Math 570A Midterm Fall 2009

Name ____

Show your work and write neatly
Only write on one side of the paper
Number your pages and put your name on each page
Avoid writing too close to the corners of the page

1) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Answer yes or no to the following. A is
(a) diagonally dominant
(b) diagonalizable
(c) positive definite
(d) orthogonal
Next find the Jacobi iteration matrix for A and determine whether Jacobi iteration converges using it.

2)
(a) Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Compute the Gauss-Seidel iteration matrix and use it to determine whether Gauss-Seidel iteration converges for A.
(b) Show that Gauss-Seidel iteration converges for every nonsingular upper triangular matrix (Hint: the eigenvalues of an upper triangular matrix are on the diagonal).

3) Let A be an $n \times n$ matrix and $A = M - N$ be a splitting for A. Let x be the solution to $Ax = b$. Consider the iterative scheme $Mx^{(k+1)} = Nx^{(k)} + b$. Let the error be given by $e^{(k)} = x - x^{(k)}$. Give a careful proof the if $\rho(G) < 1$, where $G = M^{-1}N$ and $\rho(G)$ is the spectral radius, then the iterative scheme converges (i.e. $e^{(k)} = x - x^{(k)} \rightarrow 0$). The proof should be in your notes just explain each step.

4) Explain what partial pivoting is and why it is used. For the matrix

$$A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$$

use partial pivoting to decompose it as $PA = LU$ where P is a permutation matrix and LU is the LU decomposition of PA. Use this result to solve $Ax = b$ where

$$b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}.$$

5) Count the number of flops used in computing the Cholesky decomposition of an $n \times n$ symmetric positive definite matrix. Count divisions, mutiplications, additions, subtractions, and square roots.

6) For the matrix

$$A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

(a) Determine whether Gauss-Seidel iteration converges by looking at the Jacobi iteration matrix.
(b) compute the optimal ω to be used in the SOR method. What is the spectral radius of the SOR iteration matrix?

7) Let

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors.
(b) Let

$$x^{(0)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Use the power method to compute $x^{(3)}$ then use the Rayleigh quotient $\dfrac{v^T A v}{v^T v}$ to get a approximation of the dominant eigenvalue.
(c) Do three iteration of the QR method to get approximations of the eigenvalues. Then use inverse iteration once to get approximations of the eigenvectors. I used a calculator to do this.
(d) Say something about how the results of (b) and (c) compare. What advantage does the power method have over the QR method?

8) Let $B = S^{-1}AS$ and $B + \delta B = S^{-1}(A + \delta A)S$. A and B are similar matrices. We are interested in how error in A produces error in B.
(a) Show that similar matrices have the same eigenvalues (this is not related to (b) and (c), I proved this in class).
(b) Show that $\dfrac{1}{\kappa(S)}\|A\| \leq \|B\| \leq \kappa(S)\|A\|$ (Hint: for the first inequality note $SBS^{-1} = A$)

(c) Show that

$$\frac{1}{\kappa(S)^2}\frac{\|\delta A\|}{\|A\|} \leq \frac{\|\delta B\|}{\|B\|} \leq \kappa(S)^2\frac{\|\delta A\|}{\|A\|}.$$

(Hint: Note that $B + \delta B = S^{-1}AS + \delta B = S^{-1}(A + \delta A)S = S^{-1}AS + S^{-1}\delta AS$ then use the inequality in (b)). I should mention that this inequality relates the relative error in A to the relative error in B by using S.