Practice Final Math 402A Fall 2008

1) Let $\mathbf{F} = [4x^3y^2, x^2 - y^2, yz]$. Compute div$\mathbf{F}$ and curl$\mathbf{F}$.

2) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = [xy, z^2, -y^2]$ and $\mathbf{r}(t) = [t, t^2, t^3]$ $0 \le t \le 1$.

3) For the integral

$$\int_{(1,1,2)}^{(2,1,-1)} (12xy + yz)dx + (6x^2 + xz)dy + xydz$$

show that the integrand is an exact form and evaluate the integral.

4) Use Green's theorem to evaluate the integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ over the rectangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ where $\mathbf{F} = [y^2, x^2]$.

5) Without using the divergence theorem compute directly the flux integral $\int\int_S \mathbf{F} \cdot \mathbf{n} dA$ where $\mathbf{F} = [6z, -2x, y^2]$ and the surface S is given by $\mathbf{r}(u, v) = [u, v, u^2 + v^2]$ $0 \le u \le 1$, $0 \le v \le 1$

6) Test whether the following functions are harmonic. If they are not say so. If they are find the harmonic conjugate.
(a) $u(x, y) = 2x^3 - xy + y^2$
(b) $u(x, y) = x^4 - 6x^2y^2 + y^4$

7) Let the contour C be the unit circle tranversed once counterclockwise (parametrized as $z(t) = \cos t + i \sin t$ with $0 \le t \le 2\pi$). Evaluate the integral $\oint_C \mathrm{Re} z \, dz$

8) Let the contour C be the circle $|z| = 2$ transversed once counterclockwise. Evaluate the following integrals
(a) $\oint_C e^z dz$
(b) $\oint_C \frac{z^2 + e^z}{z(z-3)} dz$
(c) $\oint_C \frac{\cos z}{(z-1)^3} dz$

9) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{n!}{(2n)!} z^n$

10) Find the Laurent expansion for $\frac{e^z}{(z-2)^2}$ centered at $z_0 = 2$.

11) Identify all the singularities of the following functions. If they are poles indicate the order of the pole.
(a) $f(z) = \frac{5}{z^3} - \frac{1}{z^2} + \frac{10}{z} + z^4 - z^5$
(b) $f(z) = \frac{e^z}{(z-1)^2(z-5)}$
(c) $f(z) = \cos(\frac{1}{z})$

12) Compute the residues of the following functions at the specified points.

(a) $f(z) = \dfrac{5}{z^3} - \dfrac{1}{z^2} + \dfrac{10}{z} + z^4$ at $z = 0$

(b) $f(z) = \dfrac{e^z}{z^2}$ at $z = 0$

(c) $f(z) = \dfrac{z^2+1}{z(z-1)}$ at $z = 1$

13) Evaluate the following two integral

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta}$

1) Compute the directional derivative $D_a f|_P$, where $a = [1, 2, 3]$, $f(x, y, z) = xy - yz^2 + xyz$, and we have the point $P(1, 1, 2)$.

2) Compute the line integral $\displaystyle\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$, where $\mathbf{F} = [-y, x, z]$, $C\colon \mathbf{r}(t) = [\cos t, \sin t, t]$, and we go from the point $(1, 0, 0)$ to the point $(1, 0, 4\pi)$.

3) Compute and simplify i^{3+i}

4) Let $\mathbf{F} = [y, 0, z]$ and consider the surface $S\colon \mathbf{r}(u, v) = [u\cos v, u\sin v, u^2]$ with $0 \le u \le 2$ and $0 \le v \le 2\pi$. Compute the surface integral $\displaystyle\int\!\!\int_S \mathbf{F} \cdot \mathbf{n}\, dA$.

5) Show the following integral is path independent and compute its value.

$$\int_{(1,1,0)}^{(2,4,\pi)} ((\cos z + yz)dx + xzdy + (xy - x\sin z + 2z)dz)$$

6) Let $\mathbf{F} = [xy, y^2, zy]$. Use the divergence theorem to compute the surface integral $\displaystyle\int\!\!\int_S \mathbf{F} \cdot \mathbf{n}\, dA$ where S is the surface of the cyllinder $x^2 + y^2 \le 4$ and $-3 \le z \le 3$. (Hint: Use polar coordinates for x and y when you do the integral, ie use cyllindrical coordinates).

7) Consider $C\colon |z| = 2$ transversed counterclockwise once. Evaluate the following integrals.

(a) $\displaystyle\oint_C \operatorname{Re} z\, dz$

(b) $\displaystyle\oint_C \sin z\, dz$

(c) $\displaystyle\oint_C \frac{z^3 + e^z}{z(z-3)}\, dz$

(d) $\displaystyle\oint_C \frac{\cos 3z}{(z-1)^4}\, dz$

(e) $\displaystyle\oint_C \frac{\cos 3z}{(z-10)^6}\, dz$

8) Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n)!} z^n$.

9) Let $f(z) = \dfrac{z}{z^2+1}$. Write a Laurent expansion for $f(z)$ centered at $z = i$ that converges

(a) for $z = i/2$.

(b) for $z = 10$.

Say in what region each series converge. (Hint: Write $z^2+1=(z-i)(z+i)$ and use partial fractions.)