1) $ABx = \lambda x \quad x \neq 0$

So λ is an eigenvalue of AB

Case 1 $\lambda \neq 0$, then $Bx \neq 0$

and multiplying by B

$BA(Bx) = \lambda(Bx)$ as $Bx \neq 0$

λ is an eigenvalue for BA

Case 2 $\lambda = 0$ then $ABx = 0$

We $\lambda = 0$ to be an eigenvalue for BA. Assume it wasn't then there is no $y \neq 0$ with $BAy = \lambda y = 0$. So BA is non singular. Hence A and B are nonsingular. Looking at $ABx = \lambda x = 0$ and multiplying by A^{-1} then B^{-1} we get $x = 0$. But $x \neq 0$. Hence $\lambda = 0$ is an eigenvalue for BA.

This shows $\lambda(AB) \subseteq \lambda(BA)$
switching A and B we get
$\lambda(BA) \subseteq \lambda(AB)$ so $\lambda(AB) = \lambda(BA)$

2) Find the eigenvectors of A

$$\det\begin{bmatrix}1-\lambda & -3\\ -2 & 2-\lambda\end{bmatrix} = (1-\lambda)(2-\lambda)-6$$

$$= \lambda^2-3\lambda+2-6 = \lambda^2-3\lambda-4$$

$= (\lambda-4)(\lambda+1)$ eigenvalues are $\lambda = 4, -1$

eigenvector for 4

$$\begin{bmatrix}-3 & -3\\ -2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$$ so we can use $x = \begin{bmatrix}1\\ -1\end{bmatrix}$

eigenvector for -1

$$\begin{bmatrix}2 & -3\\ -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 0$$ so we can use

$$x = \begin{bmatrix}3\\ 2\end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$$

3) Power method

$$\tilde{q}_1 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$q_1 = \tilde{q}_1 / \|\tilde{q}_1\|_\infty = \begin{bmatrix} 1 \\ 5/7 \end{bmatrix}$$

$$\tilde{q}_2 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5/7 \end{bmatrix} = \begin{bmatrix} 3 + 10/7 \\ 1 + 10/7 \end{bmatrix} = \begin{bmatrix} 31/7 \\ 17/7 \end{bmatrix}$$

$$q_2 = \tilde{q}_2 / \|\tilde{q}_2\|_\infty = \begin{bmatrix} 1 \\ 17/31 \end{bmatrix}$$

$$\tilde{q}_3 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 17/31 \end{bmatrix} = \begin{bmatrix} 3 + 34/31 \\ 1 + 34/31 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{128/31}\ 127/31 \\ 65/31 \end{bmatrix} \qquad q_3 = \begin{bmatrix} 1 \\ 65/127 \end{bmatrix}$$

Rayleigh Quotient

$$q_0^T q_0 = 1+4=5$$

$$A q_0 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$q_0^T A q_0 = [1, 2]\begin{bmatrix} 7 \\ 5 \end{bmatrix} = 7+10=17$$

$$\lambda = \frac{q_0^T A q_0}{q_0^T q_0} = \frac{17}{5}$$

$$A-\lambda I = \begin{bmatrix} 3-\frac{17}{5} & 2 \\ 1 & 2-\frac{17}{5} \end{bmatrix} = \begin{bmatrix} -2/5 & 2 \\ 1 & -7/5 \end{bmatrix}$$

$$(A-\lambda I)^{-1} = \frac{1}{\frac{14}{25}-2}\begin{bmatrix} -7/5 & -2 \\ -1 & -2/5 \end{bmatrix}$$

$$= \frac{-25}{36}\begin{bmatrix} -7/5 & -2 \\ -1 & -2/5 \end{bmatrix} = \begin{bmatrix} 35/36 & \frac{50}{36} \\ \frac{25}{36} & \frac{10}{36} \end{bmatrix}$$

$$\tilde{q}_1 = \frac{1}{36}\begin{bmatrix} 35 & 50 \\ 25 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 135 \\ 45 \end{bmatrix}$$

$$q_1 = \tilde{q}_1 / \|\tilde{q}_1\|_\infty = \begin{bmatrix} 1 \\ 45/135 \end{bmatrix}$$

QR method

$$c = \frac{3}{\sqrt{9+1}} \qquad s = \frac{-1}{\sqrt{10}}$$

$$Q = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

$$Q^T A = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 8 \\ 0 & 4 \end{bmatrix} = R$$

$$\cancel{A_1 = RQ = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 10 & 8 \\ 0 & 4 \end{bmatrix}}$$

$$A_1 = RQ = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 8 \\ 0 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 30 & 20 \\ 10 & 20 \end{bmatrix}$$

4) (a) Let $\|A\|_X = \|X^{-1}AX\|$

(i) c is a constant $\|cA\|_X = \|X^{-1}(cA)X\|$

$= |c|\,\|X^{-1}AX\| = |c|\,\|A\|_X$

(ii) if $\|A\|_X = 0$ then $\|X^{-1}AX\| = 0$

$\Rightarrow X^{-1}AX = 0$ multiply on the right by X^{-1} and on the left by X and we get $A = 0$.

Clearly if $A = 0$ then $\|A\|_X = \|X^{-1}0X\| = 0$

So $\|A\|_X = 0$ iff $A = 0$

(iii) $\|A+B\|_X = \|X^{-1}(A+B)X\|$

$= \|X^{-1}AX + X^{-1}BX\| \le \|X^{-1}AX\| + \|X^{-1}BX\|$

$= \|A\|_X + \|B\|_X$

(b) $\|AB\|_X = \|X^{-1}ABX\| = \|X^{-1}AXX^{-1}BX\|$

$\le \|X^{-1}AX\| \, \|X^{-1}BX\| = \|A\|_X \|B\|_X$

(c) Let $D = \begin{bmatrix} d_{11} & & O \\ & \ddots & \\ O & & d_{nn} \end{bmatrix}$ then

$Dx = \begin{bmatrix} d_{11}x_1 \\ \vdots \\ d_{nn}x_n \end{bmatrix}$ $\quad \|D\|_p = \max\limits_{\|x\| \ne 0} \dfrac{\|Dx\|_p}{\|x\|_p}$

$= \max\limits_{\|x\| \ne 0} \dfrac{[|d_{11}x_1|^p + \cdots + |d_{nn}x_n|^p]^{1/p}}{[|x_1|^p + \cdots + |x_n|^p]^{1/p}}$ (let $d = \max\limits_{1 \le i \le n} |d_{ii}|$)

$\le \max\limits_{\|x\| \ne 0} \dfrac{[|dx_1|^p + \cdots + |dx_n|^p]^{1/p}}{[|x_1|^p + \cdots + |x_n|^p]^{1/p}}$

$= \max\limits_{\|x\| \ne 0} d \dfrac{[|x_1|^p + \cdots + |x_n|^p]^{1/p}}{[|x_1|^p + \cdots + |x_n|^p]^{1/p}} = d$

So $\|D\|_p \le d = \max\limits_{1 \le i \le n} |d_{ii}|$

Next let $|d_{jj}| = \max_{1 \le i \le n} |d_{ii}|$

let $e_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ 1 in j^{th} position

then $De_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d_{jj} \\ \vdots \\ 0 \end{bmatrix}$ $\quad \dfrac{\|De_j\|_p}{\|e_j\|_p} = \dfrac{|d_{jj}|}{1} = |d_{jj}| = d$

so $\cancel{\max}$ $\dfrac{\|Dx\|_p}{\max \|x\|_p} = \|D\|_p \le d$ and for one vector e_j

$\dfrac{\|De_j\|}{\|e_j\|} = d$ hence $\max_{\|x\| \ne 0} \dfrac{\|Dx\|_p}{\|x\|_p} = \|D\|_p = d$

$= \max_{1 \le i \le n} |d_{ii}|$

(d) Choose ~~the~~ an X that diagonalizes A. the $X^{-1}AX = D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

where λ_i's are eigenvalues

from part (c) $\cancel{\|A\|_p =}$

$\|A\|_X = \|X^{-1}A\cancel{X}\|_p = \|D\|_p = \max_{1 \le i \le n} |\lambda_i|$