Tiling an m-by-n Area with Squares of Size up to k-by-k ($m \leq 5$)

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Overview

- Extension of the question of tiling rectangles with Cuisinare (1-by-r) rods

  - Tiling 1-by-n rectangles with white (1-by-1) and red (1-by-2) c-rods

    Brigham, Caron, Chinn, & Grimaldi

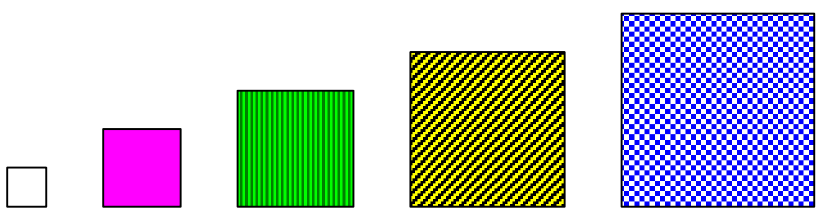
  - Tiling 2-by-n rectangles with white (1-by-1) and red (1-by-2) c-rods

    Brigham, Chinn, Holt, & Wilson

  - Tiling 2-by-n and 3-by-n rectangles with c-rods of length £ k

    Hare, and Hare & Chinn

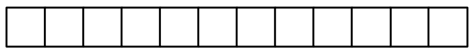
- Now tiles are squares



- 2 cases are considered:
  - only 1-by-1 and 2-by-2 squares
  - squares up to size k-by-k, with k = min{m,n}

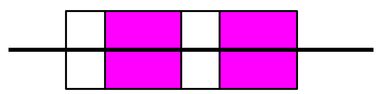
The Easy Cases

- m = 1: $T_{1,n} = 1$

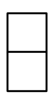


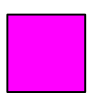
- m = 2: $T_{2,n} = f_{n+1}$ (shifted Fibonacci Sequence)

1. One-to-One Correspondence to tiling with c-rods

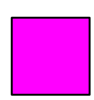
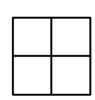


2. Recursive Formula: Tiling of size 2-by-n can have two forms:

 + any tiling of size 2-by-(n-1)

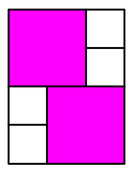
 + any tiling of size 2-by-(n-2)

” $T_{2,n} = T_{2,n-1} + T_{2,n-2}$

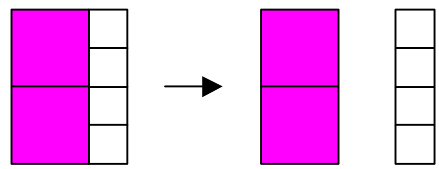
$T_{2,1} = 1$, $T_{2,2} = 2$ ( , )

Basic Blocks

- Basic blocks are tilings that cannot be vertically sliced into two or more smaller rectangles without cutting squares.



basic block

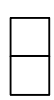
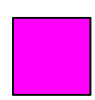


not a basic block

- Idea: Combine basic block (on the left) of size m-by-k with any tiling of size m-by-(n-k) to form tiling of size m-by-n.

$$T_{m,n} = \sum_{k=1}^{n} B_{m,k} \cdot T_{m,n-k}, \quad T_{m,0} := 1$$

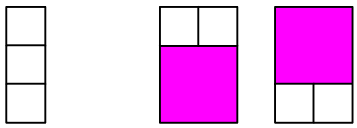
Example:

For m = 2,  and  are the only basic blocks of any size.

$B_{2,1} = B_{2,2} = 1$, and $B_{2,n} = 0$ for $n > 2$.

Case m = 3 (1-by-1 and 2-by-2 tiles)

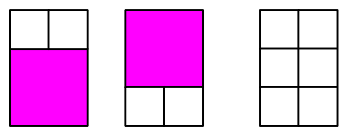
- $B_{3,1} = 1$, $B_{3,2} = 2$, $B_{3,n} = 0$ for $n > 2$



- Recursive Formula:

$$T_{3,n} = T_{3,n-1} + 2T_{3,n-2}$$

$$T_{3,1} = 1, \quad T_{3,2} = 3$$

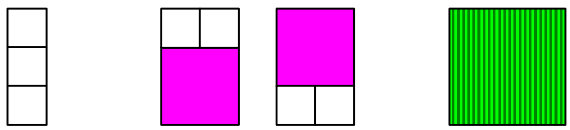


- Explicit Formula:

$$T_{3,n} = \begin{cases} \displaystyle\sum_{l=1}^{(n-1)/2} 2^{2l} + 1 & \text{if } n \text{ is odd} \\ \displaystyle\sum_{l=1}^{n/2} 2^{2l-1} + 1 & \text{if } n \text{ is even} \end{cases}$$

Case m = 3 (up to 3-by-3 tiles)

- $\tilde{B}_{3,1} = 1$, $\tilde{B}_{3,2} = 2$, $\tilde{B}_{3,3} = 1$, $\tilde{B}_{3,n} = 0$ for n > 3



- Recursive Formula:

$$\tilde{T}_{3,n} = \tilde{T}_{3,n-1} + 2\,\tilde{T}_{3,n-2} + \tilde{T}_{3,n-3}$$

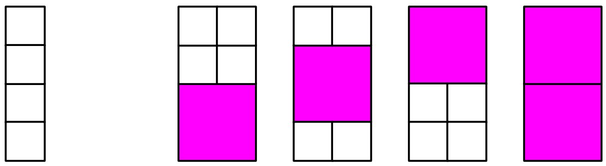
$$\tilde{T}_{3,0} = \tilde{T}_{3,1} = 1, \quad \tilde{T}_{3,2} = 2 \quad \text{(as before)}$$

- Comparison between the two cases:

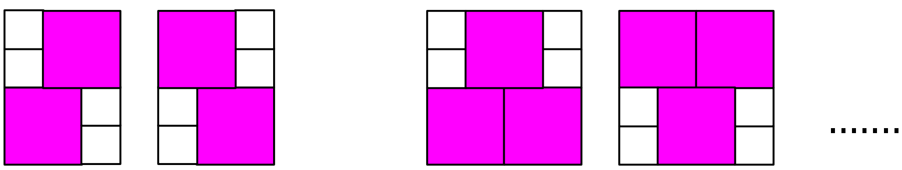
n	1	2	3	4	5	6	7	8	9	10
$T_{3,n}$	1	3	5	11	21	43	85	171	341	683
$\tilde{T}_{3,n}$	1	3	6	13	28	60	129	277	595	1,278

Case m = 4 (1-by-1 and 2-by-2 tiles)

- $B_{4,1} = 1$, $B_{4,2} = 4$



- $B_{4,n} = 2$ for n > 2



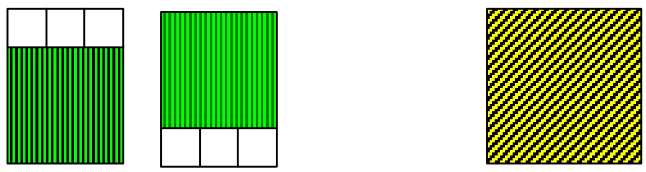
- Recursive Formula

$$T_{4,n} = T_{4,n-1} + 4 \cdot T_{4,n-2} + 2 \cdot \sum_{k=0}^{n-3} T_{4,k}$$

$$T_{4,0} = T_{4,1} = 1, \ T_{4,2} = 5$$

Case m = 4 (up to 4-by-4 tiles)

- $\tilde{B}_{4,1} = B_{4,1} = 1$, $\tilde{B}_{4,2} = B_{4,2} = 4$ (same tiles)
- $\tilde{B}_{4,3} = B_{4,3} + 2 = 4$, $\tilde{B}_{4,4} = B_{4,4} + 1 = 3$,



- $\tilde{B}_{4,n} = 2$ for n > 4 (as neither the 3-by-3 nor the 4-by-4 can be used to extend)
- Recursive formula:

$$\tilde{T}_{4,n} = \tilde{T}_{4,n-1} + 4\tilde{T}_{4,n-2} + 4\tilde{T}_{4,n-3} + 3\tilde{T}_{4,n-3} + 2\sum_{k=0}^{n-5}\tilde{T}_{4,k}$$

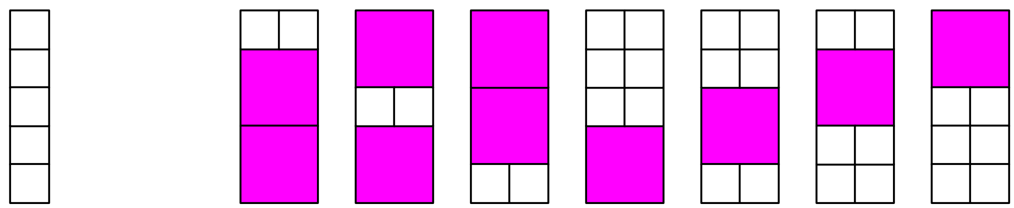
$$\tilde{T}_{4,0} = \tilde{T}_{4,1} = 1, \quad \tilde{T}_{4,2} = 2, \tilde{T}_{4,3} = 13$$

- Comparison between the two cases:

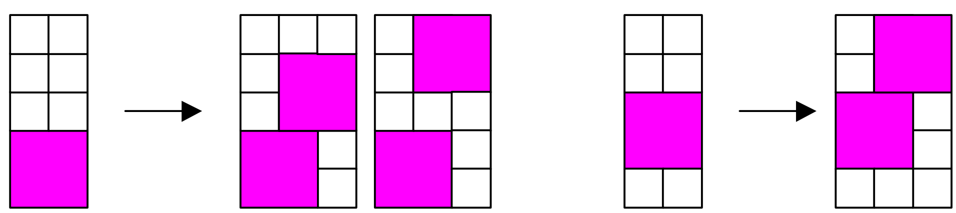
n	1	2	3	4	5	6	7	8	9	10
$T_{4,n}$	1	5	11	35	93	269	747	2,115	5,933	16,717
$\tilde{T}_{4,n}$	1	5	13	40	117	348	1,029	3,049	9,028	26,738

Case m = 5 (1-by-1 and 2-by-2 tiles)

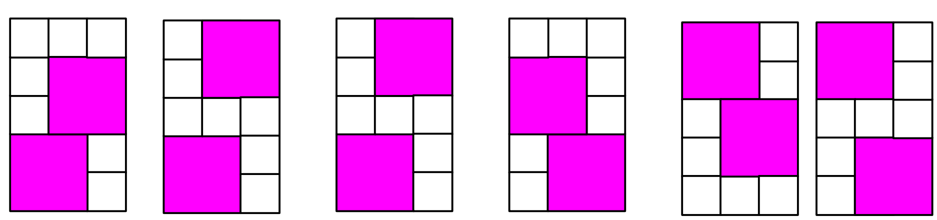
- $B_{5,1} = 1$, $B_{5,2} = 7$



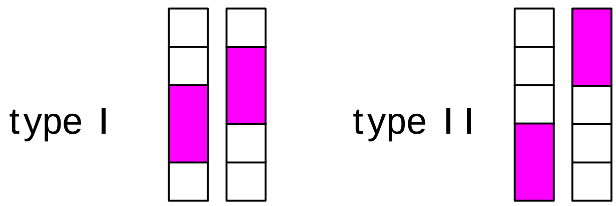
- Bigger basic blocks by extension (of last four blocks)



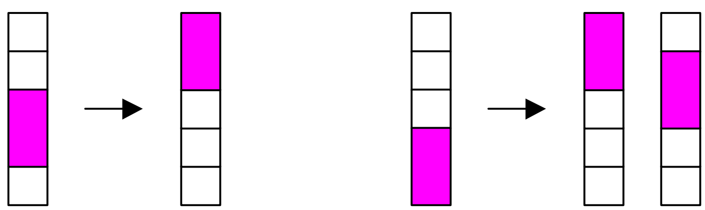
- $B_{5,3} = 6$



- Two types of basic blocks



- Type I creates one block of type II; Type II creates one block each of type I and type II



- $B^{I}_{5,n+1} = B^{II}_{5,n}$ and $B^{II}_{5,n+1} = B^{I}_{5,n} + B^{II}_{5,n}$

- Recursive Formula:

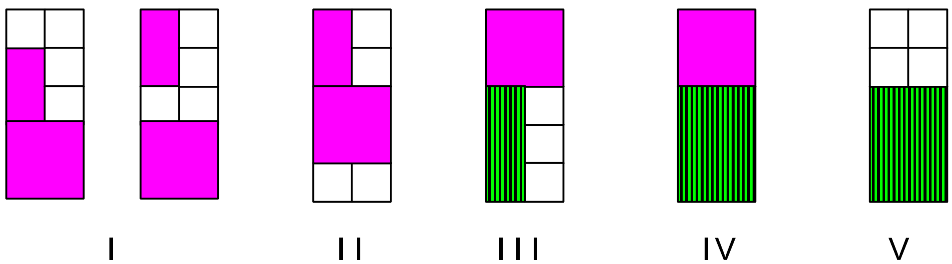
$$T_{m,n} = \sum_{k=1}^{n} B_{m,k} \cdot T_{m,n-k}$$

with $B_{5,n} = B_{5,n-1} + B_{5,n-2}$ for $n > 4$, and

$B_{5,4} = 10$, $B_{5,3} = 6$, $B_{5,2} = 7$, and $B_{5,1} = T_{5,1} = T_{5,0} = 1$.

Case m = 5 (up to 5-by-5 tiles)

- need to look at last two columns for extensions (because of 3-by-3 extension)
- more types



- Recursive Formula

$$\tilde{T}_{5,n} = \sum_{k=1}^{n} \tilde{B}_{5,k} \tilde{T}_{5,n-k}$$

$$\tilde{B}_{5,n+1} = \tilde{B}_{5,n+1}^{I} + \tilde{B}_{5,n+1}^{II} + \tilde{B}_{5,n+1}^{III} + \tilde{B}_{5,n+1}^{IV} + \tilde{B}_{5,n+1}^{V}$$

$$\begin{aligned} \tilde{B}_{5,n+1}^{I} &= \tilde{B}_{5,n}^{I} + \tilde{B}_{5,n}^{II} + \tilde{B}_{5,n}^{III} \\ \tilde{B}_{5,n+1}^{II} &= \tilde{B}_{5,n}^{I} + \tilde{B}_{5,n}^{III} \\ \tilde{B}_{5,n+1}^{III} &= \tilde{B}_{5,n}^{V} \\ \tilde{B}_{5,n+1}^{IV} = \tilde{B}_{5,n+1}^{V} &= \tilde{B}_{5,n}^{II} \end{aligned}$$

$$\tilde{B}_{5,5}^{I} = 14,\ \tilde{B}_{5,5}^{II} = 10,\ \tilde{B}_{5,5}^{III} = 2,\ \tilde{B}_{5,5}^{IV} = \tilde{B}_{5,5}^{V} = 4.$$

- Values for the basic blocks of different types

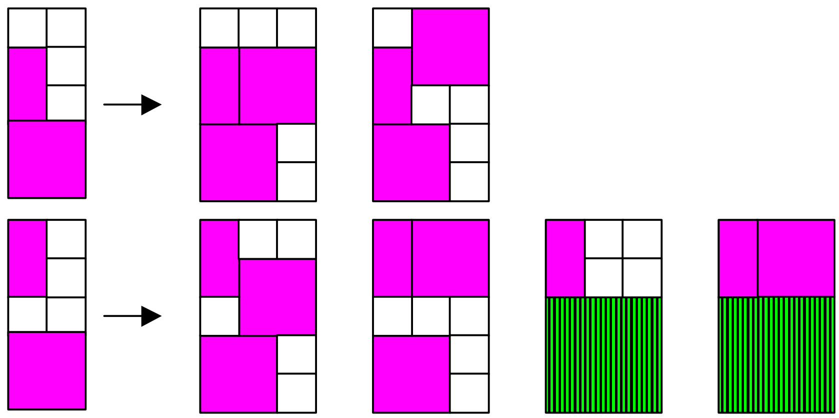
n	**5**	**6**	**7**	**8**	**9**	**10**
$\widetilde{B}^{I}_{5,n}$	14	26	46	86	158	290
$\widetilde{B}^{II}_{5,n}$	10	16	30	56	102	188
$\widetilde{B}^{III}_{5,n}$	2	4	10	16	30	56
$\widetilde{B}^{IV}_{5,n}$	4	10	16	30	56	102
$\widetilde{B}^{V}_{5,n}$	4	10	16	30	56	102

- Comparison between the two cases

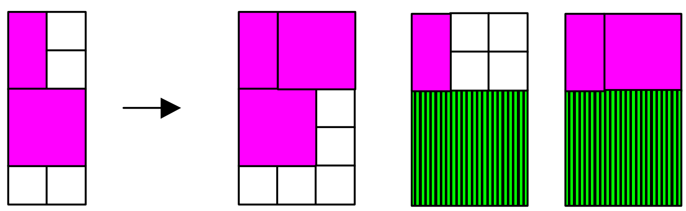
n	3	4	5	6	7	8	9	10
$B_{5,n}$	6	10	16	26	42	68	110	178
$\widetilde{B}_{5,n}$	13	20	35	66	118	218	402	738
$T_{5,n}$	21	93	314	1,213	4,375	16,334	59,925	221,799
$\widetilde{T}_{5,n}$	28	117	472	1,916	7,765	31,497	127,707	517,881

Extensions for the five types:

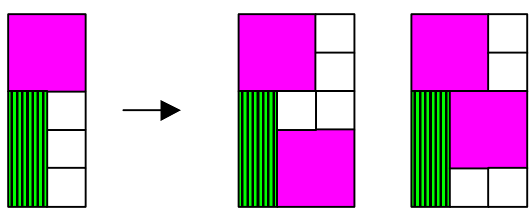
I:



II:



III:



IV: No extension

V: