Patterns Arising From Tiling Rectangles with Squares

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Overview

- Joint work with Phyllis Chinn
- Jennifer Quinn and Art Benjamin contributed an idea, Ralph Grimaldi was involved in discussions over lunch…
- Related to last year's presentation on

  $T_{m,n}$ = total number of tilings of an n-by-m rectangle with squares

- Only tiles of size 1-by-1 (white) and 2-by-2 (red) considered
- Interested in

  $T^k_{m,n}$ = # of tilings of size m-by-n that contain exactly k red tiles

  and any patterns in these numbers.

Some Basic Results

1) $T_{m,n} = \sum_{k=0}^{\lfloor mn/4 \rfloor} T^k_{m,n}$

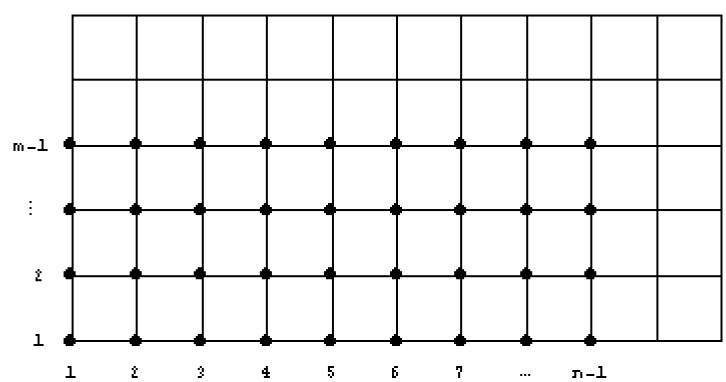
2) $T^1_{m,n} = (m-1)(n-1)$ for $m \geq 1$, $n \geq 1$

3) $T^0_{m,n} = 1$ for all values of m and n

4) $T^k_{1,n} = 0$ for $k \geq 1$ and all values of n

Proof:

2) A single red square can have its lower left endpoint in any of n-1 (horizontal) and m-1 (vertical) positions



3) & 4) In both cases the tiling consists of all white tiles and can be done in exactly one way (k = 0) or not at all (k > 0).

Case m = 1:

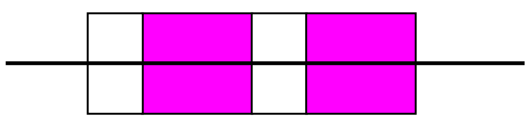
$$T_{1,n}^{0} = 1, \quad T_{1,n}^{k} = 0 \quad \text{for } k > 1$$

Case m = 2:

$$T_{2,n}^{k} = \binom{n-k}{k}$$

Proof:

- One-to-one correspondence to tiling with c-rods



Brigham, Caron, Chinn & Grimaldi: " A Tiling Scheme for the Fibonacci Numbers" , Journal of Recreational Mathematics, Vol 28, No1.

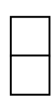
- White tiles occur in stacks ; when k red tiles are to be placed, then there are n-2k white stacks for a total of n-k objects (red tiles or white stacks) to be placed. Choose the k positions within the lineup where the red tiles are to be placed.

Table of Values for $T^k_{2,n}$

n\k	0	1	2	3	4	5	6
0	1						
1	1						
2	1	1					
3	1	2					
4	1	3	1				
5	1	4	3				
6	1	5	6	1			
7	1	6	10	4			
8	1	7	15	10	1		
9	1	8	21	20	5		
10	1	9	28	35	15	1	
11	1	10	36	56	35	6	
12	1	11	45	84	72	21	1

Patterns:

1) $T^1_{2,n} = n-1$

2) Diagonals of slope -1 contain rows of Pascal s triangle

3) Values in l^{th} diagonal of slope -2 equal values in column for k = l.

4) $T^k_{2,2k} = 1$

Proof:

1) Basic Result (2) for m = 2.

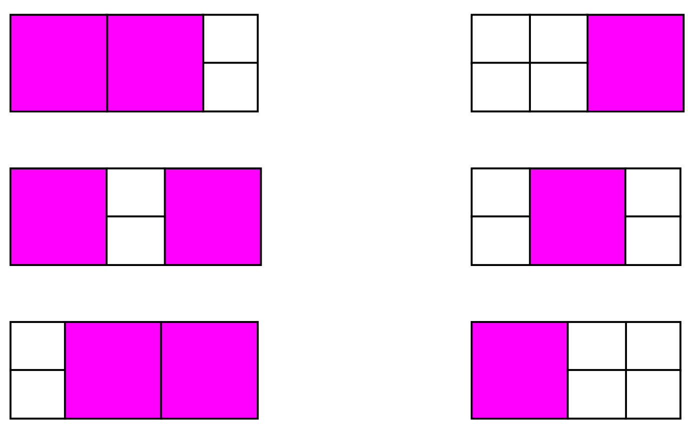
2) Entries in the l^{th} diagonal of slope -r are given by $T^k_{2,l+rk}$.

Use the general formula for $T^k_{2,n}$ to get $T^k_{2,l+k} = \binom{l+k-k}{k} = \binom{l}{k}$

3) $T^k_{2,l+2k} = T^l_{2,k+2l}$

Combinatorial Argument: Replace Squares by Stacks and vice versa.

Example (l = 1, k = 2): $T^2_{2,5} = 3 = T^1_{2,4}$



4) In this case we are placing k red tiles into an area of size 2x(2k) = 4k -> there is only one way to do this.

Case m = 3:

$$T^k_{3,n} = 2T^{k-1}_{3,n-2} + T^k_{3,n-1} \quad \text{for } n \geq 2, k \geq 1$$

Proof:

We look at the first column (from the left)

1) If there is a red tile in the first column, then k-1 red squares have to occur in a tiling of size 3x(n-2) as only one red square can be in column 1. There are two possible positions for this tile

$$\Rightarrow 2T^{k-1}_{3,n-2} \text{ tilings}$$

2) If there is no red square in the first column, then k red tiles have to occur in a tiling of size 3x(n-1)

$$\Rightarrow T^k_{3,n-1} \text{ tilings}$$

Table of Values for $T^k_{3,n}$

$_n\backslash^k$	0	1	2	3	4	5	6
0	1						
1	1						
2	1	2					
3	1	4					
4	1	6	4				
5	1	8	12				
6	1	10	24	8			
7	1	12	40	32			
8	1	14	60	80	16		
9	1	16	84	160	80		
10	1	18	112	280	240	32	
11	1	20	144	448	560	192	
12	1	22	180	672	1120	672	64

Patterns:

1) $T^1_{3,n} = 2(n-1)$

2) $T^2_{3,n} = 2(n-2)(n-3)$ for $n \geq 4$

3) $T^k_{3,2k} = 2^k$ for $k \geq 1$

4) $T^k_{3,n} = \binom{n-k}{k} 2^k$

5) $T^k_{3,l+2k} = T^l_{3,2l+k} 2^{k-l}$ (k^{th} column $\leftrightarrow$ l^{th} diagonal of slope -2)

Proof:

1) Basic Result (2) for $m = 3$.

2)& 3) are special cases of 4)

4) Combinatorial Argument:

- When placing k red tiles, there are exactly n-k possible columns available in which the lower left corner of the rd square can be placed. Select the k columns for the red tiles

  $\Rightarrow \binom{n-k}{k}$ possible choices

- Within each of the k columns, there are two possible placements for the red square

  $\Rightarrow 2^k$ possible placements

5) Same argument as in the case $m = 2$, namely replacing stacks by red tiles and vice versa. However, there are now 2 possibilities for the placement of the red square, thus the adjustment factor of 2^{k-1}

Example: k = 2

$$\underbrace{T^{k}_{3,l+2k}}_{\text{column elements}} = \underbrace{T^{l}_{3,2l+k}}_{\text{diagonal elements}} \cdot \underbrace{2^{k-l}}_{\text{adjustment factor}}$$

l	$T^{l}_{3,2l+2}$		2^{k-l}		$T^{2}_{3,l+4}$
0	1	•	4	=	4
1	6	•	2	=	12
2	24	•	1	=	24
3	80	•	1/2	=	40
4	240	•	1/4	=	60
5	672	•	1/8	=	84

Case m = 4:

$$T^{k}_{4,n} = T^{k}_{4,n-1} + 3T^{k-1}_{4,n-2} + T^{k-2}_{4,n-2} + 2\sum_{r=3}^{k+1} T^{k-r+1}_{4,n-r} \quad \text{for } k \geq 2$$

Proof:

We look again at the first column (from the left)

1) No red tile in the first column ⇒ k red squares have to occur in a tiling of size 4x(n-1)

$$\Rightarrow T^{k}_{4,n-1} \text{ tilings}$$

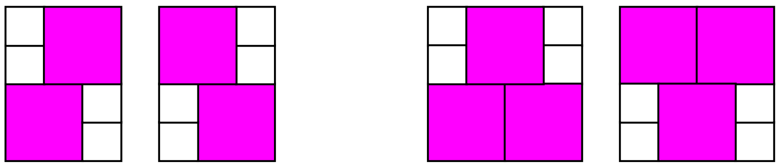
2) One red square in the first column, and no red tile whose lower left corner is in the second column ⇒ k-1 red tiles in a tiling of size 4x(n-2); 3 possible positions for the red tile in column 1

$$\Rightarrow 3T^{k-1}_{4,n-2} \text{ tilings}$$

3) Two red squares in the first column ⇒ k-2 red tiles in a tiling of size 4x(n-2)

$$\Rightarrow T^{k-2}_{4,n-2} \text{ tilings}$$

4) One red square in the first column, and one red tile whose lower left corner is in the second column ⇒ interlocking pattern that can start two ways:



Assume interlocking pattern covers the first r columns ⇒ r-1 red tiles have been placed. The remaining k-(r-1) tiles occur in a tiling of size 4x(n-r). Smallest pattern has 2 tiles (r = 3), largest has k tiles (r = k+1)

$$\Rightarrow 2\sum_{r=3}^{k+1} T_{4,n-r}^{k-(r-1)} \quad \text{tilings}$$

Table of Values for $T^k_{4,n}$

n\k	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1										
2	1	3	1								
3	1	6	4								
4	1	9	16	8	1						
5	1	12	37	34	9						
6	1	15	67	105	65	15	1				
7	1	18	106	248	250	108	16				
8	1	21	154	490	726	522	176	24	1		
9	1	24	211	858	1736	1824	994	260	25		
10	1	27	277	1379	3604	5148	4090	1770	385	35	1

Patterns:

1) $T^1_{4,n} = 3(n-1)$

2) $T^n_{4,n} = 1$ for n even

3) $T^{2l}_{4,2l+1} = (l+1)^2$ for $l \geq 1$

4) $T^{2l-1}_{4,2l} = (l+1)^2 - 1$ for $l \geq 0$

Proof:

1) Basic Result (2) for m = 4.

2) Placing n red tiles in an area of 4xn can be done in exactly one way - all tiles are red.

3) 2l red tiles cover an area of 4x(2l). Since the number of columns is odd, there must be a stack in each of the upper and lower two rows. This stack can be placed in (l+1) ways for each of the two stacks ⇒ $(l+1)^2$ possible tilings.

4) In this case one less than the maximal number of possible red tiles is being placed ⇒ 4 white tiles are being placed in the tiling. Due to the geometry of the red tiles, these white tiles occur as two white stacks, which can be placed either horizontally or vertically.
   - If stacks are placed vertically they both have to be in either the upper or lower half of the tiling. In that half there are l-1 red squares, thus $\binom{l+1}{2}$ possible locations

     ⇒ $2\binom{l+1}{2}=l^2+l$ possible tilings
   - If stacks are placed horizontally they both have to be in the same two columns, and have to be separated by a red tile (other cases already covered) ⇒ l additional tilings

$$\Rightarrow l^2+l+l = (l+1)^2 - 1 \text{ tilings}$$

Case m = 5:

$$T^{k}_{5,n} = T^{k}_{5,n-1} + 4T^{k-1}_{5,n-2} + 3T^{k-2}_{5,n-2} + 2\sum_{r=3}^{k+1} F_{r+1} T^{k-r+1}_{5,n-r} \quad \text{for } k \geq 2$$

F_r = r^{th} Fibonacci Number

Proof: (Similar to case m = 4)

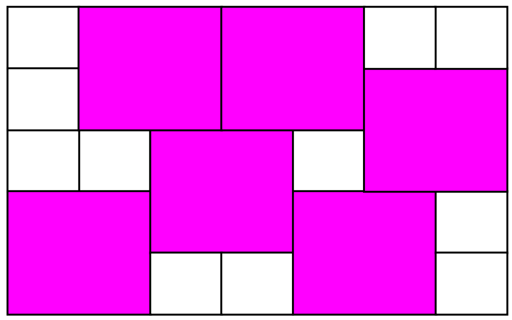
1) - 3) For the cases where there is no interlocking pattern, the argument is identical except that for 2) (one red tile) and 3) (two red tiles) the number of possible placements of the tiles in the first column is adjusted.

$$\Rightarrow T^{k}_{5,n-1} + 4T^{k-1}_{5,n-2} + 3T^{k-2}_{5,n-2} \text{ tilings}$$

4) There is a one-to-one correspondence to tilings of a 1xr area with red and white c-rods, but now to the total number of such tilings (= F_{r+1}). Each such tiling "creates" 2 interlocking tilings with red and white squares that cover the first r columns. Then combine with tilings of size 5x(n-r) which have k-(r-1) tiles

$$\Rightarrow 2\sum_{r=3}^{k+1} F_{r+1} T^{k-r+1}_{5,n-r}$$

Interlocking pattern



Middle row

"Creation"

Table of Values for $T^k_{4,n}$

n\k	0	1	2	3	4	5	6	7	8
0	1								
1	1								
2	1	4	3						
3	1	8	12						
4	1	12	37	34	9				
5	1	16	78	140	79				
6	1	20	135	382	454	194	27		
7	1	24	208	824	1566	1344	408		
8	1	28	297	1530	4103	5670	3698	926	81

Patterns:

1) $T^1_{5,n} = 4(n-1)$

2) $T^{2l}_{5,2l} = 3^l$ for $l \geq 0$

Proof:

1) Basic Result (2) for m = 5.

2) Since we are placing 2l tiles, they cannot be in an interlocking pattern (r columns -> r-1 tiles). Thus each pair of two consecutive columns contains exactly 2 red tiles, which can be arranged in 3 ways within the two columns $\Rightarrow 3^l$ possibilities

Connections with Previous Paper:

"Tiling an m-by-n Area with Squares of Size up to k-by-k ($m \leq 5$)"

- For m = 4 and m = 5, the interlocking patterns are exactly the basic blocks of size m x r
- Can combine formulas for these two cases:

$$T^k_{m,n} = T^k_{m,n-1} + (m-1)T^{k-1}_{m,n-2} + \binom{m-2}{2} T^{k-2}_{m,n-2} + \sum_{r=3}^{k+1} B_{m,r} T^{k-r+1}_{m,n-r}$$

Future Work:

- Find additional patterns within and across tables
- If patterns can be established across tables, extend to m > 5