above formula.

3 The Function ϑ of Circulant Graphs of Degree 4

Let n be an odd integer and let j be such that $1 < j \leq \frac{n-1}{2}$. Let $C(n,j)$ denote the circulant graph with vertex set $\{0, ..., n-1\}$ and edge set $\{\{i, i+1 \bmod n\}, \{i, i+j \bmod n\}, i = 0, ..., n-1\}$. By using the approach sketched in [10], we can easily obtain the following result.

Lemma 1. *Let $f_0(x,y) = n + 2x + 2y$ and, for some fixed value of j, $f_i(x,y) = 2x \cos \frac{2\pi i}{n} + 2y \cos \frac{2\pi i j}{n}$, $i = 1, ..., n-1$. Then*

$$\vartheta(C(n,j)) = \min_{x,y} \max_{i} \left\{ f_i(x,y)\,,\ i = 0,\, 1, \ldots,\, \frac{n-1}{2} \right\} . \tag{1}$$

Proof. Follows from the same arguments which lead to the known formula for the Lovász number of odd cycles [10] (i.e., taking advantage of the fact that we can restrict the set of feasible matrices within the family of circulant matrices) and observing that, for $i \geq 1$, $f_i(x,y) = f_{n-i}(x,y)$. □

3.1 A Linear Programming Formulation

Throughout the rest of this paper we will consider the following linear programming formulation of (1).

$$\begin{aligned} &\text{minimize } z \\ &\quad \text{s.t. } f_i(x,y) - z \leq 0,\ i = 0, ..., \tfrac{n-1}{2}, \\ &\qquad\qquad z \geq 0, \end{aligned} \tag{2}$$

where the $f_i(x,y)$'s are defined in Lemma 1.

Consider the intersection C of the closed halfspaces defined by $z \geq 0$ and $f_i(x,y) - z \leq 0$, $i = 1, ..., \frac{n-1}{2}$ (which is not empty, since any point $(0,0,k)$, $k \geq 0$, satisfies all the inequalities). C is a polyhedral cone with the apex at the origin. This follows from the two following facts, which can be easily verified: (1) the equations $f_i(x,y) - z = 0$, $i \geq 1$, define hyperplanes through the origin; (2) for any $z_0 > 0$, the projection Q_{z_0} of $C \bigcap \{z = z_0\}$ onto the xy plane is a polygon, i.e., Q_{z_0} is bounded[1] (see Fig. 1, which corresponds to the graph $C(13,2)$).

[1] In the appendix we shall give a rigorous proof of this fact for the case $j = 2$.

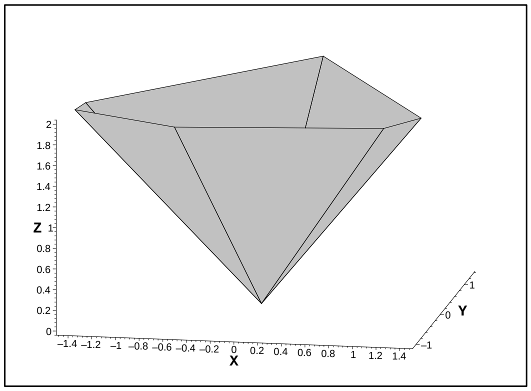

Fig. 1. The polyedral cone for $n = 13$ and $j = 2$ cut at $z = 2$

Consider now the first constraint of formulation (2). The region represented by such constraint is the halfspace above the plane A with equation $n + 2x + 2y - z = 0$. It is then easy to see that the minimum $\bar{z}$ of (2) will correspond to the point $P = (\bar{x}, \bar{y}, \bar{z})$ of C that is the last met by a sweeping line, parallel to the line $y = -x$, which moves on the surface of A towards the negative ortant (we will simply refer to these as to the *extremal* vertices). In particular, $\bar{x}$ and $\bar{y}$ are the coordinates of the extremal vertex v of the convex polygon $Q_{\bar{z}}$ in the third quadrant. The lines which define v have equations $2x\cos\alpha + 2y\cos(\alpha j) = \bar{z}$ and $2x\cos\beta + 2y\cos(\beta j) = \bar{z}$, where $\alpha = \frac{2\pi i_1}{n}$ and $\beta = \frac{2\pi i_2}{n}$, for some indices i_1 and i_2. The key property, which we will exploit both to determine a closed formula for the ϑ (in the cases $j = 2$ and $j = 3$) and to implement an efficient algorithm for the general case of circulant graphs of degree 4, is that i_1 and i_2 can be computed using "any" projection polygon Q_{z_0}, $z_0 > 0$, and determining its extremal point in the third quadrant. Once i_1 and i_2 are known, $\bar{z}$ can be computed by solving the following linear system

$$\begin{cases} 2x\cos\alpha + 2y\cos(j\alpha) - z = 0, \\ 2x\cos\beta + 2y\cos(j\beta) - z = 0, \\ \qquad\qquad 2x + 2y - z = -n. \end{cases} \tag{3}$$

3.2 The Special Case $j = 2$

The detailed proof of the following theorem is deferred to the appendix.

Theorem 1.

$$\vartheta(C(n,2)) = n\left(1 - \frac{\frac{1}{2} - \cos(\frac{2\pi}{n}\lfloor n/3\rfloor) - \cos(\frac{2\pi}{n}(\lfloor n/3\rfloor + 1))}{(\cos(\frac{2\pi}{n}\lfloor n/3\rfloor) - 1)(\cos(\frac{2\pi}{n}(\lfloor n/3\rfloor + 1)) - 1)}\right). \tag{4}$$

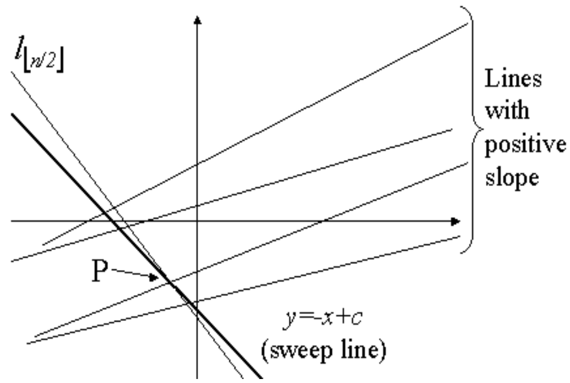

Fig. 2. Lines forming the extremal vertex P

3.3 The Special Case $j = 3$

Consider again the projection polygon Q_{z_0}, for some $z_0 > 0$. We know from Section 3.1 that the value of ϑ is the optimal value $\bar{z}$ of the objective function in the linear program (2), and that this value is achieved at the extremal vertex P of $Q_{\bar{z}}$ in the third quadrant. Also, we know that any projection polygon Q_{z_0} can be used to determine the two lines $f_i(x, y) - \bar{z} = 0$, $i \geq 1$, which form P. It turns out that finding such lines is easy when $j = 3$. In the following we will say that the line l_i has positive x-intercept (resp., y-intercept) if the intersection between l_i and the x-axis (resp., y-axis) has positive x-coordinate (resp., y-coordinate), otherwise we will say that the intercept is negative. The crucial observation is the following. Among the lines with negative x- and y-intercepts, $l_{\lfloor n/2 \rfloor}$ is the one for which these intersections are closest to the origin. It then follows that P must lay on this line and (after a moment of thought) that the second line forming P must be searched among those with positive slope. Let x_i and y_i denote the coordinates of the intersection between the line l_i and the line $l_{\lfloor n/2 \rfloor}$. Now, since $l_{\lfloor n/2 \rfloor}$ is slightly steeper than the line with equation $y = -x$, the line sought will be the one with positive x-intercept, negative y-intercept, and such that y_i is maximum (see Fig. 2). We shall prove that such line is the one with index $\bar{n} = \lceil \frac{n-3}{6} \rceil$.

To this end, observe first that the requirement of positive x-intercept and negative y-intercept implies $\frac{n}{12} < i < \frac{n}{4}$ (recall that l_i has equation $y = -\frac{\cos \frac{2\pi i}{n}}{\cos \frac{6\pi i}{n}} x + \frac{z_0}{\cos \frac{6\pi i}{n}}$). To prove that $l_{\bar{n}}$ maximizes y_i we show that, for any integer $i \neq \bar{n}$ in the interval $\frac{n}{12} < i < \frac{n}{4}$, the three points $v_i = (x_i, y_i)$, $v_{\bar{n}} = (x_{\bar{n}}, y_{\bar{n}})$ and $O = (0, 0)$ form a clockwise circuit. We already know (see the Appendix) that this amounts to proving that $d(v_i, v_{\bar{n}}, O) = x_i y_{\bar{n}} - y_i x_{\bar{n}} < 0$. This is easy; the only formal difficulty is working with the integer part of $\frac{n-3}{6}$. Clearly this might be circumvent by dealing with the three different cases, namely $n = 6k + 1$, $n = 6k + 3$, and $n = 6k + 5$, for some positive integer k. For simplicity we shall prove the first case only. Now, for $n = 6k + 1$ we have $\lceil \frac{n-3}{6} \rceil = \frac{n-1}{6}$ and, using

$z_0 = 2$,

$$x_{\bar{n}} = \frac{\cos\frac{3\pi}{n} - \cos\frac{\pi}{n}}{\cos\frac{3\pi}{n}\cos\left(\frac{\pi}{3} - \frac{\pi}{3n}\right) + \cos^2\frac{\pi}{n}}, \quad y_{\bar{n}} = \frac{-\cos\frac{\pi}{n} - \cos\left(\frac{\pi}{3} - \frac{\pi}{3n}\right)}{\cos\frac{3\pi}{n}\cos\left(\frac{\pi}{3} - \frac{\pi}{3n}\right) + \cos^2\frac{\pi}{n}},$$

$$x_i = \frac{\cos\frac{3\pi}{n} + \cos\frac{6\pi i}{n}}{\cos\frac{3\pi}{n}\cos\frac{2\pi i}{n} - \cos\frac{6\pi i}{n}\cos\frac{\pi}{n}}, \quad y_i = \frac{\cos\frac{\pi}{n} + \cos\frac{2\pi i}{n}}{\cos\frac{\pi}{n}\cos\frac{6\pi i}{n} - \cos\frac{3\pi}{n}\cos\frac{2\pi i}{n}}.$$

After some relatively simple algebra we obtain

$$d(v_i, v_{\bar{n}}, O) = \frac{\alpha + \beta + \gamma}{\left(\cos\frac{3\pi}{n}\cos\left(\frac{\pi}{3} - \frac{\pi}{3n}\right) + \cos^2\frac{\pi}{n}\right)\left(\cos\frac{3\pi}{n}\cos\frac{2\pi i}{n} - \cos\frac{\pi}{n}\cos\frac{6\pi i}{n}\right)},$$

where

$$\begin{aligned} \alpha &= \cos\tfrac{\pi}{n}\left(\cos\tfrac{6\pi i}{n} + \cos\tfrac{\pi}{n}\right), \\ \beta &= \cos(\tfrac{2\pi i}{n})\left(\cos\tfrac{\pi}{n} - \cos\tfrac{3\pi}{n}\right), \\ \gamma &= \cos\left(\tfrac{\pi}{3} - \tfrac{\pi}{3n}\right)\left(\cos\tfrac{3\pi}{n} + \cos\tfrac{6\pi i}{n}\right). \end{aligned}$$

It is easy to check that $\alpha, \beta, \gamma > 0$ while the denominator of $d(v_i, v_{\bar{n}}, O)$ is negative for the admissible values of i. We are now able to determine the value of the ϑ function of $C(n, 3)$.

Theorem 2.

$$\vartheta(C(n,3)) = n\left(1 - \frac{\cos^2\frac{\pi}{n} - \cos\frac{\pi}{n}\cos\frac{2\pi\lceil\frac{n-3}{6}\rceil}{n} + \cos^2\frac{2\pi\lceil\frac{n-3}{6}\rceil}{n} - 1}{(\cos\frac{\pi}{n} + 1)(\cos\frac{2\pi\lceil\frac{n-3}{6}\rceil}{n} - 1)(1 - \cos\frac{\pi}{n} + \cos\frac{2\pi\lceil\frac{n-3}{6}\rceil}{n})}\right). \quad (5)$$

Proof. $\vartheta(C(n,3))$ is the value of z in the solution to the linear system (3) where $j = 3$, i.e., $z = n\left(1 - \frac{\cos^2\alpha + \cos\alpha\cos\beta + \cos^2\beta - 1}{(1-\cos\alpha)(\cos\beta - 1)(\cos\alpha + \cos\beta + 1)}\right)$, where $\alpha = \frac{2\pi i_1}{n}$ and $\beta = \frac{2\pi i_2}{n}$. By the previous results we know that $i_1 = \lfloor\frac{n}{2}\rfloor$ and $i_2 = \lceil\frac{n-3}{6}\rceil$. Plugging these values into the expression for z we get the desired result. □

4 An Efficient Algorithm and Computational Results

Although the Lovász number can be computed in polynomial time, the available algorithms are far from simple and efficient (see, e.g., [1]). It is thus desirable to devise efficient algorithms tailored to the computation of ϑ for special classes of graphs. By reduction to linear programming, the theta function of circulant graphs can be computed in linear time, provided that the number of cycles is independent of n. The corresponding algorithms are not necessarily efficient in practice, though. We briefly describe a practically efficient algorithm for computing $\vartheta(C(n,j))$, i.e., in case of two cycles.

The algorithm first determines the 2 lines forming the extremal vertex of Q_1 in the third quadrant, then solves the resulting 3×3 linear system (i.e., the system (3)). More precisely, the algorithm incrementally builds the intersection of the halfplanes which define Q_1 (considering only the third quadrant) and keeps track of the extremal point. The running time is $O(n \log n)$ in the worst case (i.e., it does not improve upon the optimal algorithms for computing the intersection of n arbitrary halfplanes or, equivalently, the convex hull of n points in the plane). However, it does make use of the properties of the lines bounding the halfplanes to keep the number of vertices of the incremental intersection close to the minimum possible. In some cases (such as $C(n,2)$) this is still $\Omega(n)$, but for most values of n and j it turns out to be substantially smaller.

Using the above algorithm we have performed some preliminary experiments to get insights about the behavior of the theta function for the special class of circulant graphs considered in this abstract. Actually, since the value sandwiched by the clique and the chromatic number of $C(n,j)$ is the theta function of $\overline{C}(n,j)$ (i.e., the complementary graph of $C(n,j)$), the results refer to $\vartheta(\overline{C}(n,j)) = \frac{n}{\vartheta(C(n,j))}$.

Table 4 shows $\vartheta(\overline{C}(n,j))$ approximated to the four decimal place, for a number of values of n and j. It is immediate to note that, for a fixed value of j, the values of the theta function seem to slowly approach, as n grows, the lower bound (given by the clique number), which happens to be 2 almost always (obvious exceptions occur when 3 divides n and $j = \frac{n}{3}$.

Table 1. Some computed values of $\vartheta(\overline{C}(n,j))$

	4	5	6	7	$\lfloor\frac{n}{4}\rfloor$	$\lceil\frac{n}{4}\rceil$	$\lfloor\frac{n}{3}\rfloor$	$\lfloor\frac{n}{3}\rfloor+1$	$\frac{n-3}{2}$
51	2.2446	2.0474	2.1227	2.0838	2.1297	2.2446	3	2.0173	2.2446
101	2.2383	2.0121	2.1122	2.0228	2.2383	2.1162	2.2383	2.0044	2.2383
201	2.2366	2.0031	2.1103	2.0059	2.2366	2.1113	3	2.0011	2.2366
301	2.2363	2.0014	2.1099	2.0027	2.2363	2.1099	2.0005	2.2363	2.2363
401	2.2362	2.0008	2.11	2.0015	2.2362	2.1102	2.2362	2.0003	2.2362
501	2.2362	2.0005	2.11	2.001	2.2362	2.1102	3	2.0002	2.2362
1001	2.2361	2.0001	2.1099	2.0002	2.2361	2.1099	2.2361	2	2.2361
2001	2.2361	2	2.1099	2.0001	2.2361	2.1099	3	2	2.2361
3001	2.2361	2	2.1099	2	2.2361	2.1099	2	2.2361	2.2361
4001	2.2361	2	2.1099	2	2.2361	2.1099	2.2361	2	2.2361
5001	2.2361	2	2.1099	2	2.2361	2.1099	3	2	2.2361
10001	2.2361	2	2.1099	2	2.2361	2.1099	2.2361	2	2.2361

This is confirmed by the results in Table 2, which depicts the behavior of the relative distance d_{nj} of $\vartheta(\overline{C}(n,j))$ from the clique number. We only consider odd values of j (so that the clique number is always 2); we also rule out the cases where $j = \frac{n}{3}$, for which we know there is a (relatively) large gap between clique number and theta function. More precisely, Table 2 shows: (1) the maximum relative distance $M = \max_{j,n} d_{nj}$, where n ranges over all odd integers from 9 to $\bar{n}$; (2) the average relative distance $\mu = \frac{1}{N_{\bar{n}}} \sum_{j,n} d_{nj}$, where $N_{\bar{n}}$ is the number of admissible pairs (n,j); (3) the average quadratic distance $\sigma = \frac{1}{N_{\bar{n}}} \sum_{j,n} (d_{nj} - \mu)^2$.

The regularities presented by the value of the theta function and by the geometric structure of the optimal lines suggest the possibilities of further analytic investigations. For instance, we have observed that, for $j = 4$, the formula $i = \lfloor \frac{n}{2\pi} \arccos \frac{-1-\sqrt{5}}{4} \rfloor$ seems to correctly predict the index of the first optimal

Table 2. Relative distances of $\vartheta(\overline{C}(n,j))$ from the clique number

n	M	μ	σ
101	0.372402	0.056077	0.004343
201	0.372402	0.033712	0.002600
301	0.372402	0.024840	0.001876
401	0.372402	0.019897	0.001471
501	0.372402	0.016734	0.001214
1001	0.372402	0.009657	0.000653

line, in perfect agreement with the experimental results. In general, for j even and $j << n$, up to a value $\bar{j}$, the optimal point seems to always correspond to two consecutive indices. For j odd, the first line giving the optimal point is almost always obtained at the index $\frac{n-1}{2}$; the second line varies with j, but with a regular behaviour.

5 Conclusions

This paper has provided a first step towards extending the class of graphs for whose theta function either a formula or a very efficient algorithm is available. Work in progress by the authors [5] aims at finding an efficient algorithm for more general circulant graphs. We believe that the results of this paper together with the above mentioned more general results will contribute to shedding new lights on the properties of this fascinating function.

References

1. Alizadeh, F., Haeberly, J.-P. A., Nayakkankuppam, M., Overton, M., Schmieta, S.: SDPPACK User's Guide. http://www.cs.nyu.edu/faculty/overton/sdppack/sdppack.html
2. Alon, N.: On the Capacity of Digraphs. European J. Combinatorics, **19** (1998) 1–5
3. Alon, N., Orlitsky, A.: Repeated Communication and Ramsey Graphs. IEEE Trans. on Inf. Theory, **41** (1995) 1276–1289
4. Ashley, Siegel: A Note on the Shannon Capacity of Run-Length-Limited Codes. IEEE Trans. on Inf. Theory, **33** (1987)
5. Brimkov, V.E., Codenotti, B., Crespi, V., Leoncini, M.: Efficient Computation of the Lovász Number of Circulant Graphs. In preparation
6. Farber, M.: An Analogue of the Shannon Capacity of a Graph. SIAM J. on Alg. and Disc. Methods, **7** (1986) 67–72
7. Feige, U.: Randomized Graph Products, Chromatic Numbers, and the Lovász ϑ-Function. Proc. of the 27th STOC (1995) 635–640
8. Haemers, W.: An Upper Bound for the Shannon Capacity of a Graph. Colloq. Math. Soc. János Bolyai, **25** (1978) 267-272

9. Haemers, W.: On Some Problems of Lovász Concerning the Shannon Capacity of Graphs. IEEE Trans. on Inf. Theory, **25** (1979) 231–232
10. Knuth, D.E.: The Sandwich Theorem. Electronic J. Combinatorics, **1** (1994)
11. Lovász, L.: On the Shannon Capacity of a Graph. IEEE Trans. on Inf. Theory, **25** (1979) 1–7
12. O'Rourke, J.: Computational Geometry in C. Cambridge University Press (1994)
13. Rosenfeld, M.: On a Problem of Shannon. Proc. Amer. Math. Soc., **18** (1967) 315–319
14. Shannon, C.E.: The Zero-Error Capacity of a Noisy Channel. IRE Trans. Inform. Theory, **IT-2** (1956) 8–19
15. Szegedy, M.: A Note on the ϑ Number of Lovász and the Generalized Delsarte Bound. Proc. of the 35th FOCS, (1994) 36–41

Appendix

In this appendix we shall prove Theorem 1. Before we need to establish the following subsidiary result.

Theorem 3. *Let n be odd and $n \geq 7$. Also, as in Section 3, let C be the intersection of the halspaces defined by the inequalities $z \geq 0$ and $f_i(x,y) - z \leq 0$, $i = 1, ..., \frac{n-1}{2}$. Then, C is a polyhedral cone with the apex at the origin. The 1-dimensional faces of C (i.e., the edges of C) are the intersections (in the half-space $z \geq 0$) of "consecutive" pairs of planes P_i, $P_{s(i)}$, where P_i is defined by the equation $f_i(x,y) - z = 0$ and*

$$s(i) = \begin{cases} i+1 & \text{if } i < \frac{n-1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. To prove the result, it is sufficient to prove that (1) for any $z_0 > 0$, Q_{z_0} is bounded[2] (i.e., is a polygon), so that the polyhedron is indeed a pointed cone with the apex at the origin, and (2) Q_{z_0} has exactly $\frac{n-1}{2}$ vertices, formed by the intersections of pairs of consecutive lines l_i and $l_{s(i)}$, where l_i has equation $f_i(x,y) - z_0 = 0$, $i = 1, .., \frac{n-1}{2}$.
Now, given any two intersecting lines l and l' in the 2D space, we will say that *l' is clockwise from l* if the two lines can be overlapped by rotating l clockwise around the intersection point by an angle of less than $\pi/2$ radians.

Claim. For $n \geq 11$, l_{i+1} is clockwise from l_i, $i = 1, ..., \frac{n-1}{2} - 1$.

Proof. The equation defining l_i can be written as $y = m_j x + q_i$, where $m_t = -\frac{\cos\frac{2\pi t}{n}}{\cos\frac{4\pi t}{n}}$. Thus $-\frac{\pi}{2} < \varphi_i = \text{arctg}(m_i) < \frac{\pi}{2}$ is the angle between the positive x-axis and l_i. It is clearly sufficient to prove the following statements.

1. If $\varphi_i \varphi_{i+1} > 0$ then $\varphi_i > \varphi_{i+1}$;
2. if $\varphi_i > 0$ and $\varphi_{i+1} < 0$ then $\varphi_i - \varphi_{i+1} < \frac{\pi}{2}$;
3. if $\varphi_i < 0$ and $\varphi_{i+1} > 0$ then $\varphi_{i+1} - \varphi_i > \frac{\pi}{2}$.

[2] Recall that Q_{z_0} is the projection of $C \bigcap \{z = z_0\}$ onto the xy plane.

It is not difficult to see that Condition 1 (i.e., $\varphi_i\varphi_{i+1} > 0$) occurs if and only if $\frac{\pi}{4}(k-1) < \cos\alpha_i < \cos\alpha_{i+1} < \frac{\pi}{4}k$, for some $k \in \{1,2,3,4\}$. Since the denominator of m_t does not vanish when $t \in [i, i+1]$, then m_t is a continuous function. We shall then prove that $\varphi_i > \varphi_{i+1}$ by showing that, for $t \in [i, i+1]$, m_t is a monotone decreasing function of t. Indeed, we have

$$\begin{aligned}\frac{\mathrm{d}m_t}{\mathrm{d}t} &= -\frac{2\pi}{n}\,\frac{-\sin\alpha_t\cos 2\alpha_t + 2\cos\alpha_t\sin 2\alpha_t}{\cos^2 2\alpha_t} \\ &= \frac{2\pi}{n}\,\frac{\sin\alpha_t\cos 2\alpha_t - 4\sin\alpha_t\cos^2\alpha_t}{\cos^2 2\alpha_t} \\ &= \frac{2\pi}{n}\,\frac{\sin\alpha_t(\cos 2\alpha_t - 4\cos^2\alpha_t)}{\cos^2 2\alpha_t} \\ &= \frac{2\pi}{n}\,\frac{\sin\alpha_t(2\cos^2\alpha_t - 1 - 4\cos^2\alpha_t)}{\cos^2 2\alpha_t} \\ &= \frac{2\pi}{n}\,\frac{\sin\alpha_t(-1 - 2\cos^2\alpha_t)}{\cos^2 2\alpha_t} < 0.\end{aligned}$$

The proof of statements 2 and 3 becomes simpler if we assume that n be large enough (although only statement 3 requires that $n \geq 11$ in order to hold true). Suppose first that $\varphi_i > 0$ and $\varphi_{i+1} < 0$. This only happens if $\frac{2\pi i}{n} < \frac{\pi}{2} < \frac{2\pi(i+1)}{n}$ (i.e., if both angles are close to $\frac{\pi}{2}$). For n large enough this clearly means that both m_i and $-m_{i+1}$ are positive and close to zero, which in turn implies that $\varphi_i - \varphi_{i+1}$ is close to 0.
The proof of statement 3 is similar. The condition $\varphi_i < 0$ and $\varphi_{i+1} > 0$ occurs if either $\frac{2\pi i}{n} < \frac{\pi}{4} < \frac{2\pi(i+1)}{n}$ or $\frac{2\pi i}{n} < \frac{3\pi}{4} < \frac{2\pi(i+1)}{n}$. It both cases, $-m_i$ and m_{i+1} approach infinity as n grows, which means that $\varphi_{i+1} - \varphi_i$ approaches π. □

Following the clockwise order, in Fig. 3 we see that, for $n = 13$ and $i = 1, 2, 3$, the line l_{i+1} is indeed clockwise from l_i. □

Claim. For $i = 1, ..., \frac{n-1}{2} - 1$, let $v_i = l_i \bigcap l_{i+1}$ denote the intersection point between l_{i-1} and l_i. Then any two points v_i and v_{i+1}, for $i = 1, ..., \frac{n-1}{2} - 2$, together with the origin form a clockwise circuit.

Proof. It is well known (see, e.g., [12]) that three arbitrary points $a = (a_0, a_1)$, $b = (b_0, b_1)$, and $c = (c_0, c_1)$ form a clockwise circuit if and only if

$$d(a,b,c) = \begin{vmatrix} a_0 & a_1 & 1 \\ b_0 & b_1 & 1 \\ c_0 & c_1 & 1 \end{vmatrix} < 0.$$

In our case $c_0 = c_1 = 0$, so that the above determinant simplifies to $a_0b_1 - a_1b_0$, where a_0 and a_1 (resp., b_0 and b_1) are the coordinates of v_i (resp., v_{i+1}). To determine a_0 and a_1 we can solve the 2×2 linear system (where, for simplicity, we have set $z_0 = 1$)

$$\begin{cases} f_i(x,y) - 1 = 0 \\ f_{i+1}(x,y) - 1 = 0, \end{cases}$$

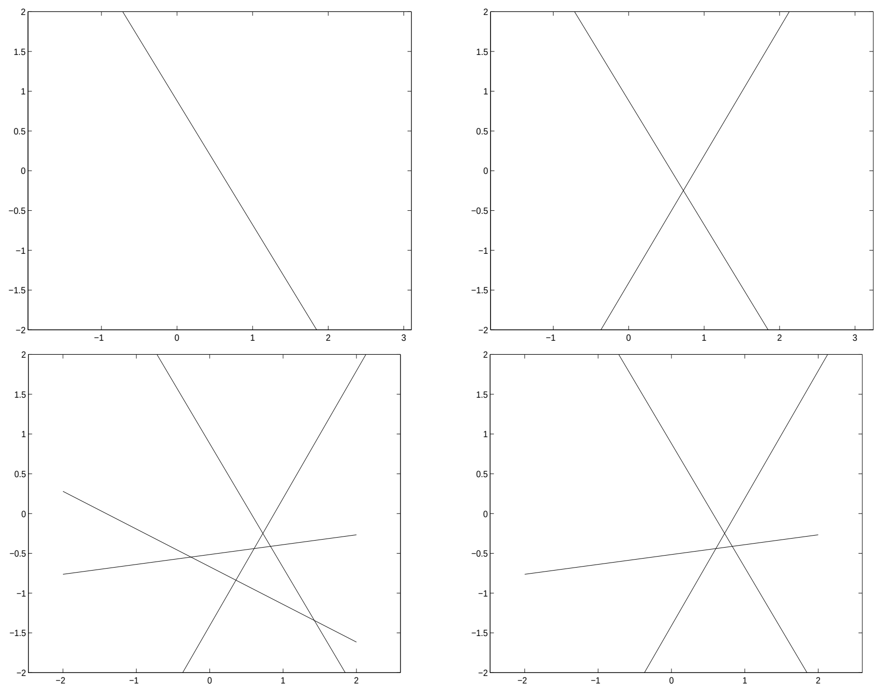

Fig. 3. Lines add in clockwise order ($n = 13$)

obtaining

$$x = \frac{\cos(2t(i+1)) - \cos(2ti)}{2(\cos(2t(i+1))\cos(ti) - \cos(2ti)\cos(t(i+1)))},$$

and

$$y = \frac{\cos(ti)\cos(2ti) - \cos(2ti)\cos(t(i+1))}{2\cos(2ti)(\cos(2t(i+1))\cos(ti) - \cos(2ti)\cos(t(i+1)))},$$

where $t = \frac{2\pi}{n}$.

For v_{i+1} we clearly obtain similar values (the correspondence being is exactly given by replacing i with $i+1$ everywhere). After some simple but quite tedious algebra, the corresponding formula for the determinant simplifies to

$$d(v_i, v_{i+1}, O) = \frac{\cos(t)\cos(t(i+1)) - \cos(ti)}{(2\cos(ti)\cos(t(i+1)) + 1)(2\cos(t(i+1))\cos(t(i+2)) + 1)}.$$

We now prove that $d = d(v_i, v_{i+1}, O) < 0$. Consider the numerator of d. Since $\cos(ti) > \cos(t(i+1))$ (recall that $0 < ti < t(i+1) < t(i+2) < \pi$), the numerator is clearly negative when $\cos(ti) > 0$. However, since $\cos(t)\cos(t(i+1)) - \cos(ti) = \cos(t(i+2)) - \cos(t)\cos(t(i+1))$ we can easily see that the numerator is also negative when $\cos(ti) < 0$. It remains to show that the denominator of d is positive. The denominator is the product of two terms, and the same argument applies to each of them. Clearly $2\cos(ti)\cos(t(i+1))+1>0$ when $\cos(ti)\cos(t(i+1))>0$. Hence the term might be negative only when $ti<\frac{\pi}{2}<t(i+1)$. In this case, however, both angles are close to $\frac{\pi}{2}$ and thus $|2\cos(ti)\cos(t(i+1))|$ is small compared to 1 (as in the proof of Claim 5, this fact is obvious for large n, although it holds for any $n\geq 7$). □

We are now able to complete the proof of Theorem 3. As in Claim 5, let v_i denote the intersection point of l_{i-1} and l_i, $i=1,...,\frac{n-1}{2}-1$. Also, let $v_{\frac{n-1}{2}}$ denote the intersection point of $l_{\frac{n-1}{2}}$ and l_1. By Claims 5 and 5, we know that any three consecutive vertices of the closed polygon $L=[v_1,...,v_{\frac{n-1}{2}}]$ make a right turn (except, possibly, for the two triples which include $v_{\frac{n-1}{2}}$ and v_1). We also know that the angle φ_i, as a function of i, changes sign three times only, starting from a negative value for $i=1$. Hence the polygon L may possibly have the three shapes depicted in Fig. 4: (1) L is convex, (2) L is simple but not convex, (3) L is not simple.

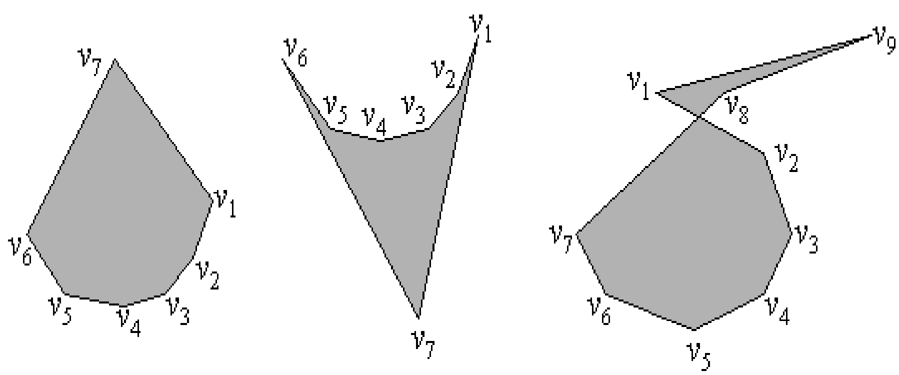

Fig. 4. Three possible forms for the polygon L of Theorem 3

Case 2 would clearly correspond to Q_{z_0} being unbounded, while case 3 would imply that the number of vertices of Q_{z_0} is less than $\frac{n-1}{2}$ and that not all of them are formed by the intersection of consecutive lines. Hence, to prove the result, we have to prove that only Case 1 indeed occurs. But this is easy. In fact, cases 2 and 3 can be ruled out simply by observing that that the three points $v_{\frac{n-1}{2}}$, v_1, and O make a left turn, while in case 1 they make a right turn. Computing the appropriate determinant d we get

$$d=\frac{-(2\alpha-1)(\alpha+1)(4\alpha^2-2\alpha-1)}{2(4\alpha^3-2\alpha-1)(32\alpha^6-48\alpha^4+20\alpha^2-1)},$$

where $\alpha=\cos\frac{\pi}{n}$. Now, the numerator is negative for $x>.8090169945$ (the largest root of $4x^2-2x-1=0$) while the denominator is positive for $x>$

.8846461772 (the unique real root of $4x^3 - 2x - 1 = 0$). But for $n = 7$ we already have $\alpha = .9009688678$; hence $d < 0$ for any $n \geq 7$ and the three points make a right turn, as required.
As the last observation, we recall that the proof holds for odd $n \geq 11$ (because of Claim 5). However, the result is true for any odd $n \geq 7$, as can be seen by directly checking the cases $n = 7$ and $n = 9$ (see Fig. 5).

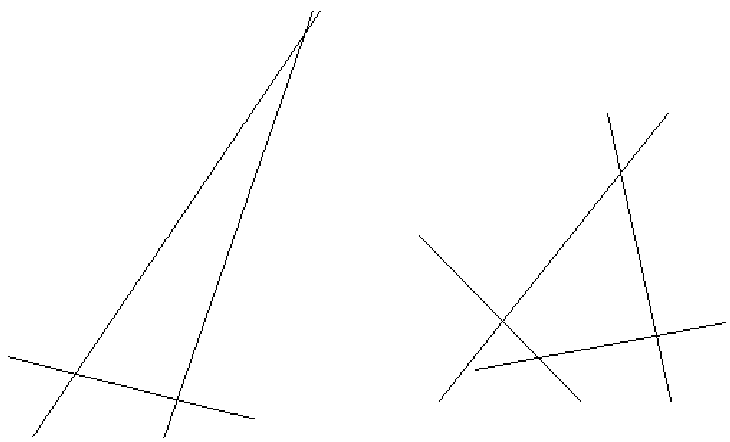

Fig. 5. Projection Q_{z_0} for $n = 7$ (left) and $n = 9$

Proof of Theorem 1. Consider the linear system (3) with $j = 2$. By Theorem 3, we know that $\alpha = \frac{2\pi i}{n}$ and $\beta = \frac{2\pi(i+1)}{n}$, for some $i \in \{1, ..., \frac{n-1}{2} - 1\}$. The solution to (3) is given by $x = \frac{n}{2} \frac{\cos\alpha + \cos\beta}{(\cos\alpha - 1)(\cos\beta - 1)}$, $y = \frac{-n}{4} \frac{1}{(\cos\alpha - 1)(\cos\beta - 1)}$, and $z = n + 2x + 2y = n\left(1 - \frac{\frac{1}{2} - \cos\alpha - \cos\beta}{(\cos\alpha - 1)(\cos\beta - 1)}\right)$. Our goal is now to determine the value of i which minimizes z. More precisely, we will compute the minimum, over the set $\{1, 2, ..., \frac{n-1}{2} - 1\}$, of the following function

$$g_n(x) = \frac{\cos\frac{2\pi x}{n} + \cos\frac{2\pi(x+1)}{n} - \frac{1}{2}}{(\cos\frac{2\pi x}{n} - 1)(\cos\frac{2\pi(x+1)}{n} - 1)}.$$

$g_n(x)$ is a continuous function in the open interval $(0, 2\pi)$, with $\lim_{x \to 0^+} = \lim_{x \to 2\pi^-} = +\infty$. Computing the derivative we obtain

$$g'_n(x) = -\frac{\pi\left(\csc\frac{\pi x}{n}\csc\frac{\pi(1+x)}{n}\right)^3}{8n} h_n(x),$$

where $h_n(x) = \sin\frac{(2x-1)\pi}{n} + \sin\frac{(2x+3)\pi}{n} + \sin\frac{(4x+1)\pi}{n} + \sin\frac{(4x+3)\pi}{n}$. Clearly, $g'_n(x) = 0$ if and only if $h_n(x) = 0$. As a first rough argument (which is useful just to locate the zero $\hat{x}$ of h_n), we note that $h_n(x) > 0$ if $\frac{(4x+3)\pi}{n} \leq \pi$. This implies that $\hat{x}$ must be greater than $\frac{n-3}{4}$. But then, for n large enough, we may "approximate" $h_n(x)$ with $\bar{h}_n(x) = 2\sin\frac{2x\pi}{n} + 2\sin\frac{4x\pi}{n}$ which vanishes at $x = \frac{n}{3}$. So $\hat{x} \approx \frac{n}{3}$ and we see that $\frac{\pi}{2} < 2\hat{x}-1 < 2\hat{x}+3 < \pi$ and $\pi < 4\hat{x}+1 < 4\hat{x}+3 < \frac{3\pi}{2}$. We now use this result to obtain tight bounds for $\hat{x}$. We observe that $h_n(x)$ is positive if

$$\begin{aligned}\sin\left(2\pi - \frac{(4x+3)\pi}{n}\right) &= \max\left\{\left|\sin\frac{(4x+3)\pi}{n}\right|, \left|\sin\frac{(4x+1)\pi}{n}\right|\right\} \\ &< \min\left\{\sin\frac{(2x-1)\pi}{n}, \sin\frac{(2x+3)\pi}{n}\right\} \\ &= \sin\frac{(2x+3)\pi}{n},\end{aligned}$$

which amounts to saying that $\hat{x}$ cannot be less than $\frac{n}{3} - 1$. Analogously, $h_n(x)$ is negative if

$$\begin{aligned}\sin\left(2\pi - \frac{(4x+1)\pi}{n}\right) &= \min\left\{\left|\sin\frac{(4x+3)\pi}{n}\right|, \left|\sin\frac{(4x+1)\pi}{n}\right|\right\} \\ &> \max\left\{\sin\frac{(2x-1)\pi}{n}, \sin\frac{(2x+3)\pi}{n}\right\} \\ &= \sin\frac{(2x-1)\pi}{n},\end{aligned}$$

which implies that $\hat{x}$ cannot be larger than $\frac{n}{3}$. This fact allows us to conclude that the integer value which minimizes $g_n(x)$ (and hence z) is one among $\lfloor\frac{n}{3}\rfloor - 1$, $\lfloor\frac{n}{3}\rfloor$ and $\lceil\frac{n}{3}\rceil$. We now prove that the value sought is $\lfloor\frac{n}{3}\rfloor$ by showing that $g_n(\lfloor\frac{n}{3}\rfloor - 1) - g_n(\lfloor\frac{n}{3}\rfloor)$ and $g_n(\lceil\frac{n}{3}\rceil) - g_n(\lfloor\frac{n}{3}\rfloor)$ are both positive. For simplicity, we shall use the following notation: $f_{\lfloor\rfloor-1} = \cos\frac{2\pi(\lfloor n/3\rfloor-1)}{n}$, $f_{\lfloor\rfloor} = \cos\frac{2\pi(\lfloor n/3\rfloor)}{n}$, $f_{\lceil\rceil} = \cos\frac{2\pi(\lceil n/3\rceil)}{n}$, and $f_{\lceil\rceil+1} = \cos\frac{2\pi(\lceil n/3\rceil+1)}{n}$. We have

$$\begin{aligned}g_n(\lceil\frac{n}{3}\rceil) - g_n(\lfloor\frac{n}{3}\rfloor) &= \frac{f_{\lceil\rceil} + f_{\lceil\rceil+1} - 1/2}{(f_{\lceil\rceil} - 1)(f_{\lceil\rceil+1} - 1)} - \frac{f_{\lfloor\rfloor} + f_{\lceil\rceil} - 1/2}{(f_{\lfloor\rfloor} - 1)(f_{\lceil\rceil} - 1)} \\ &= \frac{f_{\lceil\rceil}f_{\lfloor\rfloor} - f_{\lceil\rceil} + f_{\lceil\rceil+1}f_{\lfloor\rfloor} - f_{\lceil\rceil+1} - \frac{1}{2}f_{\lfloor\rfloor} + \frac{1}{2}}{(f_{\lfloor\rfloor} - 1)(f_{\lceil\rceil} - 1)(f_{\lceil\rceil+1} - 1)} - \\ &\quad \frac{f_{\lfloor\rfloor}f_{\lceil\rceil+1} - f_{\lfloor\rfloor} + f_{\lceil\rceil}f_{\lceil\rceil+1} - f_{\lceil\rceil} - \frac{1}{2}f_{\lceil\rceil+1} + \frac{1}{2}}{(f_{\lfloor\rfloor} - 1)(f_{\lceil\rceil} - 1)(f_{\lceil\rceil+1} - 1)} \\ &= \frac{(f_{\lfloor\rfloor} - f_{\lceil\rceil+1})(\frac{1}{2} + f_{\lceil\rceil})}{(f_{\lfloor\rfloor} - 1)(f_{\lceil\rceil} - 1)(f_{\lceil\rceil+1} - 1)}.\end{aligned}$$

The last expression is positive since the denominator is negative, $f_{\lfloor\rfloor} - f_{\lceil\rceil+1} > 0$, and $f_{\lceil\rceil} < -\frac{1}{2}$. Similarly,

$$g_n(\lfloor\frac{n}{3}\rfloor - 1) - g_n(\lfloor\frac{n}{3}\rfloor) = \frac{f_{\lfloor\rfloor-1} + f_{\lfloor\rfloor} - 1/2}{(f_{\lfloor\rfloor-1} - 1)(f_{\lfloor\rfloor} - 1)} - \frac{f_{\lfloor\rfloor} + f_{\lceil\rceil} - 1/2}{(f_{\lfloor\rfloor} - 1)(f_{\lceil\rceil} - 1)}$$

$$= \frac{f_{\lfloor\,\rfloor-1}f_{\lceil\,\rceil} - f_{\lfloor\,\rfloor-1} + f_{\lfloor\,\rfloor}f_{\lceil\,\rceil} - f_{\lfloor\,\rfloor} - \frac{1}{2}f_{\lceil\,\rceil} + \frac{1}{2}}{(f_{\lfloor\,\rfloor-1} - 1)(f_{\lfloor\,\rfloor} - 1)(f_{\lceil\,\rceil} - 1)} - \frac{f_{\lfloor\,\rfloor}f_{\lfloor\,\rfloor-1} - f_{\lfloor\,\rfloor} + f_{\lceil\,\rceil}f_{\lfloor\,\rfloor-1} - f_{\lceil\,\rceil} - \frac{1}{2}f_{\lfloor\,\rfloor-1} + \frac{1}{2}}{(f_{\lfloor\,\rfloor-1} - 1)(f_{\lfloor\,\rfloor} - 1)(f_{\lceil\,\rceil} - 1)}$$

$$= \frac{(f_{\lceil\,\rceil} - f_{\lfloor\,\rfloor-1})(\frac{1}{2} + f_{\lfloor\,\rfloor})}{(f_{\lfloor\,\rfloor-1} - 1)(f_{\lfloor\,\rfloor} - 1)(f_{\lceil\,\rceil} - 1))},$$

and again the numerator is negative since $f_{\lceil\,\rceil} - f_{\lfloor\,\rfloor-1} < 0$ and $f_{\lfloor\,\rfloor} > -\frac{1}{2}$. □