## Double Angle Formulas and Pythagorean Triples

### Introduction

Students in college trigonometry classes are often asked to find $\cos 2\theta$, $\sin 2\theta$, and $\tan 2\theta$ given the value of a trigonometric function at $\theta$ (almost always a rational number) as well as the quadrant in which $\theta$ lies. Such exercises often arise in the consideration of inverse trigonometric functions, e.g. find $\sin(2\cos^{-1}(-3/7))$. Invariably the cosine comes out as a rational number whereas the sine and tangent need not. For example $\cos(2\cos^{-1}(-3/7)) = -31/49$ where as $\sin(2\cos^{-1}(-3/7)) = -6\sqrt{40}/49$. What is going on?

In the current paper we present certain relationships between rational numbers and the double angle formulas found in trigonometry and show their relationship with Pythagorean triples. Let us begin with a special set of numbers. We call a number $r$ a *rational root* if there is a $q \in Q$, $q \geq 0$ with $r = \pm\sqrt{q}$, where $Q$ is the rationals. Note that the nonzero rational roots form a multiplicative group which we denote by $\sqrt{Q}$ in the sequel. We first note that if $f(\theta)$ is a rational root and $g$ is another trigonometric function for which $g(\theta)$ is defined, then $g(\theta)$ is also a rational root. For example if $\tan\theta = \pm q$, then using $1 + \tan^2\theta = \sec^2\theta$ we get

$$\cos\theta = \pm\sqrt{\frac{1}{1+q}}.$$

We are interested in the relationship between $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$ being rational and $\sin\theta$, $\cos\theta$, and $\tan\theta$ being rational roots. This line of inquiry is natural because $\cos 2\theta$ is rational if and only if the values of the trigonometric functions of $\theta$ are rational roots. We then investigate which rational values of $\sin 2\theta$ come from rational root values of $\cos\theta$. This leads to Pythagorean triples. Lastly we do a similar analysis for $\tan 2\theta$.

Our investigation will start with $\cos 2\theta$. We observe that $\cos 2\theta$ is rational if and only if $\cos\theta \in \sqrt{Q}$. Let us explain why. Suppose $\cos\theta = \pm\sqrt{q}$ for a rational $q$. Then by a double angle formula for cosine, $\cos 2\theta = 2\cos^2\theta - 1 = 2q - 1 \in Q$. Next suppose $\cos 2\theta = \frac{k}{n}$, where $k$ and $n$ are integers. Then by the half angle formula for cosine (see Beecher, Penna, and Bittinger [2])

$$\cos\theta = \pm\sqrt{\frac{1}{2} + \frac{k}{2n}} \in \sqrt{Q}.$$

## Rational Roots and the Sine Function

Let us turn our attention to $\sin 2\theta$ whose analysis is a bit more involved than that of $\cos 2\theta$. Unlike $\cos 2\theta$, $\sin 2\theta$ can be rational without $\cos\theta \in \sqrt{Q}$. For example, set

$$t_1 = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}, \qquad t_2 = \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}.$$

It is easy to verify that $t_1^2 + t_2^2 = 1$ and $2t_1t_2 = 2 \times \frac{1}{4} = \frac{1}{2}$, so that $t_1$ and $t_2$ are admissible values of $\cos\theta$ and $\sin\theta$ for which $\sin 2\theta$ is rational (note that these are formula for the cosine and sine of $15°$, respectively). Further it is possible for $\cos\theta \in \sqrt{Q}$ and $\sin 2\theta$ not to be rational. For example consider $\cos\theta = \sqrt{\frac{3}{5}} \in \sqrt{Q}$ and $\sin\theta = \sqrt{\frac{2}{5}}$. Then $\sin 2\theta = \frac{2\sqrt{6}}{5}$, which is not rational. As we cannot hope for the result we obtained for the $\cos 2\theta$ in the case of $\sin 2\theta$, we ask two questions.

(1) What rational root values of $\cos\theta$ and $\sin\theta$ give rational values for $\sin 2\theta$?

(2) What rational values of $\sin 2\theta$ come from rational root values of $\cos\theta$ and $\sin\theta$?

One answer to the first question is very simple. Suppose $\cos\theta = \pm\sqrt{q}$ is a rational root. Then $\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-q}$ and $\sin 2\theta = \pm 2\sqrt{q(1-q)}$. So $\sin 2\theta$ will be rational if and only if $q(1-q)$ is the square of a rational number.

We next give an answer to the second question and a second answer to the first. Assuming $\cos\theta \in \sqrt{Q}$, we claim

(1) $\sin 2\theta \in Q$ and $\cos\theta \neq 0$ if and only if $\tan\theta \in Q$;

(2) $\sin 2\theta = \frac{k}{n}$ for integers $k$ and $n$ if and only if there exists an integer $j$ with $k^2 + j^2 = n^2$.

For (1), since $\cos^2\theta$ is rational, $\sin 2\theta = 2\sin\theta\cos\theta \in Q$ and $\cos\theta \neq 0$ if and only if $\tan\theta = \sin\theta\cos\theta\dfrac{1}{\cos^2\theta} \in Q$. For (2) let $x = \cos\theta$ and $y = \sin\theta$. Then we have

$$\begin{aligned} \sin 2\theta = 2xy &= \frac{k}{n} \\ x^2 + y^2 &= 1 \end{aligned}$$

In order to satisfy these equations it is clear that we can't have $x = y = 0$. Without loss of generality, assume that $x \neq 0$. Hence

$$y = \frac{k}{2xn} \quad \text{and} \quad \frac{k^2}{(2xn)^2} + x^2 = 1.$$

We obtain the polynomial

$$4x^4 - 4x^2 + \frac{k^2}{n^2} = 0,$$

and therefore viewing this as a quadratic polynomial in $x^2$,

$$\cos^2\theta = x^2 = \frac{1 \pm \sqrt{1 - \frac{k^2}{n^2}}}{2} = \frac{1}{2} \pm \frac{\sqrt{n^2 - k^2}}{2n}.$$

It follows then if we want $\cos\theta$ to be a rational root that we must have $\sqrt{n^2 - k^2}$ equal to an integer. Hence if $\cos\theta \in \sqrt{Q}$ and $\sin 2\theta = \frac{k}{n}$ there must exist an integer $j$ such that $k^2 + j^2 = n^2$. As each step is reversible we also obtain the opposite implication, namely if $\sin 2\theta = \frac{k}{n}$, where $k$ and $n$ are integers with $k^2 + j^2 = n^2$ for some integer $j$ then $\cos\theta \in \sqrt{Q}$.

The above discussion shows that when $0 < |\sin 2\theta| < 1$, rational values for $\sin 2\theta$ (when $\cos\theta \in \sqrt{Q}$) correspond to Pythagorean triples. In fact, such Pythagorean triples arise in a canonical way as described by Theorem 5.1 of Niven and Zuckerman [5]. To see this, if $\cos\theta \in \sqrt{Q}$ and $|\sin 2\theta| \in (0,1)$, then as $\tan\theta \in Q$, we may write rationalizing the numerator

$$\cos\theta = \frac{a}{\sqrt{c}}, \quad \sin\theta = \frac{b}{\sqrt{c}}$$

for some nonzero integers $a$, $b$, and $c$ where $|a| \neq |b|$. Since $c = a^2 + b^2$, we have

$$c^2 - (2ab)^2 = (a^2 + b^2) - 4a^2b^2 = (a^2 - b^2)^2.$$

Since $\sin 2\theta = \frac{2ab}{c}$, a Pythagorean triple $k$, $j$, and $n$ as above is produced by $2ab$, $\max\{a^2 - b^2, b^2 - a^2\}$, and $a^2 + b^2$.


$\sin 2\theta < 0$ $\sin 2\theta > 0$

FIGURE 1

Let us consider the geometry of the problem for $\sin 2\theta$. With $x = \cos\theta$ and $y = \sin\theta$, we have that

$$\begin{aligned} 2xy &= \frac{k}{n} \\ x^2 + y^2 &= 1. \end{aligned}$$

So the values of $\cos\theta$ and $\sin\theta$ come from the intersection of a hyperbola and the unit circle as shown in Figure 1. Rationality of $\sin 2\theta$ means that the lines connecting each pair of antipodal solutions have rational slopes.

## Rational Roots and the Tangent Function

The function $\tan 2\theta$ can be given an analysis similar to that of $\sin 2\theta$. First we consider the geometric picture. We use the formula

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Let $x = \cos\theta$ and $y = \sin\theta$ and $\tan\theta = \frac{k}{n}$ with $k$ and $n$ being integers. We obtain

$$\tan\theta = \frac{2\frac{y}{x}}{1 - \frac{y^2}{x^2}} = \frac{k}{n},$$

so that

$$\frac{2xy}{x^2 - y^2} = \frac{k}{n},$$

and we have the polynomial equation

$$\frac{k}{n}x^2 - 2xy - \frac{k}{n}y^2 = 0.$$

We rotate the $x$ and $y$ coordinate axes to create new coordinates $x'$ and $y'$. The relationship between the original coordinates and the new coordinates after a rotation through an angle of $\phi$ is given by

$$\begin{aligned} x &= x'\cos\phi + y'\sin\phi \\ y &= y'\cos\phi - x'\sin\phi \end{aligned}$$

(see [1, pg. 688]). We write our polynomial equation in terms of $x'$ and $y'$ to obtain

$$Ax'^2 + Bx'y' + Cy'^2 = 0,$$

where

$$\begin{aligned} A &= \frac{k}{n}\cos^2\phi - 2\cos\phi\sin\phi - \frac{k}{n}\sin^2\phi \\ B &= -2(\cos^2\phi - \sin^2\phi) - 4\frac{k}{n}\sin\phi\cos\phi \\ C &= \frac{k}{n}\sin^2\phi + 2\cos\phi\sin\phi - \frac{k}{n}\cos^2\phi \end{aligned}$$

and $\phi$ is the angle of counterclockwise rotation. Choosing $\phi$ with $\sin 2\phi = \frac{k}{n}\cos 2\phi$, we get $A = C = 0$. In this case $\tan 2\phi = \frac{k}{n}$ and $x'y' = 0$. Thus our solutions are the intersection points of the axes of our new coordinate system with the unit circle, as pictured in Figure 2.

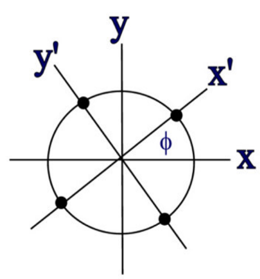


FIGURE 2

Rationality of $\tan 2\theta$ again means that the lines connecting each pair of antipodal solutions (the new coordinate axes) have rational slopes. We now characterize the integers $k$ and $n$ such that $\tan 2\theta = \frac{k}{n}$ comes from a rational root. If $\cos\theta \in \sqrt{Q}$ then

(1) $\tan 2\theta \in Q$ if and only if $\tan\theta \in Q$ and $|\tan\theta| \neq 1$;

(2) $\tan 2\theta = \frac{k}{n}$ for integers $k$ and $n$ if and only if there exists and integer $l$ with $k^2 + n^2 = l^2$.

For (1), since $\tan\theta \in \sqrt{Q}$, we have $\dfrac{2\tan\theta}{1-\tan^2\theta} \in Q$ if and only if $\tan\theta \in Q$. For (2) let $\tan 2\theta = \frac{k}{n}$ and set $\tan\theta = \alpha \in Q$. If $k = 0$ then (2) is trivially satisfied. Assume $k \neq 0$. We have then by the double angle formula

$$\tan 2\theta = \frac{2\alpha}{1-\alpha^2} = \frac{k}{n}.$$

We obtain

$$\frac{k}{n}\alpha^2 + 2\alpha - \frac{k}{n} = 0.$$

Hence

$$\alpha = -\frac{n}{k} \pm \sqrt{\frac{n^2}{k^2} + 1} = -\frac{n}{k} \pm \frac{\sqrt{n^2+k^2}}{k}.$$

As $\alpha$ is rational we must have that $\sqrt{k^2+n^2}$ is an integer. Therefore for some integer $l$ we have $k^2 + n^2 = l^2$. So we get another Pythagorean triple in this case as with the case of $\sin 2\theta$.

## Relationship between Results

Let's see if there is a relationship between exact form of $\sin 2\theta$ and that of $\tan 2\theta$. For this assume $\cos\theta \in \sqrt{Q}$, $\cos 2\theta \neq 0$, and $\sin 2\theta$ and $\tan 2\theta$ are both rational. We intend to show

$$\tan 2\theta = \frac{k}{n}, \quad \sin 2\theta = \frac{k}{l}, \text{ and } \cos 2\theta = \frac{n}{l},$$

where $k^2 + n^2 = l^2$ for some integers $l \neq 0$, $n \neq 0$, and $k$. Write $\sin 2\theta = \dfrac{j}{m}$ and $\cos 2\theta = \dfrac{p}{q}$ where $p \neq 0$. Finding the least common multiple of $m$ and $q$ we can give the fractions for $\sin 2\theta$ and $\cos 2\theta$ the same denominator: $\cos 2\theta = \dfrac{n}{l}$ and $\sin 2\theta = \dfrac{k}{l}$. We obtain

$$\cos^2 2\theta + \sin^2 2\theta = \frac{k^2+n^2}{l^2} = 1 \quad \text{and hence} \quad k^2 + n^2 = l^2.$$

Immediately, $\tan 2\theta = \dfrac{\sin 2\theta}{\cos 2\theta} = \dfrac{k}{n}$.

In the previous discussion we changed the denominators of $\sin 2\theta$ and $\cos 2\theta$ to give them the same denominator. In the case that $\sin 2\theta \neq 0$, it should be mentioned that if the fractions for $\sin 2\theta$ and $\cos 2\theta$ are completely reduced then they will automatically have the same denominator. Indeed as $k^2 + n^2 = l^2$, if $a$ is a divisor of $k$ and $l$ then $a^2$ is a divisor of $n^2 = l^2 - k^2$ so that $a|n$. In particular $\gcd(l,k)|n$. Since $\gcd(l,k)|l$ we must have $\gcd(l,k) \leq \gcd(l,n)$. Similarly $\gcd(l,n) \leq \gcd(l,k)$. Hence $\gcd(l,n) = \gcd(l,k)$. This demonstrates our claim.

## References


1. Anton H., Calculus with Analytic Geometry (2nd Edition), John Wiley and Sons, New York, 1984.
2. Beecher J., Penna J. and Bittinger M., Algebra and Trigonometry Precalculus, Addison-Wiley, New Nork, 2002.
3. Niven and H. Zuckerman, An Introduction to the Theory of Numbers (3rd Edition), Wiley, New York, 1972.