

1. Introduction

What are models, and more specifically, mathematical models?

Let's look at the first question. You all have encountered models - just think back to the little toy cars, model houses and model train sets. All these are examples of physical models, most often scale models. They are a representation of the real object, namely the car, house or train. They do not have as much detail as the original, in part because of the size, but also because capturing every last detail is very time consuming and not always necessary for the purpose at hand.

As for mathematical models, they too are a representation of a real world equivalent, and like the model trains, do not account for every detail, but capture the main aspects. Mathematical models often describe *objects or quantities that change over time*. Examples are the position of a spacecraft, the weather in Los Angeles, the number of fish in a hatchery, or the profit to be made from the sale of inventory. The simplest mathematical model consists of a single mathematical equation, but often a set of equations is required to adequately describe the underlying dynamics of change. The mathematical description is an abstraction of the actual process, and needs to capture the main features of the physical counterpart. A model that has achieved this goal can then be used in many different ways:

- to gain insight
- to simulate
- to predict
- to support decision making

Let's look at an example of a mathematical model, namely the motion of planets, to illustrate some of the points mentioned above.

1.1 Planetary Motion: The Evolution of a Famous Mathematical Model over Time

Early on in human history, people tried to explain the apparent motion of the planets. The earliest explanation, or model, stems from the Greeks in the fourth century BC and was based on the assumption that the earth is located at the center, surrounded by a succession of spheres which contained the different planets. In his books *On the Heavens* and *Physics*, Aristotle assumed perfectly circular motion of the heavenly bodies within their respective shells, i.e., the motion was

always at a constant speed. In his model, the order of the planets was Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn and finally the fixed stars. A "Prime Mover" located outside this system of spheres made everything move in this cosmos.



Christian Aristotelian Cosmos [1]

You might wonder how someone could have come up with this incorrect model (from our knowledge) of the Universe. In order to explain this fact, consider what was known at the time. First, measurements of the positions of the planets were not as accurate as they are today; in the time of Aristotle, the predictions of his model were consistent with observations. Second, the whole view of the Universe was very much colored by religion; the earth was special, hence at the center, and planetary motion came from the "Prime Mover".

Let's step back for a moment and note a few important facts:

- Aristotle's assumptions (perfectly spherical motion, earth at the center) were not based on scientific principles, but rather on the religious world view of the time.
- The geometry of the model resulted in mathematical equations describing the positions of the "seven wanderers" - sun, moon and five planets.
- Aristotle checked his model against observations (data) to validate his model.

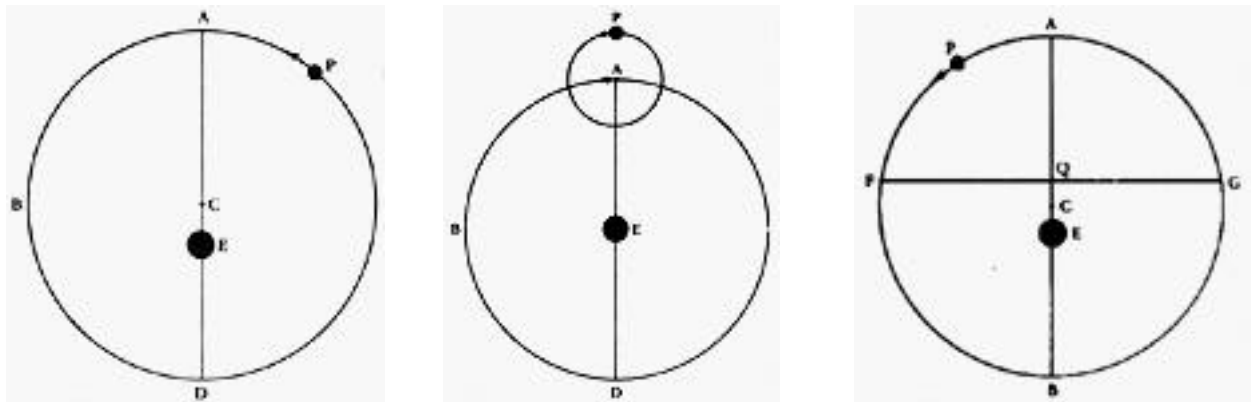
As measurements became better, the model predictions and the actual observed positions no longer agreed. Aristotle did what any good model builder would do -- he concluded that he had not taken into account enough detail. So he adjusted his model, subdividing the spheres into component spheres, for a total of fifty-five. Instead of a planet just moving in a single sphere, now the assumption was that each of the planets moved in several interconnecting spheres. This

refinement of the model gave better predictions, but had two flaws that could not be fixed by introducing additional spheres:

- As all spheres were centered at the earth, a planet's distance to the earth was fixed. This contradicted observations of the varying brightness of Venus, Mars and Jupiter.
- Improvement in measurements of the planets' positions required continual refinements of the model, indicating that not all features had been captured.

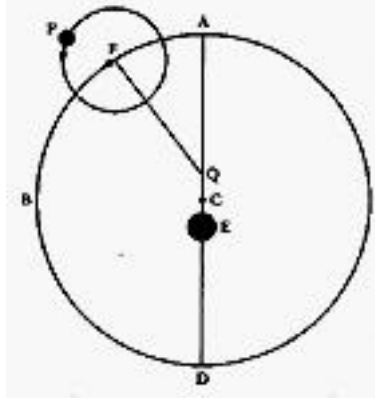
In the century after Aristotle's death, alternative constructions were proposed to explain the working of the Universe. Most of these constructions violated the physical and cosmological assumptions of Aristotle, but they succeeded in creating models that explained the observed data. Ptolemy's model, described in the *Almagest*, (2nd century AD) simplifies Aristotle's model significantly. This new model still retained circular motion, but the earth was put slightly off center, and the motion was not perfectly circular. His model was based on three (geometric) constructions:

- A large circle (deferent) with center C, with the earth placed slightly off-center at E.
- Planetary motion along a small circle (epicycle) whose center moves along the circumference of the large circle.
- An axis off center in the large circle (equant), from which the planet would appear to move in uniform motion.



Deferent, epicycle, and equant [2]

Combining these three constructs, the model predictions of Ptolemy were in good agreement with the actual observations.



Typical Ptolemaic planetary model [2]

Let's again take a step back. We have witnessed an important part of creating a model, namely to strive for as simple a model as possible which is still in agreement with the data. Furthermore, a model is not written in stone, and most likely, several adaptations have to be made until a satisfactory model is found. Generally, one starts with a simple model, then compares the model with the data. If the agreement is satisfactory, then one can stop the process and use the model for its designated purpose. If the agreement between model prediction and data is not sufficient, then the model has to be adapted. This can be achieved by either changing, refining, adding or deleting some assumptions.

Aristotle's and Ptolemy's models stayed around for a long time. They were still taught in the universities established in the 12th century in the Western hemisphere. Aristotle's model was part of philosophy courses; whereas Ptolemy's model was taught in undergraduate mathematics courses and was primarily used for astronomy and calendar related computations.

The next major development in the evolution of astronomy did not take place until the 15th and 16th centuries. The first step was a new translation of the *Almagest*, accompanied by a guide book. Two astronomers, Georg Peurbach (1423 - 1461) and his student Johannes Regiomontenus (1436 - 1476) went back to the original Greek version of the *Almagest* and weeded out the errors that had crept into the Latin translations and transcriptions over the centuries.

At the time, interest in astronomy was very high, as several important problems needed a solution:

- Tables used to predict eclipses and the Fall and Spring equinox (days when night and day are of equal length) were not accurate enough.
- The calendar instituted by Julius Caesar in 44 BC had become inaccurate (as a result of the incorrectly predicted equinox).

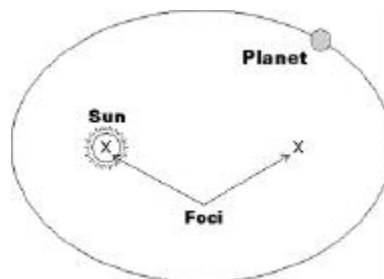
- Spanish and Portuguese explorers on expeditions to the Far East and America were out of sight of land, and used astronomy as their only means to predict their location.

Enter Nicholas Copernicus (1473-1543), who had studied from the new translation of Peurbach and Regiomontenus. He became dissatisfied with Ptolemy's model and proposed a new model of his own. In his book *De Revolutionibus Orbium Celestium* (On the Revolutions of the Celestial Orbs), he proposed a helio-centric (sun-centered) system, where the earth and the other planets moved about the sun in uniform circular orbits. This simple and elegant model explained the retrograde motion of the planets, but still needed the epicycle concept to explain the variations in brightness of the planets, just as in the Ptolemaic model. Both models were based purely on geometry, not on physics as the current model we use today. The main objections to the Copernican system was that the earth was no longer at the center of the Universe and questions were raised from the idea of a moving earth.

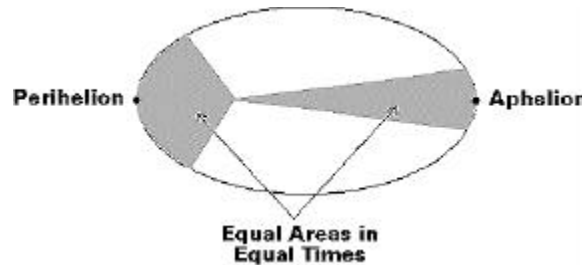
The next model update was introduced at the end of the 16th century by Johannes Kepler (1571-1630). He was an assistant to Tycho Brahe (1546-1601), a Swedish astronomer who had studied at universities in Denmark, Germany and Switzerland. Brahe had invented several astronomical instruments which he calibrated to take nightly observations of the stars. The wealth of data from these observations was used by Kepler to understand the orbit of the planet Mars, which did not match either of the previous models. Kepler, unlike Tycho Brahe, believed in the Copernican system, but eventually had to give up the idea of circular motion in favor of elliptical orbits (see Appendix A1 for more on ellipses). One reason why Kepler finally abandoned the hypothesis of circular orbits was the fact that Mars has the most elliptical orbit of all the planets.

Solely based on the data from Brahe's observations, Kepler derived his three laws of planetary motion:

First Law: The orbits of the planets are ellipses, with the sun at one focus of the ellipse (1609).



Second Law: The line joining the planet to the sun sweeps out equal areas in equal times as the planet travels along the ellipse (1609).



Third Law: The squares of the periods of revolution of any two planets have the same ratio as the cubes of their average distance from the sun (1619).

These “laws” were also not based on any physical laws, but derived purely from regularities Kepler noticed in Brahe’s observations. This changed when Isaac Newton (1643 - 1727) put forth his laws of motion and the universal law of gravity, from which Kepler’s laws can be derived. Thus, for the first time a model of planetary motion rested on a physical explanation, rather than being derived from observations without an explanation of the underlying dynamics.

With Newton's laws of motion, a first approximation of a planet's path could be determined by considering just the two-body system of the sun and the planet. Since the sun has the dominant mass, the main influence of a planet's orbit comes from the sun. Adding other planets one by one into the computations adds slight perturbations to the path predicted based on the two-body system. It turned out that not all observed perturbations were accounted for, which led to the hypothesis of the existence of additional planets. Estimating the size and location of the unknown planet(s) from the perturbations, focused searches were initiated which led to the discovery of Uranus, Neptune and Pluto [3].

In this century, the model of planetary motion has been further refined. Small perturbations in the orbit of Mercury, which could not be explained with Newtonian mechanics, provided motivation and support for the development of the theory of relativity. The quest for yet a more accurate model continues in support of our exploration of planets further away from the earth. Without an appropriate model of planetary motion, a mission such as the landing of the Pathfinder on Mars (July 4, 1997) [4] would not be possible.

References

[1] Peter Apian, *Cosmographia* (1524)

[2] Michael J. Crowe, *Theories of the World from Antiquity to the Copernican Revolution*

[3] William Dicke, *Clyde W. Tombaugh, 90, Discoverer of Pluto*, New York Times Obituaries, January 20, 1997

[4] John Noble Wilford, *Craft Arrives on Mars and Relays Images*, New York Times, July 5, 1997

William F. Lucas (Editor), *Political and Related Models*, Springer Verlag , New York Inc., 1983

Internet References

http://es.rice.edu/ES/humsoc/Galileo/Things/ptolemaic_system.html

http://es.rice.edu/ES/humsoc/Galileo/Things/copernican_system.html

http://es.rice.edu/ES/humsoc/Galileo/People/tycho_brahe.html

<http://es.rice.edu/ES/humsoc/Galileo/People/kepler.html>

<http://csep10.phys.utk.edu/astr161/lect/history/kepler.html>

<http://www.wolfram.com/~terryr/movs/planetPositions.mov> (quick time movie of planet motion)