Exact and Asymptotic Results for the Number of Tilings of an m-by-n Board with Squares

Dr. Silvia Heubach

Dept. of Mathematics and Computer Science
California State University, Los Angeles
www.calstatela.edu/faculty/sheubac

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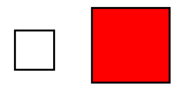
Overview

- Joint Work with Phyllis Chinn, Humboldt State University
- Interested in tilings with square tiles 

$T^{k}_{m,n}$ = # of tilings of size m-by-n with 1x1 and 2x2 tiles

$T^{k}_{m,n}$ = # of tilings of size m-by-n that contain exactly k 2x2 tiles

- ✔ General Results for $T^{k}_{m,n}$
- ✔ $T^{k}_{2,n}$ and $T^{k}_{3,n}$ - Recursion and explicit results, patterns
- ✔ $T^{k}_{4,n}$ and $T^{k}_{5,n}$ - Recursion and patterns
- ✔ Extensions to m > 5 ?????
- ✔ Generating functions for $T^{k}_{m,n}$ and $T_{m,n}$
- ✔ Asymptotic results

Notation

$T_{m,n}$ = total number of tilings

$T^k_{m,n}$ = # of tilings with k 2x2 tiles

mxn area $\leftrightarrow$ m rows and n columns

General Results

1) $T_{m,n} = \sum_{k\geq 0} T^k_{m,n}$ and $T^k_{m,n} = 0$ for $k > \left\lfloor \frac{m \cdot n}{4} \right\rfloor$

2) $T^1_{m,n} = (m-1)(n-1)$ for $m \geq 1$, $n \geq 1$

3) $T^0_{m,n} = 1$ for all values of m and n

Proof:

2) A single red square can have its lower left corner in any of m-1 rows and n-1 columns

3) In this case the tiling consists of all white tiles.

Case m = 2 and m = 3:

Recursive Formulas:

$$T^k_{2,n} = T^k_{2,n-1} + T^{k-1}_{2,n-2} \quad \text{and} \quad T^k_{3,n} = T^k_{3,n-1} + 2T^{k-1}_{3,n-2}$$

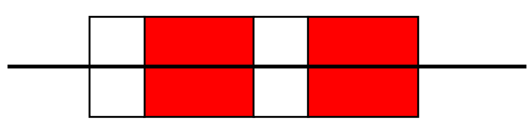
for $n \geq 2,\ k \geq 1$

Explicit Results:

$$T^k_{2,n} = \binom{n-k}{k} \quad \text{and} \quad T^k_{3,n} = \binom{n-k}{k} \cdot 2^k$$

Proof:

For m = 2, there is a one-to-one correspondence to tiling with 1x1 and 1x2 rods

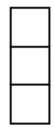


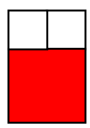
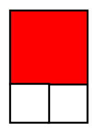
Brigham, Caron, Chinn & Grimaldi: "A Tiling Scheme for the Fibonacci Numbers", *Journal of Recreational Mathematics*, Vol 28, No 1.

For m = 3, we need to show:

$$T^{k}_{3,n} = T^{k}_{3,n-1} + 2T^{k-1}_{3,n-2} \quad \text{and} \quad T^{k}_{3,n} = \binom{n-k}{k} \cdot 2^{k}$$

Recursive formula:

- Either the tiling starts with a  $\Rightarrow T^{k}_{3,n-1}$ tilings

  or with either a  or  $\Rightarrow 2T^{k-1}_{3,n-2}$ tilings

Explicit formula:

- Line-up of white stacks and mixed stacks; if we have k red tiles, then there must be k mixed stacks and n-2k white stacks, i.e. n-k objects. Select the positions of the mixed stacks

  $\Rightarrow \binom{n-k}{k}$ possible choices

  Once positions of mixed stacks have been determined, there are two possibilities for each position

  $\Rightarrow \binom{n-k}{k} 2^{k}$ possible tilings

Patterns for $T^k_{2,n}$

$_n\backslash^k$	0	1	2	3	4	5	6
0	**1**						
1	1						
2	1	**1**					
3	1	**2**					
4	1	**3**	**1**				
5	1	**4**	**3**				
6	1	**5**	**6**	**1**			
7	1	**6**	**10**	**4**			
8	1	**7**	**15**	**10**	**1**		
9	1	**8**	**21**	**20**	**5**		
10	1	**9**	**28**	**35**	**15**	**1**	
11	1	**10**	**36**	**56**	**35**	**6**	
12	1	**11**	**45**	**84**	72	**21**	**1**

Patterns:

1) $T^1_{2,n} = n-1$
2) Diagonals of slope -1 contain rows of Pascal's triangle
3) Values in l[th] diagonal of slope -2 equal values in column for k = l.
4) $T^k_{2,2k} = 1$

Proof:

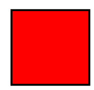
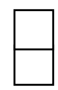
1) General Result (2) for m = 2.

Entries in the l^{th} diagonal of slope -r are given by $T^k_{2,l+rk}$.

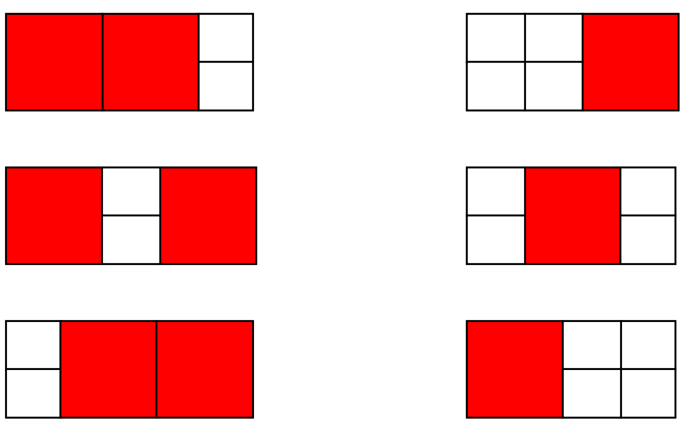
2) For l=1 this means we look at $T^k_{2,l+k}$. Thus we need to line up n-k = l+k-k = l objects . Select the positions for the k red squares

$$\Rightarrow T^k_{2,l+k} = \binom{l}{k}$$

3) $T^k_{2,l+2k} = T^l_{2,k+2l}$

Combinatorial Argument: Replace squares  by stacks  and vice versa.

Example (l = 1, k = 2): $T^2_{2,5} = 3 = T^1_{2,4}$



4) In this case we are placing k red tiles into an area of size 2x(2k) = 4k -> there is only one way to do this.

Patterns for $T^k_{3,n}$

$_n\backslash^k$	0	1	2	3	4	5	6
0	1						
1	1						
2	1	2					
3	1	4					
4	1	6	4				
5	1	8	12				
6	1	10	24	8			
7	1	12	40	32			
8	1	14	60	80	16		
9	1	16	84	160	80		
10	1	18	112	280	240	32	
11	1	20	144	448	560	192	
12	1	22	180	672	1120	672	64

Patterns:

1) $T^1_{3,n} = 2(n-1)$

2) $T^2_{3,n} = 2(n-2)(n-3)$ for $n \geq 4$

3) $T^k_{3,2k} = 2^k$ for $k \geq 1$

4) $T^k_{3,l+2k} = T^l_{3,2l+k} 2^{k-l}$ (k^{th} column $\leftrightarrow$ l^{th} diagonal of slope -2)

Proof:

1) General Result (2) for m = 3.

2) & 3) are special cases of the explicit formula for $T^k_{3,n}$.

4) Same argument as in the case m = 2, namely replacing white stacks by mixed stacks and vice versa. However, there are now two possibilities for the mixed stacks, thus an adjustment factor of 2^{k-l} is needed.

Example: k = 2

$$\underbrace{T^k_{3,l+2k}}_{\text{column elements}} = \underbrace{T^l_{3,2l+k}}_{\text{diagonal elements}} \cdot \underbrace{2^{k-l}}_{\text{adjustment factor}}$$

l	$T^2_{3,l+4}$		$T^l_{3,2l+2}$		2^{k-l}
0	4	=	1	·	4
1	12	=	6	·	2
2	24	=	24	·	1
3	40	=	80	·	1/2
4	60	=	240	·	1/4
5	84	=	672	·	1/8

Case m = 4:

$$T^{k}_{4,n} = T^{k}_{4,n-1} + 3T^{k-1}_{4,n-2} + T^{k-2}_{4,n-2} + 2\sum_{r=3}^{\min\{k+1,n\}} T^{k-r+1}_{4,n-r} \quad \text{for } k \geq 2$$

Proof:

We look at the first column (from the left)

1) No red tile in the first column $\Rightarrow$ k red squares have to occur in a tiling of size 4x(n-1)

$$\Rightarrow T^{k}_{4,n-1} \text{ tilings}$$

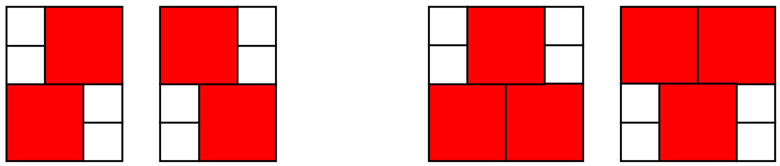
2) One red square in the first column, and no red tile whose lower left corner is in the second column $\Rightarrow$ k-1 red tiles in a tiling of size 4x(n-2); 3 possible positions for the red tile in column 1

$$\Rightarrow 3T^{k-1}_{4,n-2} \text{ tilings}$$

3) Two red squares in the first column $\Rightarrow$ k-2 red tiles in a tiling of size 4x(n-2)

$$\Rightarrow T^{k-2}_{4,n-2} \text{ tilings}$$

4) One red square in the first column, and one red tile whose lower left corner is in the second column ⇒ interlocking pattern that can start two ways:



Assume interlocking pattern covers the first r columns ⇒ r-1 red tiles have been placed. The remaining k-(r-1) tiles occur in a tiling of size 4x(n-r). Smallest pattern has 2 tiles (r = 3), largest has k tiles (r = k+1)

$$\Rightarrow 2 \sum_{r=3}^{\min\{k+1,n\}} T_{4,n-r}^{k-(r-1)} \quad \text{tilings}$$

Table of Values for $T^{k}_{4,n}$

$_n\backslash^k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1										
2	1	3	1								
3	1	6	4								
4	1	9	16	8	1						
5	1	12	37	34	9						
6	1	15	67	105	65	15	1				
7	1	18	106	248	250	108	16				
8	1	21	154	490	726	522	176	24	1		
9	1	24	211	858	1736	1824	994	260	25		
10	1	27	277	1379	3604	5148	4090	1770	385	35	1

Patterns:

1) $T^{1}_{4,n} = 3(n-1)$

2) $T^{n}_{4,n} = 1$ for n even

3) $T^{2l-1}_{4,2l} = (l+1)^2-1$ for $l \geq 0$

4) $T^{2l}_{4,2l+1} = (l+1)^2$ for $l \geq 1$

Proof:

1) General Result (2) for m = 4.

2) Placing n red tiles in an area of 4xn can be done in exactly one way - all tiles are red.

3) In this case one less than the maximal number of possible red tiles is being placed ⇒ 4 white tiles are being placed in the tiling. Due to the geometry of the red tiles, these white tiles occur as two white stacks of two squares, which can be placed either horizontally or vertically.
   - If stacks are placed vertically they both have to be in either the upper or lower half of the tiling. In that half there are l-1 red squares, thus $\binom{l+1}{2}$ possible locations

     $\Rightarrow 2\binom{l+1}{2} = l^2 + l$ possible tilings

   - If stacks are placed horizontally they both have to be in the same two columns, and have to be separated by a red tile (other cases already covered) ⇒ l additional tilings

$$\Rightarrow l^2 + l + l = (l+1)^2 - 1 \text{ tilings}$$

4) 2l red tiles cover an area of 4x(2l). Since the number of columns is odd, there must be a stack in each of the upper and lower two rows. This stack can be placed in (l+1) ways for each of the two stacks ⇒ $(l+1)^2$ possible tilings.

Case m = 5:

$$T^{k}_{5,n} = T^{k}_{5,n-1} + 4T^{k-1}_{5,n-2} + 3T^{k-2}_{5,n-2} + 2\sum_{r=3}^{k+1} F_{r+1} T^{k-r+1}_{5,n-r} \quad \text{for } k \geq 2$$

F_r = r^{th} Fibonacci Number

Proof: (Similar to case m = 4)

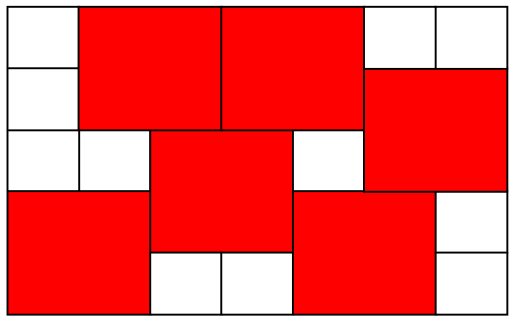
1) - 3) For the cases where there is no interlocking pattern, the argument is identical except that for 2) (one red tile) and 3) (two red tiles) the number of possible placements of the tiles in the first column is adjusted.

$$\Rightarrow T^{k}_{5,n-1} + 4T^{k-1}_{5,n-2} + 3T^{k-2}_{5,n-2} \text{ tilings}$$

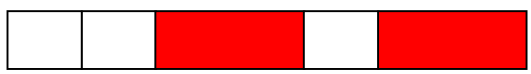
3) There is a one-to-one correspondence to the total number of tilings of a 1xr area with 1x1 and 1x2 rods, of which there are F_{r+1}. Each such tiling "creates" 2 interlocking tilings with red and white squares that cover the first r columns, which is then combined with a tiling of size 5x(n-r) which has k-(r-1) tiles

$$\Rightarrow 2\sum_{r=3}^{k+1} F_{r+1} T^{k-r+1}_{5,n-r}$$

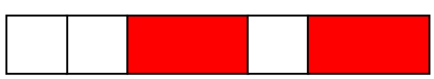
Interlocking pattern



Middle row



"Creation"



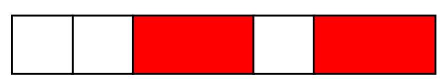


Table of Values for $T^k_{5,n}$

$_n\backslash^k$	0	1	2	3	4	5	6	7	8
0	1								
1	1								
2	1	4	3						
3	1	8	12						
4	1	12	37	34	9				
5	1	16	78	140	79				
6	1	20	135	382	454	194	27		
7	1	24	208	824	1566	1344	408		
8	1	28	297	1530	4103	5670	3698	926	81

Patterns:

1) $T^1_{5,n} = 4(n-1)$

2) $T^{2l}_{5,2l} = 3^l$ for $l \geq 0$

Proof:

1) General Result (2) for m = 5.

2) Since we are placing 2l tiles, they cannot be in an interlocking pattern (r columns -> r-1 tiles). Thus each pair of two consecutive columns contains exactly 2 red tiles, which can be arranged in 3 ways within the two columns $\Rightarrow 3^l$ possibilities

Connections with Alternative Method of Counting

Tiling an m-by-n Area with Squares of Size up to k-by-k ($m \leq 5$), *Congressus Numerantium* 140 (1999), 43 - 64.

- For m = 4 and m = 5, the interlocking patterns are exactly the basic blocks of size m x r

$$T^{k}_{m,n} = T^{k}_{m,n-1} + (m-1)T^{k-1}_{m,n-2} + \binom{m-2}{2} T^{k-2}_{m,n-2} + \sum_{r=3}^{k+1} B_{m,r} T^{k-r+1}_{m,n-r}$$

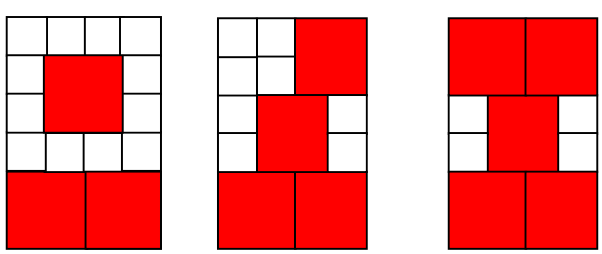
Extensions to m > 5

Combined formula looks promising - but there is a BIG problem

- First three terms generalize to $T^{k}_{m,n-1} + \sum_{l=1}^{\lfloor m/2 \rfloor} \binom{m-l}{l} T^{k-l}_{m,n-2}$

BUT

- Interlocking patterns cannot be generalized, because the width of the pattern does no longer determine the number of red tiles within the pattern!!



Generating Functions

Idea: Create a function (in series representation) whose coefficients count the quantity of interest

$T_{m,n}$ = total # of tilings of mxn board with 1x1 and 2x2 tiles

$$G_m(x) = \sum_{n\geq 0} T_{m,n} x^n$$ generating function

$T^k_{m,n}$ = # of tilings of mxn board containing exactly k 2x2 tiles

$$g_m(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} T^k_{m,n} x^n y^k$$ generating function

Note:

$$g_m(x,1) = \sum_{n\geq 0}\sum_{k\geq 0} T^k_{m,n} x^n = \sum_{n\geq 0} x^n \sum_{k\geq 0} T^k_{m,n} = \sum_{n\geq 0} T_{m,n} x^n = G_m(x)$$

generating function for $T_{m,n}$

Generating Functions

- Multiply the recursion equation by $x^n y^k$
- Sum over appropriate values of n and k for which the recursion holds
- Use initial conditions to obtain equation in the generating function and solve for it

Results

- $g_2(x,y)=(1-x-x^2y)^{-1}$ and $G_2(x)=(1-x-x^2)^{-1}$
- $g_3(x,y)=(1-x-2x^2y)^{-1}$ and $G_3(x)=(1-x-2x^2)^{-1}$
- $g_4(x,y)=\dfrac{1-xy}{(1-xy)(1-x-3x^2y-x^2y^2)-2x^3y^2}$ and

  $G_4(x)=\dfrac{1-x}{1-2x-3x^2+2x^3}$
- $g_5(x,y)=\dfrac{1-xy-x^2y^2}{1-x-xy-3x^2y-4x^2y^2-x^3y^2+3x^3y^3+3x^4y^4}$ and

  $G_5(x)=\dfrac{1-x-x^2}{1-2x-7x^2+2x^3+3x^4}$

Example of Expansion for $T^k_{5,n}$

In[2]:=
```
Normal[
 Series[(1 - y x - y^2 x^2) /
   (1 - x - y x - 3 y x^2 - 4 y^2 x^2 - y^2 x^3 + 3 y^3 x^3 + 3 y^4 x^4),
  {y, 0, 8}, {x, 0, 8}]]
```

Out[2]= $1 + x + x^2 (1 + 4y + 3y^2) + x^3 (1 + 8y + 12y^2) + x^4 (1 + 12y + 37y^2 + 34y^3 + 9y^4) + x^5 (1 + 16y + 78y^2 + 140y^3 + 79y^4) + x^6 (1 + 20y + 135y^2 + 382y^3 + 454y^4 + 194y^5 + 27y^6) + x^7 (1 + 24y + 208y^2 + 824y^3 + 1566y^4 + 1344y^5 + 408y^6) + x^8 (1 + 28y + 297y^2 + 1530y^3 + 4103y^4 + 5670y^5 + 3698y^6 + 926y^7 + 81y^8)$

In[3]:=
```
Normal[Series[(1 - x - x^2) / (1 - 2 x - 7 x^2 + 2 x^3 + 3 x^4),
  {x, 0, 8}]]
```

Out[3]= $1 + x + 8x^2 + 21x^3 + 93x^4 + 314x^5 + 1213x^6 + 4375x^7 + 16334x^8$

n\k	0	1	2	3	4	5	6	7	8		sum
0	1										1
1	1										1
2	1	4	3								8
3	1	8	12								21
4	1	12	37	34	9						93
5	1	16	78	140	79						314
6	1	20	135	382	454	194	27				1213
7	1	24	208	824	1566	1344	408				4375
8	1	28	297	1530	4103	5670	3698	926	81		16334

Asymptotics

$$f(z)=\sum_{n\geq 0} a_n z^n \; ; \; [z^n] = \text{coefficient of } z^n = a_n$$

Facts from Complex Analysis:

- If f(z) is analytic for |z| < R, then $a_n = O(\frac{1}{R}+\varepsilon)^n$

- If f(z) is meromorphic (only poles) and z_0 is a simple pole of f(z), then

$$f(z)=\frac{a_{-1}}{z-z_0}+\sum_{j\geq 0} a_j (z-z_0)^j$$

$$PP(f;z_0):=\frac{a_{-1}}{z-z_0}$$ Principal Part of f at z_0

$$\sum_{j\geq 0} a_j (z-z_0)^j$$ analytic at z_0

- $a_{-1} = \lim_{z\to z_0} f(z)(z-z_0)$

Case m = 4

$G_4(z)$ has poles at

$$z_1 = \frac{1}{2} - \sqrt{\frac{7}{3}}\, \mathrm{Cos}\left[\frac{1}{3}\left(2\pi - \mathrm{ArcCos}\left[-\frac{3\sqrt{\frac{3}{7}}}{7}\right]\right)\right] =$$

0.35541572677584501545866127091571630591695803226 6...

$$z_2 = \frac{1}{2} - \sqrt{\frac{7}{3}}\, \mathrm{Cos}\left[\frac{1}{3}\, \mathrm{ArcCos}\left[-\frac{3\sqrt{\frac{3}{7}}}{7}\right]\right] =$$

-0.74464428590503938139646826522742462057728311471 7...

$$z_3 = \frac{1}{2} + \sqrt{\frac{7}{3}}\, \mathrm{Cos}\left[\frac{1}{3}\, \mathrm{ArcCos}\left[\frac{3\sqrt{\frac{3}{7}}}{7}\right]\right] =$$

1.88922855912919436593780699431170831466032494764 8.

$$[z^n]G_4(z) = [z^n]\left\{PP(G_4;z_1)+PP(G_4;z_2)\right\}+O\left(\left(\frac{1}{|z_3|}+\varepsilon\right)^n\right)$$
$$\approx \frac{0.191012}{z_1^{\,n+1}} - \frac{0.301069}{z_2^{\,n+1}}$$

Case m = 5

$G_5(z)$ has poles at

$z_1 = 0.2709802057537904152008174679265051829599027662177912\ldots$

$z_2 = -0.5361047619344666715566582613028149502998433349161892\ldots$

$z_3 = 1.3272430955725595250992701091153328870491888508624863\ldots$

$z_4 = -1.7287852060585499354100959824056897863759149488830755\ldots$

$$[z^n]G_5(z) = [z^n]\left\{PP(G_5;z_1)+PP(G_5;z_2)\right\}+O\left(\left(\frac{1}{|z_3|}+\varepsilon\right)^n\right)$$

$$\approx \frac{0.128186}{z_1^{\,n+1}} + \frac{0.115759}{z_2^{\,n+1}}$$

Approximation

- For m = 4, approximation matches exact value until n at least 2000 using the exact solutions
- For m = 5, approximation matches exact value until n = 85 (using a precision of 40 for numerical computations)
- Asymptotics can be used to compute values, but be prepared for round-off errors when dealing with numerical solutions
- Asymptotics can be used to compare growth rates for the two cases