Counting Rises, Levels and Drops in Compositions

Silvia Heubach

Department of Mathematics
California State University Los Angeles

joint work with

Toufik Mansour

Department of Mathematics, Haifa University, Israel

Various authors have investigated specific sets A:

- Alladi and Hoggatt, $A = \{1, 2\}$
  Fibonacci Quarterly **13** (1975) No. 3, 233–239.
- Grimaldi, $A = \{2k + 1, k \geq 0\}$
  Congressus Numerantium **142** (2000), 113–127.
- Grimaldi, $A = \mathbb{N} - \{1\}$
  Congressus Numerantium **152** (2001), 33–43.
- Chinn, Grimaldi, and Heubach, $A = \mathbb{N}$
  Fibonacci Quarterly **41** (2003) No. 3, 229–239.
- Chinn and Heubach, $A = \mathbb{N} - \{k\}$
  Congressus Numerantium, **164** (2003), pp. 33–51.
- Chinn and Heubach, $A = \{1, k\}$
  Congressus Numerantium, **164** (2003), pp. 183–194.

Definitions

- A composition $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n.
- The number of summands, namely m, is called the number of parts of the composition.
- A palindromic composition of $n \in \mathbb{N}$ is a composition for which $\sigma_1\sigma_2\ldots\sigma_m = \sigma_m\sigma_{m-1}\ldots\sigma_1$.
- A Carlitz composition is a composition of $n \in \mathbb{N}$ in which no two consecutive parts are the same.

- rise = summand followed by larger summand
- level = summand followed by itself
- drop = summand followed by smaller summand

We define the generating functions

$$C_A(x;y;r,\ell,d) = \sum_{n\geq 0} \sum_{\sigma \in C_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}$$

$$P_A(x;y;r,\ell,d) = \sum_{n\geq 0} \sum_{\sigma \in P_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}$$

Main Result

Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$.

(i) The generating function $C_A(x; y; r, \ell, d)$ is given by

$$\frac{1 + (1-d)\sum_{j=1}^{k}\left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y(\ell - r)}{1 - x^{a_i} y(\ell - d)}\right)}{1 - d\sum_{j=1}^{k}\left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y(\ell - r)}{1 - x^{a_i} y(\ell - d)}\right)}$$

(ii) The generating function $P_A(x;y;r,\ell,d)$ is given by

$$\frac{1+\displaystyle\sum_{i=1}^{k}\frac{x^{a_i}y+x^{2a_i}y^2(\ell-d\,r)}{1-x^{2a_i}y^2(\ell^2-d\,r)}}{1-\displaystyle\sum_{i=1}^{k}\frac{x^{2a_i}y^2d\,r}{1-x^{2a_i}y^2(\ell^2-d\,r)}}$$

Proof Outline for (i)

- Define the g.f. $C_A(s_1 s_2 \ldots s_e | x; y; r, \ell, d)$ for compositions σ that start with $s_1 s_2 \ldots s_e$.
- $C_A(x; y; r, \ell, d) = 1 + \sum_{i=1}^{k} C_A(a_i | x; y; r, \ell, d)$
- Lemma: The g.f. $C_A(a_i | x; y; r, \ell, d)$ is given by

$$x^{a_i} y \left(1 + d \sum_{j=1}^{i-1} C_A(a_j | x; y; r, \ell, d) \right.$$

$$\left. + \ell C_A(a_i | x; y; r, \ell, d) + r \sum_{j=i+1}^{k} C_A(a_j | x; y; r, \ell, d) \right)$$

- 2. & 3. give a set of $k+1$ equations in $k+1$ variables $C_A(x; y; r, \ell, d)$ and $C_A(a_j | x; y; r, \ell, d)$.

- Use Cramer's rule; needs some ingenuity to get closed form for the respective determinants

Easier proof for (ii), as the corresponding lemma has a different structure. Due to the symmetry of palindromic compositions, we need to distinguish only $i = j$ and $i \neq j$. No need for Cramer's rule.

Rises and Drops in Compositions

Setting $l = d = 1$ in the main result gives

$$C_A(x;y;r,1,1) = \frac{1}{1 - \sum_{j=1}^{k} \left(x^{a_j} y \prod_{i=1}^{j-1} (1 - x^{a_i} y(1-r)) \right)}$$

Computing

$$\frac{\partial}{\partial r} C_A(x;y;r,1,1) \bigg|_{r=1} = \frac{y^2 \sum_{k \geq j > i \geq 1} x^{a_i + a_j}}{\left(1 - y \sum_{j=1}^{k} x^{a_j}\right)^2}$$

and expressing this function as a power series about $y = 0$ gives

Corollary: Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)} =$$

$$\left(\sum_{k \geq j > i \geq 1} x^{a_i + a_j} \right) \sum_{m \geq 0} (m+1) \left(\sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}$$

Levels in Compositions

Setting $r = d = 1$ in the main result and computing the respective partial derivative gives:

Corollary: Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then

$$\sum_{n\geq 0} \sum_{\sigma \in C_n^A} \text{levels}(\sigma) x^n y^{\text{parts}(\sigma)} =$$

$$\left(\sum_{j=1}^{k} x^{2a_j} \right) \sum_{m\geq 0} (m+1) \left(\sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}$$

Carlitz Compositions

- σ is a Carlitz composition $\Leftrightarrow$ $\text{levels}(\sigma) = 0$
- g.f. for Carlitz compositions is given by $C_A(x; y; r, \mathbf{0}, d)$

Few results known for Carlitz compositions - we will look in particular at the set $A = \{a, b\}$.

If $A = \{a, b\}$, then compositions consist of alternating a's and b's.

n	Carlitz compositions of n
$k(a+b)$	$abab\ldots ab$ and $baba\ldots ba$
$k(a+b)+a$	$abab\ldots aba$
$k(a+b)+b$	$babab\ldots bab$

Thus, the number of Carlitz compositions of $n > 0$ is

$$\begin{cases} 2, & \text{if } n \equiv 0 \pmod{(a+b)}; \\ 1, & \text{if } n \equiv a \pmod{(a+b)} \text{ or } n \equiv b \pmod{(a+b)}; \\ 0, & \text{otherwise.} \end{cases}$$

Rises in Carlitz compositions of $A = \{a, b\}$

G.f. is given by

$$\frac{x^{a+b}(1 + x^a)(1 + x^b)}{(1 - x^{a+b})^2}.$$

Specifically, the number of rises in all Carlitz compositions of $n \geq (a + b)$ is

$$\begin{cases} 2k - 1, & \text{if } n = k(a + b). \\ k, & \text{if } n = k(a + b) + a \text{ or } n = k(a + b) + b; \end{cases}$$

Partitions

- σ is a partition (unordered composition) $\Leftrightarrow$ $\text{rises}(\sigma) = 0$
- G.f. for partitions is given by $C_A(x; y; \mathbf{0}, \ell, d)$

Special case $A = \{a, b\}$: generating function for the number of partitions of n with parts in A

$$\frac{1}{(1 - x^a)(1 - x^b)}$$

If $A = \{1, k\}$, then the number of partitions of n with parts in A is given by

$$\lfloor (n+k)/k \rfloor$$

For $n \in [n'k, (n'+1)k)$, the only partitions are

$$11\ldots11$$

$$k11\ldots11$$

$$kk11\ldots11$$

$$\vdots$$

$$\underbrace{kk\ldots kk}_{n'}11\ldots11$$

for a total of $n' + 1 = \lfloor (n+k)/k \rfloor$ partitions.

Future Research

- rises, levels and drops are two-letter patterns
- extend to three-letter patterns

Example:

- $123 \Leftrightarrow$ rise followed by rise $\Leftrightarrow\uparrow\uparrow$
- $121, 132, 231 \Leftrightarrow$ rise followed by drop = peak $\Leftrightarrow\uparrow\downarrow$

Thanks for Listening

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