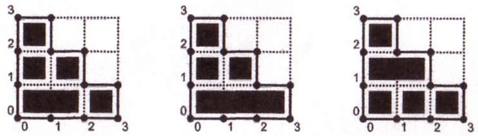
Staircase Tilings and Lattice Paths



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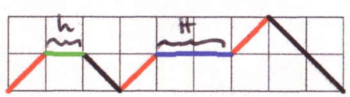
Overview

- Define a staircase tiling
- Describe direct bijections to Dyck paths, Motzkin paths, and Schröder paths
- Connect structures in the lattice paths to structures in the tilings
- Use results about characteristics of the lattice paths to enumerate certain types of tilings or characteristics of tilings

Quick Review

Lattice paths starting at $(0,0)$ and returning to the x-axis without going below the x-axis



U, D	$\leftrightarrow$	Dyck paths of length $2n$	$\leftrightarrow$	$\mathcal{D}_n$
U, D, h	$\leftrightarrow$	Motzkin paths of length n	$\leftrightarrow$	$\mathcal{M}_n$
U, D, H	$\leftrightarrow$	Schröder paths of length $2n$	$\leftrightarrow$	$\mathcal{S}_n$

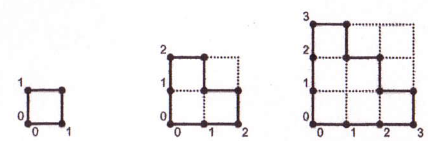
$|\mathcal{D}_n| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (A000108)

$|\mathcal{M}_n| = M_n$, the n-th Motzkin number (A000106)

$|\mathcal{S}_n| = s_n$, the n-th (large) Schröder number (A006318)

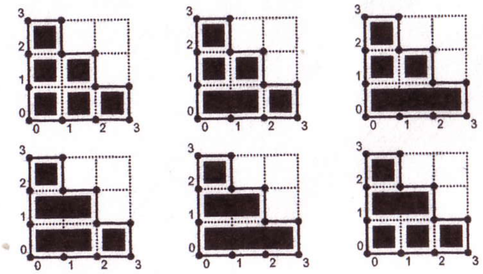
Definitions

- $\mathcal{A}_n = \{(x,y) : 0 \leq x \leq n, 0 \leq y \leq \lfloor n-x \rfloor + 1\}$



- j^{th} row $= \{(x,y) : j-1 \leq y \leq j\}$; border $= \{(x,y) \in \mathcal{A}_n, x = 0\}$
- row-tiling has rows with tiles of size or length m; large tile = tile of size ≥ 2; complete row = row in which a single tile fills row completely
- row-tiling is border if it there is at most one large tile in each row which is adjacent to the border
- row-tiling is heap if no tile is above a tile of smaller size

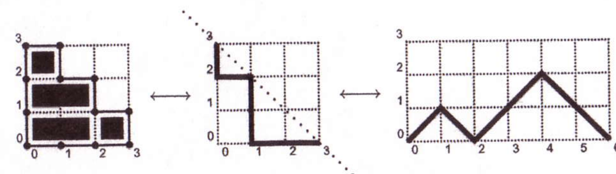
Border row-tilings of the staircase $\mathcal{A}_3$



Path Creation Algorithm for BHR tilings

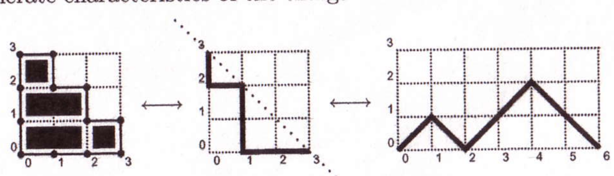
To create an associated tiling path

- Start at position $(0, n)$.
- If the path is at position (i, j) and the tile adjacent to the border in row j ends at $x = k$, then continue the path to $(k-1, j)$, and then to $(k-1, j-1)$.
- Continue until the path is at position $(i, 0)$, then complete the path to position $(n, 0)$.



Proposition: There is a bijection between BHR_n, the BHR-tilings of $\mathcal{A}_n$, and the set of Dyck paths $\mathcal{D}_n$.

Dyck paths have been studied widely. Translating features of Dyck paths into features of the staircase tilings will allow us to enumerate characteristics of the tilings.



Block in Dyck path	$\leftrightarrow$ complete row
DU (valley)	$\leftrightarrow$ size of consecutive border tile changes
UDU	$\leftrightarrow$ change of size one

Corollary: The number of BHR-tilings of $\mathcal{A}_n$ with exactly k complete rows is given by $\frac{k}{2n-k}\binom{2n-k}{n}$.

Corollary: The number of tilings in BHR_n with exactly k changes (of any size) equals $\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$, the Narayana numbers [2] and [4].

Corollary: The number of tilings in BHR_n with exactly k changes of size one equals $\binom{n-1}{k}M_{n-1-k}$, where M_n denotes the n-th Motzkin number.

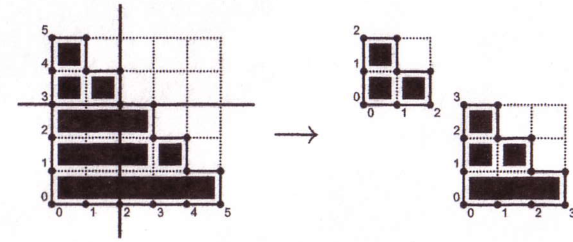
The last corollary indicates that there is a bijection between Motzkin paths and BHR tilings with no change of size one (setting $k = 0$). We will provide two algorithms that create Motzkin paths from BHR tilings with certain properties.

Motzkin Paths

To create Motzkin paths we define these operations on tilings:

- Reduction of a tiling (without complete row of size ≥ 2) means taking away the tiles of size one at the right end of each row.
- Split applies to tilings T of $\mathcal{A}_n$ that have a complete row of size ≥ 2. If the topmost complete row is at row j, $1 \leq j \leq n-1$, then we cut horizontally above row j, and vertically at $x = n - j$. This creates two smaller tilings T_u and T_l, of $\mathcal{A}_{n-(j-1)}$ and $\mathcal{A}_{j-1}$, respectively, where the line $x = n - j$ becomes the border of T_l.

The Split Operation



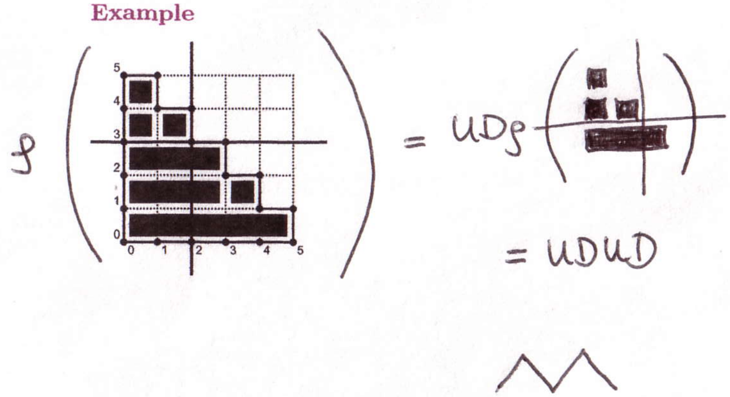
Motzkin Path Creation Algorithm I

To create a Motzkin path associated with BHR-tilings that have no change of size one, we use the following recursive algorithm based on the first return decomposition method:

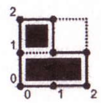
- $\rho(\blacksquare) = \emptyset$.
- If T has no complete row with tile of size $m \geq 2$, then $\rho(T) = h\rho(R(T))$.
- If T has at least one complete row with tile of size $m \geq 2$, then $\rho(T) = U\rho(R(T_u))D\rho(T_l)$.
- Apply the algorithm until the tiling has been transformed completely.

Proposition: The BHR-tilings of $\mathcal{A}_n$ that have no change of size one are in one-to-one correspondence with $\mathcal{M}_{n-1}$.

Example



Proof: By induction on n, where we check that a proper Motzkin path of the correct length is created. The condition that the tiling cannot have a change of size one comes in very subtly. If there was a change of size one, then eventually a staircase $\mathcal{A}_2$ with two complete rows would remain, and then the algorithm does no longer apply - we would need $\rho(\emptyset)$, which is not defined.



For the reverse operation, decompose the Motzkin path according to the first return into either $M = U\,M^1 D\,M^2$, or if the path starts with a horizontal step, into $M = hM^1$, and define an extension operation that is the reverse of the reduction. ∎

Motzkin Path Creation Algorithm II

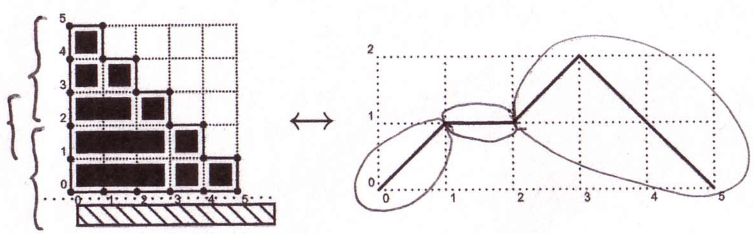
Motzkin path associated with a BHR-tiling that has at most two large tiles of the same size. Start reading from the top row of the tiling.

- If row i and $i-1$ both contain a tile of size m, and row $i-2$ contains a tile of size $m+k$, append $U\,\underbrace{D\cdots D}_{k-1}$ to the path.

  Move to row $i-2$.
- If row i contains a tile of size m, and row $i-1$ contains a tile of size $m+k$, then append $h\,\underbrace{D\cdots D}_{k-1}$ to the path. Move to row $i-1$.

To apply the algorithm to row one, imagine that there is a row zero with a tile of length $n+1$.

Example:

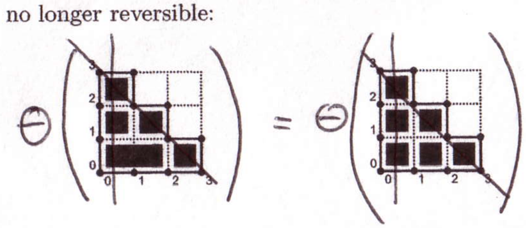


Proposition: The BHR-tilings of $\mathcal{A}_n$ that have at most two tiles adjacent to the border of the same size m $(1 \le m \le n-1)$ are in one-to-one correspondence with $\mathcal{M}_n$.

Proof: Induction on n using a recursive algorithm. Let $\theta(\emptyset) = \emptyset$ and $\theta(\blacksquare) = h$. We distinguish 2 cases:

- T has no complete row of length $\ge 2 \leftrightarrow \theta(T) = U\theta(T')D$, where T' is created from T by deleting the first column of T, and by deleting the tiles of size 1 at the end of each row.
- T has a complete row of length $m \ge 2 \leftrightarrow \theta(T) = \theta(T_u)\theta(T_l)$, where we use the Split operation.

If there are more that two adjacent tiles of the same size, then θ is no longer reversible:



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