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Two Variational Inequality Problems for the Wave Equation in a Half-space

Randolph G. Cooper, Jr. III

Department of Mathematics, University of California,
Los Angeles, California 90095-1555

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1. INTRODUCTION

An important problem in the theory of elasticity is the dynamic plane-strain type crack problem. The subject of dynamic cracks in solids has received much attention in the past two decades (see Freund [4], Poruchikov [11], and Walton and Herrmann [17]). Most of the work done, however, has not explicitly imposed the physical condition that the two faces of the crack do not interpenetrate. This type of condition is known as a unilateral constraint. Imposing this type of condition amounts to imposing certain inequalities on the displacements or tractions to the problem. Problems in elasticity involving the contact of two elastic bodies give another example of problems with unilateral constraints imposed on boundaries (see Kikuchi and Oden [7]). A particular case of a contact problem is the punch problem which consists of a rigid body (the punch) indenting an elastic half-space.

While static contact problems have been studied extensively (see Kikuchi and Oden [7]) the case of dynamic contact problems has not been studied nearly as much. There is no work (to our knowledge) on the dynamic crack problem taking into account the condition of noninterpenetration. The mathematical formulation of the dynamic crack and punch problems with unilateral constraints involve the dynamic equations of linear elasticity. In this paper we consider two simplified dynamic problems with unilateral constraints obtained when the elasticity equations are replaced by a single wave equation. Although these simplified problems have no physical meaning, we will still call them the “crack” and “punch” problems, respectively.

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Our motivation to study these simplified problems is that they represent a new and interesting class of hyperbolic variational inequalities, and we hope that the methods developed for the study of these problems will be useful in the study of the real dynamic crack and punch problems. We shall formulate now the mathematical "crack" problem.

For the space $\{(x_0, x_1, \dots, x_n) \in \mathbb{R} \times \mathbb{R}^n\}$, where x_0 is the time variable and $(x_1, \dots, x_n)$ the spatial variable, consider a "crack" as an $(n-1)$-dimensional bounded open spatial domain G contained in the hyperplane defined by $\{(x_1, \dots, x_{n-1}, x_n = 0) \in \mathbb{R}^{n-1}\}$. Let w be the displacement vector of the space $\{(x_1, \dots, x_n) \in \mathbb{R}^n\}$, w^+ the displacement of the upper face of the "crack" (with respect to x_n) and w^- that of the lower face. Similarly let τ^+ be the tractions on the upper face and τ^- those on the lower face. We impose the unilateral conditions

$$\begin{cases} w^+ - w^- \geq 0 & \text{noninterpenetrability} \\ w^+ - w^- > 0 \Rightarrow \tau^+ = \tau^- = 0. \end{cases}$$

Assuming that the "crack" is symmetric with respect to $x_n = 0$, applying these conditions to the model we obtain

$$\begin{cases} \dfrac{\partial^2 w}{\partial x_0^2} = c_s^2 \Delta w & x_n > 0, \quad -\infty < x_0 < +\infty \\ w|_{x_n=0} = 0 & x' \notin G, \quad -\infty < x_0 < +\infty \\ w|_{x_n=0} \geq 0 & x' \in G, \quad -\infty < x_0 < +\infty \\ -\mu \left.\dfrac{\partial w}{\partial x_n}\right|_{x_n=0} \geq h & x' \in G, \quad -\infty < x_0 < +\infty \\ \left(-\mu \dfrac{\partial w}{\partial x_n} - h\right) \cdot w\Big|_{x_n=0} \equiv 0 & -\infty < x_0 < +\infty \\ w \equiv 0 & x_0 < 0, \end{cases} \tag{1.0}$$

where $x' = (x_1, \dots, x_{n-1})$. We intend to solve (1.0) in a space derived from the Sobolev space $H_1(\mathbb{R}_+^{n+1})$ where $\mathbb{R}_+^{n+1} = \{x \in \mathbb{R}^{n+1}\colon x_n > 0\}$ (we will give exact definitions of the Sobolev spaces used later). After a reduction of the problem (1.0) to the boundary $\{x_n = 0\}$ we obtain a variational inequality for the Dirichlet-to-Neumann operator $\Gamma(D_0 + i\tau, D')$. We will use the Fourier–Laplace transform in time and the Fourier transform in space. Because $\Gamma(D_0 + i\tau, D')$ is a pseudo-differential operator it is more convenient to treat it in a domain without boundary (i.e., for $-\infty < x_0 < +\infty$). The introduction of a parameter $\tau > 0$ in $\Gamma(D_0 + i\tau, D')$ is a natural thing when we consider the Fourier–Laplace transform in time as we get the analyticity of the dual time variable $\xi_0 + i\tau$ (we will exploit this).

In order to treat our variational inequality which has a hyperbolic pseudo-differential operator we introduce elliptic regularization using a parameter $\varepsilon > 0$. The thus obtained operator is not symmetric but it is possible to use known results (see Kinderlehrer and Stampacchia [8], Lions [9], Lions and Stampacchia [10], and Troianiello [16]) to prove existence of a solution u_ε of the regularized problem in Theorem 3.8. In the next step we show that the solutions u_ε converge in some weak sense to the solution of the hyperbolic variational inequality as $\varepsilon \to 0$ (Theorem 3.11). However, since we loose causality when we introduce the elliptic regularization (i.e., $\operatorname{supp} u_\varepsilon$ will not be contained in $\{x_0 \geq 0\}$) we need to prove that the limiting map has the causality property (i.e., $\operatorname{supp} u \subset \{x_0 \geq 0\}$, Theorem 3.13). This will imply that u satisfies the zero initial conditions. It is very convenient to pose the property of causality as $u \equiv 0$ for $x_0 < 0$ because the existence of the restriction to $\{x_0 = 0\}$ in the Sobolev space used is questionable. This is another reason for posing our reduced problem on the domain $(-\infty, \infty) \times \mathbb{R}^{n-1}$. Note also that since we are dealing with a variational inequality it is not obvious that the solution of the inequality has a causal property in any case.

The solution of the variational inequality (the existence of which is proven in Theorem 3.11) has a weak regularity. Therefore it is important to study what additional regularity the solution has. This is done in Section 4.

The equations case of (1.0) was studied both numerically and theoretically by Sako [13]. Lebeau and Schatzman [14] studied an “obstacle” type problem with unilateral constraints similar to our punch problem with $h \equiv 0$ and nonzero initial conditions. They posed their problem in the spatial domain $G = \mathbb{R}^{n-1}$ and the restricted time domain $(0, T)$ (for $T > 0$ being arbitrary). The problem was then reduced to the boundary and an variational inequality involving a form of the Dirichlet-to-Neumann operator was obtained. The unique solvability of this variational inequality was shown using penalization and semigroup methods. The Dirichlet-to-Neumann operator is a pseudo-differential operator and is most naturally treated in the time domain $\mathbb{R}$. This greatly simplifies proof of unique solvability as shown in our paper. In the current work this is done. We use elliptic regularization instead of penalization and also obtain better regularity results.

We begin with the formulation of the problem as a variational inequality. We prove the unique solvability and causality of this variational inequality in Sections 2 and 3. In Section 4 we consider questions of regularity. In Section 5 we relate the solution of the reduced problem to the original one. In Sections 6–8 we consider what would naturally be called a simplified model of a dynamic rigid body indenting a half-space. We shall call it the dynamic “punch” problem (for a reference on the “punch” problem see Eskin [3], Galin [5], and Poruchikov [11]). We follow the same approach for this problem as with the “crack” problem.

2. THE VARIATIONAL INEQUALITY FORMULATION OF THE "CRACK" PROBLEM

We quickly review the functional spaces that will be used in this paper (for a more thorough review see Eskin [3] and Sakamoto [12]). As the reduction to the boundary must be done on the domain $x_0 \in (-\infty, \infty)$ we define function spaces using the Fourier–Laplace transform in time and the Fourier transform in spatial variables. Denote by $H_{r,s}(\mathbb{R}^n)$ the space of functions $\phi(x)$ satisfying

$$||\phi||_{r,s}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+|\xi_0|)^{2r}(1+|\xi_0|+|\xi'|)^{2s} \left|\widehat{\phi}(\xi_0, \xi')\right|^2 d\xi_0\, d\xi' < +\infty,$$

where $\widehat{\phi}$ is the Fourier transform of ϕ in time and space variables, ξ_0 is the Fourier time variable and $\xi' = (\xi_1, \ldots, \xi_{n-1})$ is the Fourier space variable. Given an open $\Omega \subset \mathbb{R}^n$ denote by $\mathring{H}_{r,s}(\Omega)$ the space consisting of functions $\phi \in H_{r,s}(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset \Omega$. Denote by $H_{r,s}(\Omega)$ the restriction of $H_{r,s}(\mathbb{R}^n)$ to Ω with norm

$$||\phi||_{r,s}^* = \inf_{l\phi} ||l\phi||_{r,s},$$

where the infimum is taken over all extensions $l\phi$ of ϕ. In the case where $r = 0$ we will write $H_s(\mathbb{R}^n)$ in place of $H_{r,s}(\mathbb{R}^n)$ and similarly for $H_s(\Omega)$ and $\mathring{H}_s(\Omega)$. Let $H_{r,s,\tau}(\mathbb{R}^n)$ denote the Sobolev space dependent on the parameter $\tau > 0$ consisting of all $\phi \in \mathcal{D}'(\Omega)$ (the space of distributions, here $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$) such that $e^{-\tau x_0}\phi \in H_{r,s}(\mathbb{R}^n)$, endowed with the norm

$$||\phi||_{r,s,\tau}^2 = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1+\tau+|\xi_0|)^{2r}(1+\tau+|\xi_0|+|\xi'|)^{2s} \times |\widehat{\phi}(\xi_0+i\tau, \xi')|^2\, d\xi_0\, d\xi'.$$

Given $\Omega \subset \mathbb{R}^n$ define $\mathring{H}_{r,s,\tau}(\Omega)$ and $H_{r,s,\tau}(\Omega)$ similarly. In the special case $u \in H_{r,s,\tau}(\mathbb{R}^n)$ we will understand u_τ to mean $e^{-\tau x_0}u$. We have the dualities

$$(H_{r,s}(\mathbb{R}^n))^* \cong H_{-r,-s}(\mathbb{R}^n)$$
$$(\mathring{H}_{r,s}(\Omega))^* \cong H_{-r,-s}(\Omega)$$
$$(H_{r,s}(\Omega))^* \cong \mathring{H}_{-r,-s}(\Omega).$$

The first isomorphism is set up by the natural pairing

$$[\phi, \psi] = \int_{\mathbb{R}^n} \phi(x)\overline{\psi}(x)\, dx \qquad \phi \in H_{-r,-s}(\mathbb{R}^n), \quad \psi \in H_{r,s}(\mathbb{R}^n).$$

The next two are set up by the natural pairing

$$\langle\phi,\psi\rangle=\int_{\mathbb{R}^n} l\phi(x)\overline{\psi}(x)\,dx \qquad \phi\in H_{-r,-s}(\Omega),\quad \psi\in \mathring{H}_{r,s}(\Omega).$$

We note that as $\operatorname{supp}\psi\subset\overline{\Omega}$ and all extensions $l\phi$ of ϕ agree on Ω this natural pairing is well defined. We will add the superscript "+" to a Sobolev space to indicate its positive cone of nonnegative distributions. Further we will use the notation

$$D_j=-i\frac{\partial}{\partial x_j}\quad\text{for}\quad 0\le j\le n$$

$$D'=\left(-i\frac{\partial}{\partial x_1},\dots,-i\frac{\partial}{\partial x_n}\right).$$

DEFINITION 2.0. A pseudo-differential operator $A(D_0,D')$ on $\mathcal{D}(\mathbb{R}^n)$ is defined according to

$$A(D_0,D')\phi(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} A(\xi_0,\xi')\widehat{\phi}(\xi_0,\xi')e^{ix_0\xi_0-ix'\cdot\xi'}\,d\xi_0\,d\xi',$$

where $\widehat{\phi}$ is the Fourier transform of ϕ and $A(\xi_0,\xi')$ is called the symbol of A. Note that in the symbol $A(\xi_0,\xi')$, D_j is replaced by ξ_j. We assume that $A(\xi_0,\xi')$ satisfies a polynomial bound of the form $|A(\xi_0,\xi')|\le C(1+|\xi_0|+|\xi'|)^p$ for some real p.

A pseudo-differential operator is then defined on a Sobolev space by extending it by continuity. A symbol $A(\xi_0,\xi')$ that has an analytic extension $A(\xi_0+i\tau,\xi')$ in $\xi_0+i\tau$, for $\tau>0$, is called a plus symbol. If $\operatorname{supp}\phi(x_0,x')\subset\{x_0\ge 0\}$ and the pseudo-differential operator $A(D_0,D')$ has a plus symbol then by the Paley–Wiener theorem (see, for example, Eskin [3]) $\operatorname{supp}A(D_0,D')\phi(x_0,x')\subset\{x_0\ge 0\}$.

As w vanishes along with its first derivative in time at $x_0=0$ we consider classes of functions such that $w(x_0,x',x_n)\equiv 0$ for $x_0<0$. The initial conditions will then be automatically satisfied. To this end we extend any given $h(x_0,x')$ into negative time with support contained in $\{x_0>-1\}$, requiring that $h\le 0$ for $x_0<0$, and call this new function again $h(x_0,x')$. We note that as $w(x_0,x',x_n)\equiv 0$ for $x_0<0$ that

$$0=-\mu\left.\frac{\partial w}{\partial x_n}\right|_{x_n=0}\ge h(x_0,x')\qquad\text{for } x_0<0.$$

This amounts to a compatibility condition for the data h.

In order to avoid singularities in our symbol and apply variational methods we complexify the Fourier time variable letting $w_\tau=e^{-\tau x_0}w$ and define

h_τ similarly for $\tau > 0$. We obtain by multiplying (1.0) by $e^{-\tau x_0}$

$$\begin{cases} \left(\dfrac{\partial}{\partial x_0} + \tau\right)^2 w_\tau = c_s^2 \Delta_{(x',x_n)} w_\tau & x_n > 0, \quad -\infty < x_0 < +\infty \\ w_\tau\big|_{x_n=0} = 0 & x' \notin G, \quad -\infty < x_0 < +\infty \\ w_\tau\big|_{x_n=0} \geq 0 & x' \in G, \quad -\infty < x_0 < +\infty \\ -\mu \left.\dfrac{\partial w_\tau}{\partial x_n}\right|_{x_n=0} \geq h_\tau(x_0, x') & x' \in G, \quad -\infty < x_0 < +\infty \\ \left(-\mu \dfrac{\partial w_\tau}{\partial x_n} - h_\tau\right) \cdot w_\tau\Big|_{x_n=0} \equiv 0 & -\infty < x_0 < +\infty \\ w_\tau \equiv 0 & x_0 < 0, \end{cases} \tag{2.1}$$

where c_s is the wave speed. If we let the superscript "$\widehat{\ }$" denote the Fourier–Laplace transform in x_0 and the Fourier transform in x' (and $\mathscr{F}^{-1}$ its inverse) we have, in the sense of tempered distributions,

$$\frac{d^2\widehat{w}}{dx_n^2} + \left(c_s^{-2}(\xi_0 + i\tau)^2 - |\xi'|^2\right)\widehat{w} = 0.$$

This has the classical solution

$$\widehat{w}(\xi_0 + i\tau, \xi', x_n) = C_1(\xi_0 + i\tau, \xi')e^{i\Gamma x_n} + C_2(\xi_0 + i\tau, \xi')e^{-i\Gamma x_n}, \tag{2.2}$$

where $\Gamma(\xi_0 + i\tau, \xi') = \sqrt{c_s^{-2}(\xi_0 + i\tau)^2 - |\xi'|^2}$ and we take the branch of the square root such that

$$|\xi_0 + i\tau| \gg |\xi'| \qquad \text{implies} \quad \sqrt{c_s^{-2}(\xi_0 + i\tau)^2 - |\xi'|^2} \approx c_s^{-1}(\xi_0 + i\tau).$$

As we would like to have $w(x)$ in some Sobolev space we impose the boundedness condition $C_2 \equiv 0$.

Let $u_\tau(x_0, x') = w_\tau(x_0, x', 0^+) = \mathscr{F}^{-1}(C_1(\xi_0 + i\tau, \xi'))$; then

$$\widehat{w}(\xi_0 + i\tau, \xi', x_n) = \widehat{u}(\xi_0 + i\tau, \xi')e^{i\Gamma x_n}. \tag{2.3}$$

Hence letting $\Gamma(D_0 + i\tau, D')$ denote the pseudo-differential operator with the symbol $-i\mu\Gamma(\xi_0 + i\tau, \xi')$ we obtain

$$-\mu \left.\frac{dw_\tau}{dx_n}\right|_{x_n=0} = \Gamma(D_0 + i\tau, D')u_\tau.$$

We obtain then, in the sense of $\mathscr{D}'(Q)$, the following reduction of (2.1) to the boundary $x_n = 0$

$$\begin{cases} \operatorname{supp} u_\tau \subset \overline{Q}_0 & \\ u_\tau \geq 0 & (x_0, x') \in Q \\ \text{(a)}\ \Gamma(D_0 + i\tau, D')u_\tau \geq h_\tau & (x_0, x') \in Q \\ \text{(b)}\ (\Gamma(D_0 + i\tau, D')u_\tau - h_\tau)u_\tau \equiv 0 & (x_0, x') \in Q, \end{cases} \tag{2.4}$$

where $Q_0 = (0, +\infty) \times G$ and $Q = (-\infty, +\infty) \times G$. We note that by the form of $w(x)$ in (2.3) the trace $w(\,\cdot\,, x_n = 0)$ is well defined. The problem (2.4) may be reformulated as a variational inequality. We first choose an appropriate Sobolev space $H(Q)$ (i.e., one on which $\langle p_Q \Gamma(D_0 + i\tau, D')\,\cdot\,, \,\cdot\,\rangle$ is finite) for the solution u_τ. Integrating (2.4b) over $\mathbb{R}^n$ and multiplying (2.4a) by an arbitrary nonnegative $\phi \in H(Q)$ and integrating we obtain

$$\begin{cases} \operatorname{supp} u_\tau \subset \overline{Q}_0 \\ u_\tau \geq 0 \qquad (x_0, x') \in Q \\ \text{(a)}\ \big\langle p_Q \Gamma(D_0 + i\tau, D')u_\tau, \phi \big\rangle \geq \big\langle h_\tau, \phi \big\rangle \\ \qquad \text{for all } \phi \in H(Q), \quad \phi \geq 0 \\ \text{(b)}\ \big\langle p_Q \Gamma(D_0 + i\tau, D')u_\tau, u_\tau \big\rangle = \big\langle h_\tau, u_\tau \big\rangle, \end{cases} \tag{2.5}$$

where p_Q is the restriction to Q operator.

Clearly (2.5a) and (2.5b) imply

$$\big\langle p_Q \Gamma(D_0 + i\tau, D')u_\tau, \phi - u_\tau \big\rangle \geq \big\langle h_\tau, \phi - u_\tau \big\rangle \qquad \text{for all } \phi \in H(Q), \quad \phi \geq 0.$$

Further letting $\phi = 0, 2u_\tau$ in the previous expression implies (2.5b) and adding (2.5b) implies (2.5a). Hence we have equivalently the variational inequality

$$\begin{cases} \operatorname{supp} u_\tau \subset \overline{Q}_0 \\ u_\tau \geq 0 \\ \big\langle p_Q \Gamma(D_0 + i\tau, D')u_\tau, \phi - u_\tau \big\rangle \geq \big\langle h_\tau, \phi - u_\tau \big\rangle \\ \qquad \text{for all } \phi \in H(Q), \quad \phi \geq 0. \end{cases} \tag{2.6}$$

The proof of the unique solvability of the variational inequality (2.6) in the hyperbolic case (corresponding to the dynamic problem) is more difficult than in the elliptic case (corresponding to the static problem) because of the following three reasons:

(1) $\Gamma(D_0 + i\tau, D')$ is not symmetric. Therefore we cannot associate with $\Gamma(D_0 + i\tau, D')$ a functional such that the solution of the problem (2.5) will be the minimum of this functional.

(2) $\Gamma(D_0 + i\tau, D')$ is not coercive in the hyperbolic case.

(3) We must show causality (i.e., we must take care of the initial conditions).

We shall overcome the second difficulty by perturbing our operator to a coercive one and thereby reduce the problem to the coercive case. The first difficulty is handled using the Lions and Stampacchia [10] theorem, reducing the problem to demonstrating the existence of a fixed point. The last will be dealt with by employing the compatibility condition on the data h.

3. THE UNIQUE SOLVABILITY OF THE VARIATIONAL INEQUALITY (2.6)

We rewrite $\Gamma(D_0 + i\tau, D')$ as an integro-differential operator. For $\phi(x_0, x') \in \mathscr{D}(\mathbb{R}^n)$

$$\begin{aligned}\Gamma(D_0 + i\tau, D')\phi(x_0, x') \\ = \Box_\tau \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} -i\mu\Gamma^{-1}(\xi_0 + i\tau, \xi')\widehat{\phi}(\xi_0, \xi') e^{-ix_0\xi_0 + ix'\cdot\xi'} \, d\xi_0 \, d\xi',\end{aligned}$$

where $\Box_\tau = c_s^{-2}(D_0 + i\tau)^2 - (D_1^2 + \cdots + D_{n-1}^2)$.

LEMMA 3.0 (Sako [13]). *For $n = 2$*

$$\mathscr{F}^{-1}\Gamma^{-1}(\xi_0 + i\tau, \xi') = \frac{i}{\pi} \frac{e^{-\tau x_0}\Theta(x_0 - |x'|)}{\sqrt{x_0^2 - x_1^2}}$$

and for $n = 3$

$$\mathscr{F}^{-1}\Gamma^{-1}(\xi_0 + i\tau, \xi') = \frac{i}{2\pi} \frac{e^{-|x'|\tau}\delta(x_0 - |x'|)}{|x'|},$$

where Θ is the Heaviside function and δ is the delta function.

Remark 3.1. In convolution form $\Gamma(D_0 + i\tau, D')$ is given by

$$\begin{aligned}\Gamma(D_0 + i\tau, D')\phi(x_0, x') \\ = \Box_\tau \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} -i\mu K_\tau(x_0 - y_0, x' - y')\phi(y_0, y') \, dy_0 \, dy',\end{aligned}$$

where $K_\tau(x_0, x') = \mathscr{F}^{-1}(\Gamma^{-1}(\xi_0 + i\tau, \xi'))$. As the kernel for $\Gamma(D_0 + i\tau, D')$ is real for $n = 2$ and 3, $\Gamma(D_0 + i\tau, D')$ maps real-valued functions to real-valued functions in these dimensions. For arbitrary dimensions it suffices to show that

$$-i\mu\Gamma(-\xi_0 + i\tau, -\xi') = \overline{-i\mu\Gamma(\xi_0 + i\tau, \xi')}. \tag{3.2}$$

By elementary complex analysis we have

$$\begin{cases} \mathfrak{Re}\,\Gamma(\xi_0 + i\tau, \xi') = \dfrac{\beta}{|\beta|}\sqrt{\dfrac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} \\ \mathfrak{Im}\,\Gamma(\xi_0 + i\tau, \xi') = \sqrt{\dfrac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}, \end{cases}$$

where

$$\begin{cases} \alpha = c_s^{-2}(\xi_0^2 - \tau^2) - |\xi'|^2 \\ \beta = 2c_s^{-2}\xi_0\tau. \end{cases}$$

The equality (3.2) follows from this. ∎

We will use the following estimates in demonstrating the unique solvability of (2.6).

LEMMA 3.3 (Bennish [1]). *For* $\tau > 0$

(a) $\mathfrak{Im}\,\Gamma(\xi_0 + i\tau, \xi') \geq \tau/c_s$

(b) $\mathfrak{Im}\,\Gamma(\xi_0 + i\tau, \xi') \geq \tau/c_s(1 + \tau + |\xi_0|)^{-1}(1 + \tau + |\xi_0| + |\xi'|)$

As $\Gamma(D_0 + i\tau, D')$ is not coercive we first consider the perturbed pseudo-differential operator

$$\Gamma_\varepsilon(D_0 + i\tau, D') = \Gamma(D_0 + i\tau, D') - \varepsilon\left(\frac{\partial^2}{\partial x_0^2} - \tau^2 + \Delta\right).$$

We note adding any perturbation of the form

$$\varepsilon\left(-\frac{\partial^2}{\partial x_0^2} + \tau^2 - \Delta\right)^\alpha$$

with $\alpha > \frac{1}{2}$ would suffice. We obtain from the definitions of $\Gamma(\xi_0 + i\tau, \xi')$ and $\Gamma_\varepsilon(\xi_0 + i\tau, \xi')$ that

$$\begin{cases} |\Gamma(\xi_0 + i\tau, \xi')| \leq C(1 + \tau + |\xi_0| + |\xi'|) \\ |\Gamma_\varepsilon(\xi_0 + i\tau, \xi')| \leq C(1 + \tau + |\xi_0| + |\xi'|)^2. \end{cases}$$

It follows, therefore, recalling the definition of $||\cdot||^*_{r,s}$ and applying Parseval's identity, that for $\phi \in \mathscr{D}(Q)$

$$\begin{cases} ||p_Q\Gamma(D_0+i\tau, D')\phi||^*_{r,s-2} \le ||p_Q\Gamma(D_0+i\tau, D')\phi||^*_{r,s-1} \\ \qquad \le ||\Gamma(D_0+i\tau, D')\phi||_{r,s-1} \le C||\phi||_{r,s} \\ ||p_Q\Gamma_\varepsilon(D_0+i\tau, D')\phi||^*_{r,s-2} \le ||\Gamma_\varepsilon(D_0+i\tau, D')\phi||_{r,s-2} \le C||\phi||_{r,s} \\ \lim_{\varepsilon\to 0}||(\Gamma_\varepsilon-\Gamma)\phi||^*_{r,s-2} \le \lim_{\varepsilon\to 0}||(\Gamma_\varepsilon-\Gamma)\phi||_{r,s-2} \\ \qquad \le \lim_{\varepsilon\to 0}\varepsilon\left|\left|\left(\dfrac{\partial^2}{\partial x_0^2}+\tau^2-\Delta\right)\phi\right|\right|_{r,s-2} \\ \qquad \le \lim_{\varepsilon\to 0}\varepsilon C||\phi||_{r,s} = 0. \end{cases}$$

We obtain then the corollary of lemma (3.3).

COROLLARY 3.4. *For* $\phi \in \mathscr{D}(Q)$ *and* $0 < \varepsilon \le 1$

(a) $p_Q\Gamma(D_0+i\tau, D'), p_Q\Gamma_\varepsilon(D_0+i\tau, D')\colon \mathring{H}_{r,s}(Q) \to H_{r,s-2}(Q)$

(b) $p_Q\Gamma_\varepsilon(D_0+i\tau, D') \to p_Q\Gamma(D_0+i\tau, D')$ *for* $\varepsilon \to 0$ *in operator norm of operators from* $\mathring{H}_{r,s}(Q)$ *to* $H_{r,s-2}(Q)$

(c) $(\mu\tau/c_s)||\phi||^2_{-1/2,1/2} \le \langle p_Q\Gamma(D_0+i\tau, D')\phi, \phi\rangle \le C_\tau||\phi||^2_{1/2}$

(d) $\varepsilon c_\tau||\phi||^2_1 \le \langle p_Q\Gamma_\varepsilon(D_0+i\tau, D')\phi, \phi\rangle \le C_\tau||\phi||^2_1$,

where $c_\tau, C_\tau > 0$ *are constants depending on* τ.

We pose the analogous variational inequality for $\Gamma_\varepsilon(D_0+i\tau, D')$,

$$\left\langle p_Q\Gamma_\varepsilon(D_0+i\tau, D')u_{\tau,\varepsilon}, \phi-u_{\tau,\varepsilon}\right\rangle \ge \left\langle h_\tau, \phi-u_{\tau,\varepsilon}\right\rangle \qquad \text{for all } \phi \in \mathring{H}^+_1(Q). \tag{3.5}$$

THEOREM 3.6. *For any real-valued* $h_\tau \in H_{-1}(Q)$ *and* $\varepsilon > 0$ *there exists a unique* $u_{\tau,\varepsilon} \in \mathring{H}^+_1(Q)$ *satisfying the variational inequality* (3.5).

Proof. It is first noticed that a solution to (3.5) in $\mathring{H}^+_1(Q)$ is unique. Indeed suppose $u_{\tau,\varepsilon}$ and $\widetilde{u}_{\tau,\varepsilon}$ both solved (3.5). We would have then

$$\begin{cases} \left\langle p_Q\Gamma_\varepsilon(D_0+i\tau, D')u_{\tau,\varepsilon}, \widetilde{u}_{\tau,\varepsilon}-u_{\tau,\varepsilon}\right\rangle \ge \left\langle h_\tau, \widetilde{u}_{\tau,\varepsilon}-u_{\tau,\varepsilon}\right\rangle \\ \left\langle p_Q\Gamma_\varepsilon(D_0+i\tau, D')\widetilde{u}_{\tau,\varepsilon}, u_{\tau,\varepsilon}-\widetilde{u}_{\tau,\varepsilon}\right\rangle \ge \left\langle h_\tau, u_{\tau,\varepsilon}-\widetilde{u}_{\tau,\varepsilon}\right\rangle. \end{cases}$$

Adding we obtain

$$\left\langle p_Q\Gamma_\varepsilon(D_0+i\tau, D')(\widetilde{u}_{\tau,\varepsilon}-u_{\tau,\varepsilon}), \widetilde{u}_{\tau,\varepsilon}-u_{\tau,\varepsilon}\right\rangle \le 0.$$

Therefore by Corollary 3.4(d) $u_{\tau,\varepsilon} = \widetilde{u}_{\tau,\varepsilon}$.

This variational inequality may be solved by standard methods (see, for example, Lions and Stampacchia [10]). To this end we rewrite (3.5) as

$$\big(\sigma\pi(p_Q\Gamma_\varepsilon(D_0+i\tau, D')u_{\tau,\varepsilon}-h_\tau), \phi-u_{\tau,\varepsilon}\big)_{H_1(Q)} \geq 0 \quad \text{for all } \phi\in \mathring{H}_1^+(Q), \tag{3.7}$$

where $\sigma > 0$ is arbitrary, $(\cdot, \cdot)_{H_1(Q)}$ is the inner product of $H_1(\mathbb{R}^n)$, and π: $H_{-1}(Q)\to \mathring{H}_1(Q)$ is the Riesz map satisfying

$$\text{for all } \psi\in H_{-1}(Q), \quad \phi\in \mathring{H}_1(Q) \qquad \langle\psi,\phi\rangle = (\pi\psi,\phi)_{H_1(Q)}.$$

Define the affine map

$$T_\sigma\phi = \phi-\sigma\pi(p_Q\Gamma_\varepsilon(D_0+i\tau, D')\phi-h_\tau);$$

then

$$\big(u_{\tau,\varepsilon}-T_\sigma u_{\tau,\varepsilon}, \phi-u_{\tau,\varepsilon}\big)_{H_1(Q)}\geq 0 \qquad \text{for all } \phi\in\mathring{H}_1^+(Q). \tag{3.8}$$

From (3.7) it follows equivalently that

$$P\circ T_\sigma u_{\tau,\varepsilon} = u_{\tau,\varepsilon},$$

where P is the orthogonal projection onto $\mathring{H}_1^+(Q)$. Here we have made use of the fact that projections onto closed convex subsets may be characterized in terms of variational inequalities (see, for example, Kinderlehrer and Stampacchia [8]). We will show that $P\circ T_\sigma$ is a contraction and hence has a unique fixed point by the contraction mapping theorem.

Noting that orthogonal projections are nonexpansive we obtain for ϕ, $\psi\in\mathring{H}_1(Q)$ by Corollary 3.4(a) and (d)

$$\begin{aligned}||P\circ T_\sigma\phi-P\circ T_\sigma\psi||_1^2 &\leq ||T_\sigma\phi-T_\sigma\psi||_1^2\\ &= ||(\phi-\psi)-\sigma\pi p_Q\Gamma_{\tau,\varepsilon}(\phi-\psi)||_1^2\\ &= ||\phi-\psi||_1^2-2\sigma\big(\pi p_Q\Gamma_{\tau,\varepsilon}(\phi-\psi),\phi-\psi\big)_{H_1(Q)}\\ &\quad +\sigma^2||p_Q\Gamma_{\tau,\varepsilon}(\phi-\psi)||_{-1}^2\\ &\leq (1-2\sigma\varepsilon c_\tau+\sigma^2C_\tau^2)||\phi-\psi||_1^2.\end{aligned}$$

Letting $\sigma=\varepsilon c_\tau/C_\tau^2$ we obtain

$$||P\circ T_\sigma\phi-P\circ T_\sigma\psi||_1^2\leq\left(1-\left(\frac{\varepsilon c_\tau}{C_\tau}\right)^2\right)||\phi-\psi||_1^2.$$

Noting $1-(\varepsilon c_\tau/C_\tau)^2<1$ we are done. ∎

We next demonstrate that our perturbation scheme is convergent in a generalized sense.

LEMMA 3.9. *The sequence* $\{u_{\tau,\varepsilon}\}_{\varepsilon>0}$ *is bounded in* $\mathring{H}_{-1/2,1/2}(Q)$.

Proof. We have from (3.5) letting $\phi = 0, 2u_{\tau,\varepsilon}$ that

$$\frac{\mu\tau}{c_s}||u_{\tau,\varepsilon}||^2_{-1/2,1/2} \leq \langle p_Q\Gamma_\tau u_{\tau,\varepsilon}, u_{\tau,\varepsilon}\rangle \leq \langle p_Q\Gamma_{\tau,\varepsilon} u_{\tau,\varepsilon}, u_{\tau,\varepsilon}\rangle$$
$$= \langle h_\tau, u_{\tau,\varepsilon}\rangle \leq ||h_\tau||^*_{1/2,-1/2}||u_{\tau,\varepsilon}||_{-1/2,1/2}.$$

Hence $||u_{\tau,\varepsilon}||_{-1/2,1/2} \leq (c_s/\mu\tau)||h_\tau||^*_{1/2,-1/2}$. ∎

As the sequence $\{u_{\tau,\varepsilon}\}_{\varepsilon>0}$ is bounded in $\mathring{H}_{-1/2,1/2}(Q)$ one would expect to find a solution to (2.6) in $\mathring{H}_{-1/2,1/2}(Q)$. However, the bilinear form $\langle p_Q\Gamma(D_0 + i\tau, D')\cdot, \cdot\rangle$ diverges at some points in $\mathring{H}_{-1/2,1/2}(Q)$. We define therefore the new Sobolev space $\mathring{H}(Q; \Gamma_{R,\tau})$.

Denote by $\Gamma_R(D_0 + i\tau, D')$ and $\Gamma_I(D_0 + i\tau, D')$ pseudo-differential operators with symbols

$$\mu\, \Im m\, \Gamma(\xi_0 + i\tau, \xi') \quad \text{and} \quad -\mu\, \Re e\, \Gamma(\xi_0 + i\tau, \xi'),$$

respectively. We have then $\Gamma(D_0 + i\tau, D') = \Gamma_R(D_0 + i\tau, D') + i\Gamma_I(D_0 + i\tau, D')$. The bilinear form $\langle p_Q\Gamma_R(D_0 + i\tau, D')\cdot, \cdot\rangle$ is a positive definite, sesquilinear form on $\mathscr{D}(Q)$ and hence an inner-product. Define the Sobolev space $\mathring{H}(Q; \Gamma_{R,\tau})$ as the closure of $\mathscr{D}(Q)$ in the norm $||\cdot||_{\Gamma_{R,\tau}}$ where

$$(\phi, \psi)_{\Gamma_{R,\tau}} = \langle p_Q\Gamma_R(D_0 + i\tau, D')\phi, \psi\rangle, \qquad ||\phi||^2_{\Gamma_{R,\tau}} = (\phi, \phi)_{\Gamma_{R,\tau}}.$$

Then $\mathring{H}(Q; \Gamma_{R,\tau})$ is a Hilbert space between $\mathring{H}_{-1/2,1/2}(Q)$ and $\mathring{H}_1(Q)$:

$$\mathring{H}_1(Q) \subset \mathring{H}(Q; \Gamma_{R,\tau}) \subset \mathring{H}_{-1/2,1/2}(Q).$$

By Remark 3.1 if $\langle p_Q\Gamma(D_0 + i\tau, D')\phi, \phi\rangle$ is defined we have

$$\langle p_Q\Gamma(D_0 + i\tau, D')\phi, \phi\rangle = \Re e\,\langle p_Q\Gamma(D_0 + i\tau, D')\phi, \phi\rangle$$
$$= \langle p_Q\Gamma_R(D_0 + i\tau, D')\phi, \phi\rangle.$$

Note that we can think of this process as forming the Friedrichs extension of $\Gamma_R(D_0 + i\tau, D')$. We now indicate in what sense we will demonstrate the existence of a solution to the variational inequality (2.6).

DEFINITION 3.10. We say that u is a classical solution in a Sobolev space H for the variational inequality (2.6) if for $\tau > 0$

$$\begin{cases} u_\tau = e^{-\tau x_0} u \\ \operatorname{supp} u_\tau \subset \overline{Q}_0 \\ u_\tau \in H^+ \\ \langle p_Q\Gamma(D_0 + i\tau, D')u_\tau, \phi - u_\tau\rangle \geq \langle h_\tau, \phi - u_\tau\rangle \qquad \text{for all } \phi \in H^+. \end{cases}$$

We say u_τ is a generalized solution of (2.6) in H if

$$\begin{cases} u_\tau \in H^+ \\ \langle p_Q \Gamma(D_0 + i\tau, D')u_\tau, \phi\rangle \geq \langle h_\tau, \phi\rangle \qquad \text{for all } \phi \in \mathscr{D}(Q), \quad \phi \geq 0 \\ \langle p_Q \Gamma_R(D_0 + i\tau, D')u_\tau, u_\tau\rangle \leq \langle h_\tau, u_\tau\rangle, \end{cases}$$

where $H^+ \subset H$ consists of the nonnegative functions in H. ∎

Note that for generalized solutions u_τ the exact dependence on τ is unknown.

We first consider the existence of generalized solutions.

THEOREM 3.11. *Given any real-valued* $h_\tau \in H_{1/2,-1/2}(Q)$ *there exists a generalized solution* $u_\tau \in \mathring{H}(Q; \Gamma_{R,\tau})$ *of* (2.6).

Proof. We have as in Lemma 3.9

$$\begin{aligned} \|u_{\tau,\varepsilon}\|^2_{\Gamma_R,\tau} &= \langle p_Q \Gamma_R(D_0 + i\tau, D')u_{\tau,\varepsilon}, u_{\tau,\varepsilon}\rangle \\ &\leq \langle p_Q \Gamma_\varepsilon(D_0 + i\tau, D')u_{\tau,\varepsilon}, u_{\tau,\varepsilon}\rangle \\ &= \langle h_\tau, u_{\tau,\varepsilon}\rangle \leq \|h_\tau\|^*_{1/2,-1/2}\|u_{\tau,\varepsilon}\|_{-1/2,1/2} \\ &\leq \sqrt{\frac{\mu\tau}{c_s}}\|h_\tau\|^*_{1/2,-1/2}\|u_{\tau,\varepsilon}\|_{\Gamma_R,\tau}. \end{aligned}$$

Hence the sequence $\{u_{\tau,\varepsilon}\}_{\varepsilon>0}$ is bounded in $\mathring{H}(Q; \Gamma_{R,\tau})$. We extract a weakly convergent subsequence $u_{\tau,\varepsilon_k} \rightharpoonup u_\tau$ in $\mathring{H}(Q, \Gamma_{R,\tau})$ ("$\rightharpoonup$" indicates weak convergence). By Theorem 3.8

$$\begin{cases} \text{(a)} \ \langle p_Q \Gamma_{\varepsilon_k}(D_0 + i\tau, D')u_{\tau,\varepsilon_k}, \phi\rangle \geq \langle h_\tau, \phi\rangle \\ \qquad \text{for all } \phi \in \mathscr{D}(Q), \quad \phi \geq 0 \\ \text{(b)} \ \langle p_Q \Gamma_{\varepsilon_k}(D_0 + i\tau, D')u_{\tau,\varepsilon_k}, u_{\tau,\varepsilon_k}\rangle = \langle h_\tau, u_{\tau,\varepsilon_k}\rangle. \end{cases} \tag{3.12}$$

As $h_\tau \in H_{1/2,-1/2}(Q) \subset (\mathring{H}(Q; \Gamma_{R,\tau}))^*$ (with respect to the pairing $\langle \cdot, \cdot\rangle$) and by Corollary 3.4(b)

$$p_Q \Gamma_{\varepsilon_k}(D_0 + i\tau, D')u_{\tau,\varepsilon_k} \rightharpoonup p_Q \Gamma(D_0 + i\tau, D')u_\tau \qquad \text{in } H_{-1/2,-5/2}(Q),$$

taking the limit $\varepsilon_k \to 0$ in (3.12a) we obtain

$$\langle p_Q \Gamma(D_0 + i\tau, D')u_\tau, \phi\rangle \geq \langle h_\tau, \phi\rangle \qquad \text{for all } \phi \in \mathscr{D}(Q), \quad \phi \geq 0.$$

Further taking the lim inf of (3.5b) we obtain

$$\begin{aligned} \langle h_\tau, u_\tau\rangle &= \liminf \langle p_Q \Gamma_{\varepsilon_k}(D_0 + i\tau, D')u_{\tau,\varepsilon_k}, u_{\tau,\varepsilon_k}\rangle \geq \liminf \|u_{\tau,\varepsilon_k}\|^2_{\Gamma_R,\tau} \\ &\geq \|u_\tau\|^2_{\Gamma_R,\tau} = \langle p_Q \Gamma_R(D_0 + i\tau, D')u_\tau, u_\tau\rangle. \end{aligned}$$

Hence u_τ is a generalized solution. ∎

Theorem 3.11 only guarantees the existence of a generalized solution in $\mathring{H}(Q;\Gamma_{R,\tau}) \subset \mathring{H}_{-1/2,1/2}(Q)$. Assuming sufficient regularity for the generalized solution u_τ we obtain a classical solution and uniqueness.

THEOREM 3.13. *If a generalized solution u_τ from Theorem* 3.11 *belongs to the space $\mathring{H}^+_{1/2,1/2}(Q)$, then $u = e^{\tau x_0}u_\tau$ is a classical solution of* (2.5) *in $\mathring{H}_{1/2,1/2,\tau}(Q)$ and is unique among such solutions.*

Proof. Suppose u_τ is a generalized solution with $u_\tau \in \mathring{H}^+_{1/2,1/2}(Q)$; then $\Gamma(D_0 + i\tau, D')u_\tau \in H_{1/2,-1/2}(Q)$ (as $|\Gamma| \leq C(1+|\xi_0|+|\xi'|)$) and hence a continuous functional on $\mathring{H}_{1/2}(Q)$. It follows then from (3.12a) that

$$\langle p_Q\Gamma(D_0+i\tau, D')u_\tau, \phi\rangle \geq \langle h_\tau, \phi\rangle \qquad \text{for all } \phi \in \mathring{H}^+_{1/2}(Q).$$

Letting $\phi = u_\tau$ we obtain from (3.12b)

$$\begin{cases} \text{(a) } \langle p_Q\Gamma(D_0+i\tau, D')u_\tau, \phi\rangle \geq \langle h_\tau, \phi\rangle \qquad \text{for all } \phi \in \mathring{H}^+_{1/2}(Q) \\ \text{(b) } \langle p_Q\Gamma(D_0+i\tau, D')u_\tau, u_\tau\rangle = \langle p_Q\Gamma_R(D_0+i\tau, D')u_\tau, u_\tau\rangle \\ \qquad\qquad = \langle h_\tau, u_\tau\rangle. \end{cases} \tag{3.14}$$

Uniqueness follows as in Theorem 3.8 applying Corollary 3.4(c).

We next show that $u_\tau = 0$ for $x_0 < 0$. We have that

$$\begin{cases} u_\tau \in \mathring{H}_{1/2,1/2}(Q) \\ p_Q\Gamma(D_0+i\tau, D')u_\tau \in H_{1/2,-1/2}(Q). \end{cases}$$

Further as

$$\begin{cases} (1+|\xi_0|)^{1/2}(1+|\xi_0|+|\xi'|)^{1/2} \geq (1+|\xi_0|)^{1/2}(1+|\xi'|)^{1/2} \\ (1+|\xi_0|)^{1/2}(1+|\xi_0|+|\xi'|)^{1/2} \geq (1+|\xi_0|) \\ (1+|\xi_0|)^{1/2}(1+|\xi_0|+|\xi'|)^{-1/2} \geq (1+|\xi'|)^{-1/2}, \end{cases}$$

we obtain

$$\begin{cases} u_\tau \in H_{1/2}\big(\mathbb{R}; \mathring{H}_{1/2}(G)\big) \cap H_1\big(\mathbb{R}, \mathring{H}_0(G)\big) \\ p_Q\Gamma(D_0+i\tau, D')u_\tau \in H_0(\mathbb{R}; H_{-1/2}(G)). \end{cases}$$

We have, then, letting $(\cdot\,,\,\cdot)$ denote the $H_0(\mathbb{R}^{n-1})$ inner-product in x',

$$\int_{-\infty}^{\infty} ((p_Q\Gamma(D_0+i\tau, D')u_\tau, u_\tau) - (h_\tau, u_\tau))\, dx_0 = 0. \tag{3.15}$$

We introduce the following family of cutoff functions that will allow us to use the Fourier transform in time while only considering the solution on the interval $x_0 \in (-\infty, 0)$. For $\delta > 0$ define

$$\Theta_\delta(x_0) = \begin{cases} 1, & x_0 < -\delta \\ 0, & x_0 > -\delta. \end{cases}$$

If we let $\Theta_\delta\cdot$ indicate multiplication by $\Theta_\delta(x_0)$ we obtain

$$\Theta_\delta\,\cdot : H_0(\mathbb{R};\mathring{H}_{1/2}(G)) \to H_0(\mathbb{R};\mathring{H}_{1/2}(G)).$$

As $p_Q\Gamma(D_0+i\tau, D')u_\tau \in H_0(\mathbb{R};H_{-1/2}(G))$ (3.14a) holds for any $\phi \in H_0(\mathbb{R};\mathring{H}_{1/2}(G))$. In particular we can let $\phi = \Theta_\delta u_\tau$. We obtain then from (3.14a)

$$\begin{cases} \langle p_Q\Gamma(D_0+i\tau, D')u_\tau, \Theta_\delta u_\tau\rangle \geq \langle h_\tau, \Theta_\delta u_\tau\rangle \\ \langle p_Q\Gamma(D_0+i\tau, D')u_\tau, (1-\Theta_\delta)u_\tau\rangle \geq \langle h_\tau, (1-\Theta_\delta)u_\tau\rangle. \end{cases}$$

This together with (3.15) and the compatibility condition on h_τ leads to

$$\langle p_Q\Gamma(D_0+i\tau, D')u_\tau, \Theta_\delta u_\tau\rangle = \langle h_\tau, \Theta_\delta u_\tau\rangle \leq 0.$$

The crux of the proof of causality lies in the inequality

$$\begin{aligned} &\langle p_Q\Gamma(D_0+i\tau, D')u_\tau, \Theta_\delta u_\tau\rangle \\ &\quad = \frac{\mu}{2}(u_\tau(-\delta,\ \cdot), u_\tau(-\delta,\ \cdot)) \\ &\qquad + \mu\langle p_Q\Gamma(D_0+i\tau, D')\Theta_\delta u_\tau, \Theta_\delta u_\tau\rangle \\ &\quad \geq \frac{\mu\tau}{c_s}||\Theta_\delta u_\tau||^2_{-1/2,1/2}. \end{aligned} \tag{3.16}$$

From this and the previous inequality we obtain that $u_\tau \equiv 0$ for $x_0 \leq -\delta$. As $\delta > 0$ is arbitrary $u_\tau \equiv 0$ for $x_0 < 0$.

To prove (3.16) we note that

$$-i\mu\Gamma(\xi_0+i\tau, \xi') = -i\frac{\mu}{c_s}(\xi_0+i\tau) + \mu\Gamma^{(1)}(\xi_0+i\tau, \xi'),$$

where $\Gamma^{(1)}(\xi_0+i\tau, \xi')$ is a plus-symbol bounded in $\xi_0+i\tau$: $\Gamma^{(1)}(\xi_0+i\tau, \xi') = O(1+|\xi'|)$. Hence

$$\begin{aligned} &\langle p_Q\Gamma(D_0+i\tau, D')u_\tau, \Theta_\delta u_\tau\rangle \\ &\quad = \frac{\mu}{c_s}\left\langle\left(\frac{\partial}{\partial x_0}+\tau\right)u_\tau, \Theta_\delta u_\tau\right\rangle \\ &\qquad + \mu\langle p_Q\Gamma^{(1)}(D_0+i\tau, D')u_\tau, \Theta_\delta u_\tau\rangle \\ &\quad = \frac{\mu}{2c_s}(u_\tau(-\delta,\ \cdot), u_\tau(-\delta,\ \cdot)) + \frac{\mu\tau}{c_s}\langle\Theta_\delta u_\tau, \Theta_\delta u_\tau\rangle \\ &\qquad + \mu\langle p_Q\Gamma^{(1)}(D_0+i\tau, D')\Theta_\delta u_\tau, \Theta_\delta u_\tau\rangle \\ &\qquad + \mu\langle p_Q\Gamma^{(1)}(D_0+i\tau, D')(u_\tau - \Theta_\delta u_\tau), \Theta_\delta u_\tau\rangle. \end{aligned}$$

However, $p_Q\Gamma^{(1)}(D_0+i\tau, D')(u_\tau - \Theta_\delta u_\tau) \equiv 0$ for $x_0 < -\delta$ as $p_Q\Gamma^{(1)}(D_0 + i\tau, D')$ is a plus symbol. Further

$$\Re e - i\mu\Gamma(\xi_0 + i\tau, \xi') = \frac{\mu\tau}{c_s} + \mu\, \Re e\, \Gamma^{(1)}(\xi_0 + i\tau, \xi'),$$

hence (3.16).

Lastly to demonstrate the dependence on the parameter τ we write

$$\begin{cases} (p_Q\Gamma(D_0+i\tau, D')u_\tau, \phi) \geq (h_\tau, \phi) & \text{for all } \phi \in \mathscr{D}(Q), \quad \phi \geq 0 \\ (p_Q\Gamma(D_0+i\tau, D')u_\tau, u_\tau) = (h_\tau, u_\tau). \end{cases}$$

Multiplying the inequality by $e^{-\eta x_0}$ and the equality by $e^{-2\eta x_0}$ for $\eta > 0$ we obtain

$$\begin{cases} \big(p_Q\Gamma(D_0+i(\tau+\eta), D')(e^{-\eta x_0}u_\tau), \phi\big) \geq (h_{\tau+\eta}, \phi) \\ \qquad \text{for all } \phi \in \mathscr{D}(Q), \phi \geq 0 \\ \big(p_Q\Gamma(D_0+i(\tau+\eta), D')(e^{-\eta x_0}u_\tau), \qquad (e^{-\eta x_0}u_\tau)\big) = (h_{\tau+\eta}, (e^{-\eta x_0}u_\tau)). \end{cases}$$

Integrating in time and noting that $e^{-\eta x_0}u_\tau \in \mathring{H}_{1/2,1/2}(Q)$, $\eta > 0$ and the uniqueness of solution we obtain that $u_{\tau+\eta} = e^{-\eta x_0}u_\tau$ for $\eta, \tau > 0$. Dependence on $\tau > 0$ follows from this. ∎

The argument given for $\operatorname{supp} u_\tau \subset \{x_0 \geq 0\}$ also shows that the solution u_τ is independent of how we choose our extension h_τ.

We may naturally define, for $\phi \in \mathscr{D}(Q)$

$$\begin{cases} \langle p_Q\Gamma(D_0, D')u, \phi\rangle = \langle p_Q\Gamma(D_0+i\tau, D')u_\tau, e^{\tau x_0}\phi\rangle \\ \langle p_Q\Gamma(D_0, D')u \cdot u, \phi\rangle = \langle p_Q\Gamma(D_0+i\tau, D')u_\tau, e^{2\tau x_0}\phi u_\tau\rangle \\ \langle h \cdot u, \phi\rangle = \langle h_\tau, e^{2\tau x_0}\phi u_\tau\rangle. \end{cases}$$

From Theorem 3.13 we obtain then, in the sense of $\mathscr{D}'(Q)$, a solution to the free boundary problem

$$\begin{cases} \operatorname{supp} u \subset \overline{Q}_0 \\ u \in \mathring{H}^+_{1/2,1/2,\tau}(Q) \\ p_Q\Gamma(D_0, D')u \geq h \\ (p_Q\Gamma(D_0, D')u - h)u \equiv 0. \end{cases}$$

4. REGULARITY OF SOLUTIONS

We consider the regularity of generalized solutions to (2.6) using the Brézis-Stampacchia method (see Lions [9]). The proof for regularity will follow closely that presented by Eskin [3] in showing a regularity result for the static "punch" problem.

THEOREM 4.0. *Given a real-valued* $h \in H_{3/2,1/2,\tau}(Q)$ *the generalized solution* u_τ *of Theorem* 3.11 *satisfies* $u_\tau = e^{-\tau x_0} u$ *for a* $u \in \mathring{H}_{1/2,1/2,\tau}(Q)$. *That* u *is classical and unique follows from Theorem* 3.13.

Proof. Let $u_{\tau,\varepsilon} \in \mathring{H}_1^+(Q)$ be the unique solution to (3.5) given by Theorem 3.6. We have then for all $\phi \in \mathring{H}_1^+(Q)$

$$\begin{aligned}&\langle p_Q \Gamma_\varepsilon(D_0 + i\tau, D')\phi - h_\tau, \phi - u_{\tau,\varepsilon}\rangle \\ &\quad = \langle p_Q \Gamma_\varepsilon(D_0 + i\tau, D')u_{\tau,\varepsilon} - h_\tau, \phi - u_{\tau,\varepsilon}\rangle \\ &\qquad + \langle p_Q \Gamma_\varepsilon(D_0 + i\tau, D')(\phi - u_{\tau,\varepsilon}), \phi - u_{\tau,\varepsilon}\rangle \geq 0.\end{aligned} \tag{4.1}$$

For fixed $\tau, \varepsilon > 0$ and any $\nu > 0$ define the sequence $\{\phi_\nu\}_{\nu>0}$ through

$$u_{\tau,\varepsilon} = \phi_\nu - \nu\left(\frac{\partial^2}{\partial x_0^2} - \tau^2\right)\phi_\nu.$$

We have then

$$\widehat{\phi}_\nu = \frac{\widehat{u}_{\tau,\varepsilon}}{1 + \nu(\xi_0^2 + \tau^2)}$$

or in convolution form

$$\phi_\nu(x_0, x') = \frac{\pi}{1 + \nu\tau^2}\int_{-\infty}^{\infty} e^{-(\frac{1}{\nu}+\tau^2)|x_0 - y_0|} u_{\tau,\varepsilon}(y_0, x')\, dy_0.$$

We have then that $\phi_\nu \geq 0$ as $u_{\tau,\varepsilon} \geq 0$, and hence $\phi_\nu \in \mathring{H}_1^+(Q)$. We obtain substituting ϕ_ν into (4.1)

$$-\left\langle p_Q \Gamma_\varepsilon(D_0 + i\tau, D')\phi_\nu, \left(\frac{\partial^2}{\partial x_0^2} - \tau^2\right)\phi_\nu\right\rangle \leq -\left\langle h_\tau, \left(\frac{\partial^2}{\partial x_0^2} - \tau^2\right)\phi_\nu\right\rangle,$$

and hence

$$\begin{aligned}&\frac{\mu\tau}{c_s}||\Lambda_0\phi_\nu||^2_{-1/2,1/2} \\ &\quad \leq ||\Lambda_0\phi_\nu||^2_{\Gamma_R} = -\left\langle p_Q \Gamma_R(D_0 + i\tau, D')\phi_\nu, \left(\frac{\partial^2}{\partial x_0^2} - \tau^2\right)\phi_\nu\right\rangle \\ &\quad \leq ||\Lambda_0\phi_\nu||_{-1/2,1/2}||\Lambda_0 h_\tau||^*_{1/2,-1/2},\end{aligned}$$

where Λ_0 is the pseudo-differential operator with symbol $(\xi_0^2 + \tau^2)^{1/2}$. We obtain therefore

$$\begin{aligned}\frac{\mu\tau}{c_s}\int_{\mathbb{R}^n}&(1+|\xi_0|)^{-1}(1+|\xi_0|+|\xi'|)\Lambda_0^2|\widehat{\phi_\nu}|^2\,d\xi_0\,d\xi'\\ &\le \frac{\mu\tau}{c_s}\int_{\mathbb{R}^n}(1+|\xi_0|)^{-1}(1+|\xi_0|+|\xi'|)\Lambda_0^2\frac{|\widehat{u}_{\tau,\varepsilon}|^2}{(\nu(\xi_0^2+\tau^2)+1)^2}\,d\xi_0\,d\xi'\\ &\le ||\Lambda_0 h_\tau||^*_{1/2,-1/2}.\end{aligned}$$

Consequently by the monotone convergence theorem letting $\nu \to 0$ we obtain

$$c_\tau ||u_{\tau,\varepsilon}||_{1/2,1/2} \le ||\Lambda_0 h_\tau||^*_{1/2,-1/2} \le ||h_\tau||^*_{3/2,-1/2}$$

and hence the sequence $\{u_{\tau,\varepsilon}\}_{\varepsilon>0}$ is uniformly bounded in $\mathring{H}_{1/2,1/2}(Q)$. Extracting a weakly convergent (in $\mathring{H}_{1/2,1/2}(Q)$) subsequence from the subsequence $\{u_{\tau,\varepsilon_k}\}_{k=0}^\infty$ in Theorem 3.11 and calling it again $\{u_{\tau,\varepsilon_k}\}_{k=0}^\infty$ we obtain a generalized solution u_τ. Further $u_{\tau,\varepsilon_k} \rightharpoonup u_\tau$ in $\mathring{H}_{1/2,1/2}(Q)$. Following Theorem 3.11 we obtain a generalized solution $u_\tau \in \mathring{H}_{1/2,1/2}(Q)$. ∎

5. THE UNIQUE SOLVABILITY OF THE FREE BOUNDARY PROBLEM (2.0)

We now relate the classical solution to the reduced problem (2.6) to a solution of (2.1).

THEOREM 5.0. *Given any $h \in H_{3/2,-1/2,\tau}(Q)$ there exists a*

$$w(\cdot, x_n) \in C\big(\overline{\mathbb{R}}_+; H_{1/2,1/2,\tau}(\mathbb{R}^n)\big) \cap H_{1,\tau}(\mathbb{R}^{n+1}_+)$$

satisfying (2.1). *Further w is unique among such solutions.*

Proof. As $h \in H_{3/2,-1/2,\tau}(Q)$ we have a $u \in \mathring{H}_{1/2,1/2,\tau}(Q)$ satisfying (2.6). Let

$$w_\tau(x_0, x', x_n) = \int_{\mathbb{R}^n} e^{-ix_0\xi_0 - ix'\cdot\xi'} e^{i\Gamma(\xi_0+i\tau,\xi')x_n}\widehat{u}(\xi_0+i\tau,\xi')\,d\xi_0\,d\xi'.$$

For fixed $x_n \ge 0$

$$\begin{aligned}||w(\cdot, x_n)||^2_{1/2,1/2,\tau} &= \int_{\mathbb{R}^n}(1+\tau+|\xi_0|)(1+\tau+|\xi_0|+|\xi'|)\\ &\qquad\times|e^{i\Gamma(\xi_0+i\tau,\xi')x_n}\widehat{u}(\xi_0+i\tau,\xi')|^2\,d\xi_0\,d\xi'\\ &\le e^{-(2\mu\tau x_n/c_s)}\int_{\mathbb{R}^n}(1+\tau+|\xi_0|)\\ &\qquad\times(1+\tau+|\xi_0|+|\xi'|)|\widehat{u}|^2\,d\xi_0\,d\xi'\\ &= e^{-(2\mu\tau x_n/c_s)}||u||^2_{1/2,1/2,\tau}.\end{aligned}$$

For $x_n, y_n \geq 0$

$$\begin{aligned}||w(\,\cdot\,,x_n)-w(\,\cdot\,,y_n)||^2_{1/2,1/2,\tau} &= \int_{\mathbb{R}^n}(1+\tau+|\xi_0|)(1+\tau+|\xi_0|+|\xi'|)\\ &\qquad\times|e^{ix_n\Gamma}-e^{iy_n\Gamma}|^2|\widehat{u}|^2\,d\xi_0\,d\xi'\\ &\leq \int_{\mathbb{R}^n}(1+\tau+|\xi_0|)(1+\tau+|\xi_0|+|\xi'|)\\ &\qquad\times|e^{-x_n\,\Im m\,\Gamma}-e^{-y_n\,\Im m\,\Gamma}|^2|\widehat{u}|^2\,d\xi_0\,d\xi'.\end{aligned}$$

We obtain then by the Lebesgue dominated convergence theorem

$$\lim_{x_n\to y_n}||w(\,\cdot\,,x_n)-w(\,\cdot\,,y_n)||^2_{1/2,1/2,\tau}=0.$$

Further

$$\begin{aligned}||w||^2_{H_{1,\tau}(\mathbb{R}^{n+1}_+)} &= \int_{\mathbb{R}^n}\int_0^\infty(1+\tau+|\xi_0|+|\xi'|)^2|\widehat{w}(\xi_0+i\tau,\xi')|^2\\ &\qquad+\left|\frac{\partial w_\tau}{\partial x_n}\right|^2 dx_n\,d\xi_0\,d\xi'\\ &\leq \int_{\mathbb{R}^n}\int_0^\infty e^{-2x_n\,\Im m\,\Gamma}((1+\tau+|\xi_0|+|\xi'|)^2+|\Gamma|^2)|\widehat{u}|^2\,dx_n d\xi_0\,d\xi'\\ &= \int_{\mathbb{R}^n}\frac{1}{2\,\Im m\,\Gamma}((1+\tau+|\xi_0|+|\xi'|)^2+|\Gamma|^2)|\widehat{u}|^2\,d\xi_0\,d\xi'\\ &\leq \int_{\mathbb{R}^n}\frac{c_s(1+\tau+|\xi_0|+|\xi'|)^2}{\mu\tau(1+\tau+|\xi_0|)^{-1}(1+\tau+|\xi_0|+|\xi'|)}|\widehat{u}|^2\,d\xi_0\,d\xi'\\ &= \frac{c_s}{\mu\tau}||u||^2_{1/2,1/2,\tau}.\end{aligned}$$

Therefore $w(\,\cdot\,,x_n)\in C(\overline{\mathbb{R}}_+;H_{1/2,1/2,\tau}(\mathbb{R}^n))\cap H_{1,\tau}(\mathbb{R}^{n+1}_+)$.

As the symbol $e^{i\Gamma(\xi_0+i\tau,\xi')x_n}$ is analytic in $\xi_0+i\tau$ for $\tau>0$ (i.e., it is a plus symbol) and $\operatorname{supp} u\subset\{x_0\geq 0\}$ it follows that $\operatorname{supp} w_\tau\subset\{x_0\geq 0\}$. Further as w_τ satisfies the second order equation

$$\left(\frac{\partial}{\partial x_0}+\tau\right)^2 w_\tau = c_s^2\Delta_{(x',x_n)}w_\tau \qquad -\infty<x_0<\infty$$

by construction, it is infinitely smooth in time, i.e., $w(x_0,\cdot)\in C^\infty(\mathbb{R},\mathscr{D}'(\mathbb{R}^n))$ (see, for example, Sakamoto [12]). Hence we obtain the initial conditions

$$w_\tau\big|_{x_0=0}=\frac{\partial w_\tau}{\partial x_0}\bigg|_{x_0=0}=0.$$

∎

6. VARIATIONAL INEQUALITY FORMULATION OF THE PUNCH PROBLEM

We consider the unique solvability in an appropriate Sobolev space of a function $w(x_0, x', x_n)$ satisfying

$$\begin{cases} \left(\dfrac{\partial}{\partial x_0} + \tau\right)^2 w_\tau = c_s^2 \Delta_{(x',x_n)} w_\tau & x_n > 0, -\infty < x_0 < \infty \\ -\mu \left.\dfrac{\partial w_\tau}{\partial x_n}\right|_{x_n=0} \geq 0 & -\infty < x_0 < \infty \\ w_\tau\big|_{x_n=0} \geq h_\tau & -\infty < x_0 < \infty \\ (w_\tau - h_\tau) \cdot \left.\dfrac{\partial w_\tau}{\partial x_n}\right|_{x_n=0} \equiv 0 & -\infty < x_0 < \infty \\ w_\tau\big|_{x_0=0} = \left.\dfrac{\partial w_\tau}{\partial x_0}\right|_{x_0=0} = 0, & \end{cases} \tag{6.0}$$

with the compatibility condition

$$h_\tau \leq 0 \qquad \text{for} \quad x_0 < 0.$$

We suppose that the domain of contact between the rigid punch and the half-space is contained in the open set $\{|x'| < R\}$ for some $R > 0$. We hence impose the condition that $h_\tau \leq 0$ in a neighborhood of $\{|x'| = R\}$. By (2.3) we have that

$$\Gamma^{-1}(D_0 + i\tau, D')\left(-\mu \left.\frac{\partial w_\tau}{\partial x_n}\right|_{x_n=0}\right) = w_\tau\big|_{x_n=0},$$

where $\Gamma^{-1}(D_0 + i\tau, D')$ is a pseudo-differential operator with symbol $(i/\mu)\Gamma^{-1}(\xi_0 + i\tau, \xi')$. Letting $u_\tau = -\mu\,(\partial w_\tau/\partial x_n)\big|_{x_n=0}$ and making the natural assumption that the support of u_τ is contained in the domain of contact we obtain the reduction to the boundary $x_n = 0$ of (6.0)

$$\begin{cases} \operatorname{supp} u_\tau \subset \overline{Q}_0 & \\ u_\tau \geq 0 & (x_0, x') \in Q \\ \text{(a)}\ \ \Gamma^{-1}(D_0 + i\tau, D')u_\tau \geq h_\tau & (x_0, x') \in Q \\ \text{(b)}\ \ (\Gamma^{-1}(D_0 + i\tau, D')u_\tau - h_\tau)u_\tau \equiv 0 & (x_0, x') \in Q, \end{cases} \tag{6.1}$$

where $Q_0 = (0, +\infty) \times \{|x'| < R\}$ and $Q = (-\infty, +\infty) \times \{|x'| < R\}$.

Following the same reasoning as with the "crack" problem in Section 2 we obtain the variational inequality formulation

$$\begin{cases} \operatorname{supp} u_\tau \subset \overline{Q}_0 \\ \langle p_Q \Gamma^{-1}(D_0 + i\tau, D')u_\tau, \phi - u_\tau \rangle \geq \langle h_\tau, \phi - u_\tau \rangle \\ \qquad \forall \phi \in H(Q), \quad \phi \geq 0, \end{cases} \tag{6.2}$$

where $H(Q)$ is an appropriate Sobolev space.

7. UNIQUE SOLVABILITY OF THE VARIATIONAL INEQUALITY (6.2)

We will use the following estimates in demonstrating the unique solvability of (6.2).

LEMMA 7.0. *For* $\tau > 0$

(a) $$-\Im m\, \Gamma^{-1}(\xi_0 + i\tau, \xi') \geq \frac{\mu\tau}{c_s}(1 + \tau + |\xi_0|)^{-1}(1 + \tau + |\xi_0| + |\xi'|)^{-1}$$

(b) $$|\Gamma^{-1}(\xi_0 + i\tau, \xi')| \leq \frac{c_s}{\mu\tau}(1 + \tau + |\xi_0|)(1 + \tau + |\xi_0| + |\xi'|)^{-1}.$$

Proof. For the first estimate we have

$$\begin{aligned} -\Im m\, \Gamma^{-1}(\xi_0 + i\tau, \xi') &= -\Im m\, \frac{\overline{\Gamma}(\xi_0 + i\tau, \xi')}{|\Gamma(\xi_0 + i\tau, \xi')|^2} = \Im m\, \frac{\Gamma(\xi_0 + i\tau, \xi')}{|\Gamma(\xi_0 + i\tau, \xi')|^2} \\ &\geq \frac{\mu\tau(1 + \tau + |\xi_0|)^{-1}(1 + \tau + |\xi_0| + |\xi'|)}{c_s(1 + \tau + |\xi_0| + |\xi'|)^2} \\ &= \frac{\mu\tau}{c_s}(1 + \tau + |\xi_0|)^{-1}(1 + \tau + |\xi_0| + |\xi'|)^{-1}. \end{aligned}$$

The second estimate follows immediately from Lemma 3.3(b). ∎

As $\Gamma^{-1}(D_0 + i\tau, D')$ is not coercive we first consider the perturbed pseudo-differential operator

$$\Gamma_\varepsilon^{-1}(D_0 + i\tau, D') = \Gamma^{-1}(D_0 + i\tau, D') - \varepsilon\left(\frac{\partial^2}{\partial x_0^2} - \tau^2 + \Delta\right).$$

We note that adding any perturbation of the form

$$\varepsilon\left(-\frac{\partial^2}{\partial x_0^2} + \tau^2 - \Delta\right)^\alpha$$

with $\alpha > 0$ would suffice. Proceeding as in Corollary (3.4) we obtain

COROLLARY 7.1. *For $\phi \in \mathcal{D}(Q)$ and $0 < \varepsilon \le 1$*

(a) $p_Q\Gamma^{-1}(D_0 + i\tau, D'),\ p_Q\Gamma_\varepsilon^{-1}(D_0 + i\tau, D')\colon \mathring{H}_{r,s}(Q) \to H_{r,s-2}(Q)$

(b) $p_Q\Gamma_\varepsilon^{-1}(D_0 + i\tau, D') \to p_Q\Gamma^{-1}(D_0 + i\tau, D')$ *for* $\varepsilon \to 0$ *in operator norm of operators from* $\mathring{H}_{r,s}(Q)$ *to* $H_{r,s-2}(Q)$

(c) $(\mu\tau/c_s)||\phi||^2_{-1/2,-1/2} \le \langle p_Q\Gamma^{-1}(D_0 + i\tau, D')\phi, \phi\rangle \le (c_s/\mu\tau)||\phi||^2_{1/2,-1/2}$

(d) $\varepsilon c_\tau||\phi||_1^2 \le \langle p_Q\Gamma_\varepsilon^{-1}(D_0 + i\tau, D')\phi, \phi\rangle \le C_\tau||\phi||_1^2,$

where $c_\tau, C_\tau > 0$ *are positive constants depending on* τ.

Proceeding as in Theorem 3.8 we obtain

THEOREM 7.2. *For any real-valued* $h_\tau \in H_{-1}(Q)$ *and* $\varepsilon > 0$ *there exists an unique* $u_{\tau,\varepsilon} \in \mathring{H}_1(Q)$ *satisfying the variational inequality*

$$\begin{cases} u_{\tau,\varepsilon} \in \mathring{H}_1^+(Q) \\ \langle p_Q\Gamma_\varepsilon^{-1}(D_0 + i\tau, D')u_{\tau,\varepsilon}, \phi - u_{\tau,\varepsilon}\rangle \ge \langle h_\tau, \phi - u_{\tau,\varepsilon}\rangle \qquad \forall\phi \in \mathring{H}_1^+(Q). \end{cases}$$

Denote by $\Gamma_R^{-1}(D_0 + i\tau, D')$ and $\Gamma_I^{-1}(D_0 + i\tau, D')$ the pseudo-differential operators with symbols

$$-\frac{i}{\mu}\,\Im m\,\Gamma^{-1}(\xi_0 + i\tau, \xi') \quad \text{and} \quad \frac{i}{\mu}\,\Re e\,\Gamma^{-1}(\xi_0 + i\tau, \xi'),$$

respectively. We note $\Gamma^{-1}(D_0 + i\tau, D') = \Gamma_R^{-1}(D_0 + i\tau, D') + i\Gamma_I^{-1}(D_0 + i\tau, D')$. Define the Sobolev space $\mathring{H}(Q; \Gamma_{R,\tau}^{-1})$ as the closure of $\mathcal{D}(Q)$ in the norm $||\cdot||_{\Gamma_R^{-1},\tau}$ where

$$(f, g)_{\Gamma_R^{-1},\tau} = \langle p_Q\Gamma_R^{-1}(D_0 + i\tau, D')f, g\rangle, \qquad ||f||^2_{\Gamma_R^{-1},\tau} = (f, f)_{\Gamma_R^{-1},\tau}.$$

We then have

$$\mathring{H}_1(Q) \subset \mathring{H}(Q; \Gamma_{R,\tau}^{-1}) \subset \mathring{H}_{-1/2,-1/2}(Q).$$

Utilizing the space $\mathring{H}(Q; \Gamma_{R,\tau}^{-1})$ and proceeding as in Theorem 3.11 we obtain

THEOREM 7.3. *Given any real-valued* $h_\tau \in H_{1/2,1/2}(Q)$ *there exists a generalized solution* $u_\tau \in \mathring{H}(Q; \Gamma_{R,\tau}^{-1}) \subset \mathring{H}_{-1/2,-1/2}(Q)$ *of* (6.2) *satisfying*

$$\begin{cases} u_\tau \in \mathring{H}^+(Q; \Gamma_{R,\tau}^{-1}) \\ \langle p_Q\Gamma^{-1}(D_0 + i\tau, D')u_\tau, \phi\rangle \ge \langle h_\tau, \phi\rangle \qquad \forall\phi \in \mathcal{D}(Q), \qquad \phi \ge 0 \\ \langle p_Q\Gamma_R^{-1}(D_0 + i\tau, D')u_\tau, u_\tau\rangle \le \langle h_\tau, u_\tau\rangle. \end{cases}$$

Next, noting that if $\langle p_Q\Gamma^{-1}(D_0+i\tau, D')\phi, \phi\rangle$ is defined we have

$$\langle p_Q\Gamma^{-1}(D_0+i\tau, D')\phi, \phi\rangle = \langle p_Q\Gamma_R^{-1}(D_0+i\tau, D')\phi, \phi\rangle;$$

we can obtain a classical solution to (6.2) assuming sufficient regularity for u_τ. Indeed assuming $u_\tau \in \mathring{H}_{1/2,-1/2}$ existence and uniqueness follow as in Theorem 3.13. We further have that

$$\begin{cases} u_\tau \in \mathring{H}_{1/2,-1/2}(Q) \\ p_Q\Gamma^{-1}(D_0+i\tau, D')u_\tau \in H_{1/2,1/2}(Q). \end{cases}$$

Hence as

$$\begin{cases} (1+|\xi_0|)^{1/2}(1+|\xi_0|+|\xi'|)^{-1/2} \geq (1+|\xi'|)^{-1/2} \\ (1+|\xi_0|)^{1/2}(1+|\xi_0|+|\xi'|)^{1/2} \geq (1+|\xi_0|)^{1/2}(1+|\xi'|)^{1/2} \\ (1+|\xi_0|)^{1/2}(1+|\xi_0|+|\xi'|)^{1/2} \geq (1+|\xi_0|), \end{cases}$$

we obtain

$$\begin{cases} u_\tau \in H_0\big(\mathbb{R}; \mathring{H}_{-1/2}(G)\big) \\ p_Q\Gamma^{-1}(D_0+i\tau, D')u_\tau \in H_{1/2}(\mathbb{R}; H_{1/2}(G)) \cap H_1(\mathbb{R}; H_0(G)). \end{cases}$$

We may also derive proceeding as in Theorem 3.13 that

$$\langle p_Q\Theta_\delta\Gamma^{-1}(D_0+i\tau, D')u_\tau, u_\tau\rangle \geq \frac{\mu\tau}{c_s}||\Theta_\delta u_\tau||^2_{-1/2.-1/2}.$$

From this causality and dependence on the parameter $\tau > 0$ follows. We obtain then the following theorem.

THEOREM 7.4. *If a generalized solution u_τ from Theorem 7.3 belongs to $\mathring{H}_{1/2,-1/2}(Q)$ then $u = e^{-\tau x_0}u_\tau$ is a classical solution in $\mathring{H}_{1/2,-1/2,\tau}(Q)$ and is unique u among such solutions.*

From Theorem 7.4 we obtain then, in the sense of $\mathscr{D}'(Q)$, a solution to the free boundary problem

$$\begin{cases} \operatorname{supp} u \subset \overline{Q}_0 \\ u \in \mathring{H}^+_{1/2,-1/2,\tau}(Q) \\ p_Q\Gamma^{-1}(D_0, D')u \geq h \qquad (x_0, x') \in Q \\ (p_Q\Gamma^{-1}(D_0, D')u - h)u \equiv 0 \qquad (x_0, x') \in Q. \end{cases}$$

8. REGULARITY OF SOLUTIONS

We proceed in demonstrating regularity estimates for the "punch" problem in much the same way as was done for the "crack" problem. We have immediately the following theorem.

THEOREM 8.0. *Given any* $h \in H_{3/2,1/2,\tau}(Q)$ *the generalized solution from Theorem* 7.3 *satisfies* $u_\tau = e^{-\tau x_0} u$ *for some* $u \in \mathring{H}^{+}_{1/2,-1/2,\tau}(Q)$. *That* u *is classical and unique follows from Theorem* 7.4.

In the case of the "punch" problem we may get a further global regularity result using the properties of $\Gamma^{-1}(D_0 + i\tau, D')$. We obtain from Lemma 3.0 that $\Gamma^{-1}(D_0 + i\tau, D')$ has a positive kernel when written in convolution form for $n = 2, 3$ from which we may obtain the following theorem.

THEOREM 8.1. *For the dimensions* $n = 2, 3$ *given any* $h \in H_{1/2,3/2,\tau}(Q)$ *the generalized solution from Theorem* 7.3 *satisfies* $u \in \mathring{H}^{+}_{-1/2,1/2,\tau}(Q)$.

Proof. Assuming $h_\tau \in H_{1/2,3/2}(Q) \subset H_{3/2,1/2}(Q)$ we have a classical solution u_τ to (6.2) by Theorem 8.0. As $h_\tau \le 0$ in a neighborhood of $(-\infty, \infty) \times \{|x'| = R\}$ there exists an extension f of h_τ with $f < 0$ on $\mathbb{R}^n \backslash (-\infty, \infty) \times \{|x'| \le R\}$ and $\|f\|_{1/2,3/2,\tau} \le 2\|h\|^{*}_{1/2,3/2,\tau}$. As $u_\tau \ge 0$ by Lemma 3.0 $\Gamma^{-1}(D_0 + i\tau, D')u_\tau \ge 0$ and hence

$$\Gamma^{-1}(D_0 + i\tau, D')u_\tau \ge f \qquad \text{for all} \quad (x_0, x') \in \mathbb{R}^n.$$

Further as $\operatorname{supp} u_\tau \subset \overline{Q}$

$$\left[\Gamma^{-1}(D_0 + i\tau, D')u_\tau, u_\tau\right] = [f, u_\tau],$$

where $[\,\cdot\,,\,\cdot\,]$ is the pairing between $H_{r,s}(\mathbb{R}^n)$ and $H_{-r,-s}(\mathbb{R}^n)$ (functions defined on all of $\mathbb{R}^n$). Therefore

$$\left[\Gamma^{-1}(D_0 + i\tau, D')u_\tau - f, \phi - u_\tau\right] \ge 0 \qquad \forall \phi \in H_{1/2,-1/2}(\mathbb{R}^n)$$

and hence

$$\begin{aligned} &\left[\Gamma^{-1}(D_0 + i\tau, D')\phi - f, \phi - u_\tau\right] \\ &\quad = \left[\Gamma^{-1}(D_0 + i\tau, D')u_\tau - f, \phi - u_\tau\right] \\ &\qquad + \left[\Gamma^{-1}(D_0 + i\tau, D')(\phi - u_\tau), \phi - u_\tau\right] \ge 0 \\ &\qquad\qquad \forall \phi \in H_{1/2,-1/2}(\mathbb{R}^n). \end{aligned} \tag{8.2}$$

Define the sequence, for $\varepsilon > 0$

$$u_\tau = v_\varepsilon - \varepsilon\left(\frac{\partial^2}{\partial x_0^2} - \tau^2 + \Delta\right)v_\varepsilon.$$

We note as in Theorem 4.2 $v_\varepsilon \geq 0$ (see for example Stein [15]). Letting $\phi = v_\varepsilon$ in (8.1) we obtain

$$\frac{\mu\tau}{c_s}||v_\varepsilon||^2_{1/2,-1/2} \leq \left[\Gamma^{-1}(D_0 + i\tau, D')v_\varepsilon, \left(\frac{\partial^2}{\partial x_0^2} - \tau^2 + \Delta\right)v_\varepsilon\right]$$
$$\leq -\left[f, \left(\frac{\partial^2}{\partial x_0^2} - \tau^2 + \Delta\right)v_\varepsilon\right] \leq ||f||_{1/2,3/2,\tau}||v_\varepsilon||_{-1/2,1/2}.$$

Applying the Lesbegue monotone convergence theorem we obtain $u_\tau \in \mathring{H}_{-1/2,1/2}(Q)$. ∎

From the proof of Theorem 8.1 we have that u_τ is the unique solution of the variational inequality

$$\left[\Gamma^{-1}(D_0 + i\tau, D')u_\tau - f, \phi - u_\tau\right] \geq 0 \qquad \forall \phi \in H_{1/2,-1/2}(\mathbb{R}^n). \tag{8.3}$$

However for (8.3) we have made no assumptions on the support of u_τ. Hence that $\operatorname{supp} u_\tau \subset \overline{Q}_0$ follows directly from the condition $h_\tau \leq 0$ on $(-\infty, \infty) \times \{|x'| = R\}$.

Proceeding as in Theorem 5.0 we may relate the classical solution of (6.2) to a solution of (6.0).

THEOREM 8.4. *Given any $h \in H_{3/2,-1/2,\tau}(Q)$ let*

$$w_\tau(x_0, x', x_n) = \int_{\mathbb{R}^n} e^{-ix_0\xi_0 - ix'\cdot\xi'}\Gamma^{-1}(\xi_0 + i\tau, \xi')$$
$$\times e^{i\Gamma(\xi_0+i\tau,\xi')x_n}\widehat{u}(\xi_0 + i\tau, \xi')\, d\xi_0\, d\xi',$$

where u_τ is the solution to (6.2) *with data h_τ. We obtain then that w satisfies the free boundary problem* (6.0) *with*

$$w(\,\cdot\,, x_n) \in C\big(\overline{\mathbb{R}}_+; H_{1/2,1/2,\tau}(\mathbb{R}^n)\big) \cap H_{1,\tau}(\mathbb{R}^{n+1}_+).$$

Further w is unique among such solutions.

APPENDIX: GLOSSARY OF NOTATIONS

$\mathbb{R}$	reals
$H_{r,s}(Q)$	Sobolev space of functions restricted to Q
$\mathring{H}_{r,s}(Q)$	Sobolev space of functions supported in Q
$\|\|\cdot\|\|_{r,s}$, $\|\|\cdot\|\|^*_{r,s}$	Sobolev space norms
$\mathscr{D}(\Omega)$	space of infinitely differentiable functions supported in Ω
$\mathscr{D}'(\Omega)$	space of distributions over Ω
$[\cdot,\cdot]$	pairing over whole space
$\langle\cdot,\cdot\rangle$	pairing over restricted space

D_j	derivative in the j'th coordinate multiplied by $-i$
$A(D_0, D')$	pseudo-differential operator
$\mathscr{F}$, $\mathscr{F}^{-1}$	Fourier and inverse Fourier transforms, respectively
$\square_\tau$	wave operator
$\Im m$	imaginary part
$\Re e$	real part
$\blacksquare$	used to indicate the end of a proof or remark

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