SIMPLE EXAMPLES OF THE COMBINATORIAL TRACE METHOD IN ACTION

MIKE KREBS AND NATALIE C. MARTINEZ

Abstract. For any finite graph X, the number of closed walks in X of length k is equal to the sum of the kth powers of the eigenvalues of any adjacency matrix of X. This simple observation forms the basis for the so-called "combinatorial trace method," wherein one attempts to count (or bound) the number of closed walks of a given length so as to obtain information about the graph's eigenvalues. In this paper, we give a brief overview of the subject and discuss a few simple but interesting examples. For example: by counting closed walks in certain two-vertex graphs, we obtain the classical closed formulae for the kth Fibonacci and Lucas numbers. Applying this method to other small graphs, one can easily generate interesting but accessible undergraduate research projects.

The so-called "combinatorial trace method" is a technique for establishing equalities between combinatorial expressions on the one hand and power sums on the other. The method relies on the following observation.

Theorem 0.1. *In any finite graph X, the number of closed walks in X of length n equals the sum of the nth powers of the eigenvalues of any adjacency matrix of X.*

(Note that we are using the word "graph" loosely here; we allow our graphs to have multiple edges and loops.)

To prove this theorem, let X be a finite graph. Order the vertices $v_1, v_2, \ldots, v_n$. Let A be the corresponding adjacency matrix. Observe that the (i, j)-entry of A^n equals the number of walks in X of length n from v_i to v_j. Hence the number of closed walks of length n in X is equal to the trace of A^n. On the other hand, the eigenvalues of A^n are the nth powers of the eigenvalues of A, and the trace is the sum of the eigenvalues. The theorem follows.

While elementary, the combinatorial trace method can be used to prove non-trivial theorems, by enabling us to go back and forth between the combinatorics of closed walks on the one hand, and graph eigenvalues on the other. The primary purpose of this paper is to show some simple but

---

Date: November 28, 2011.

Key words and phrases. Lucas numbers, Fibonacci numbers, adjacency matrix, finite graph, combinatorial trace method, Binet formula.

interesting examples of the method in action; at the end, we'll briefly sketch a general overview of the topic.

1. Simple examples of the method in action: Binet formulae for Fibonacci and Lucas nunbers

We start by giving a simple example of this technique in action. Namely, we will use it to derive the classical closed formulae (the "Binet formulae") for the Fibonacci and Lucas nunbers.

We take the Fibonacci numbers F_n to be defined by $F_{-1} = 1$, $F_0 = 0$, $F_{n+2} = F_{n+1} + F_n$. (It will become clear in the sequel why we find it convenient to begin the indexing at -1.) Lucas numbers are defined recursively as follows: $L_0 = 2, L_1 = 1, L_{n+2} = L_n + L_{n+1}$. Intimately related to the Fibonacci numbers, Lucas numbers occur naturally in many counting problems.

The classical Binet formula for Lucas numbers states that

$$L_n = \varphi^n + \overline{\varphi}^n, \tag{1}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\overline{\varphi} = \frac{1-\sqrt{5}}{2}$.

The form of this equation—with a combinatorial expression on the left and a power sum on the right—immediately suggests that it should be approachable via the combinatorial trace method, for an appropriate choice of graph X. We shall furnish just such a graph, thereby giving a graph-theoretic proof of (1). By applying Theorem 0.1 to a related graph, we will then derive the renowned Binet formula for Fibonacci numbers.

Let X be the graph shown in Figure 1.

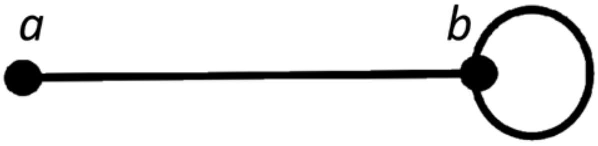

Figure 1.

The graph X has two vertices labeled a and b, with a single edge between them, as well as a loop at vertex b.

We claim that the number of closed walks in X of length n is exactly L_n. To show this, let B_n be the number of closed walks of length n based at vertex b. We have $B_0 = 1$ (the trivial walk) and $B_1 = 1$ (traversing the loop). For $n \geq 2$, a walk of length n based at b must either end with going around the loop or else with traveling back and forth across the non-loop edge. Hence,

$B_n = B_{n-1} + B_{n-2}$. Therefore $B_n = F_{n+1}$. Now, let A_n be the number of closed walks of length n based at vertex a. We have $A_0 = 1 = F_{-1}$ and $A_1 = 0 = F_0$. For $n \geq 2$, a closed walk of length n based at a must consist of the non-loop edge from a to b, followed by a closed walk of length $n-2$ based at b, followed by the non-loop edge from b back to a. Hence $A_n = B_{n-2} = F_{n-1}$. We recall the well-known (and easily verified) identity $L_n = F_{n+1} + F_{n-1}$. Therefore, the total number of closed walks in X of length n is $B_n + A_n = L_n$.

An adjacency matrix for X is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which has eigenvalues φ and $\overline{\varphi}$. (Note that a loop sometimes counts as a 2 in the adjacency matrix, but we need it to count as a 1 in order for Theorem 0.1 to hold.) Equation (1) now follows from Theorem 0.1.

We now turn our attention to Fibonacci numbers. The Binet formula in this case is:

$$F_n = (\varphi^n - \overline{\varphi}^n)/\sqrt{5}. \tag{2}$$

Prima facie, it would seem that this equation is not amenable to the combinatorial trace method, because unlike (1), neither side of (2) is a power sum. However, squaring both sides and taking advantage of the fact that $\varphi \cdot \overline{\varphi} = -1$, we find that (2) is equivalent to:

$$5F_n^2 + 2(-1)^n = (\varphi^2)^n + (\overline{\varphi}^2)^n. \tag{3}$$

Now the right-hand side is a power sum, so now we are in combinatorial trace method territory. (Indeed, we offer our profuse thanks to the referee who suggested that this technique might well work here, despite our original insistence that it would not.)

So we're on the lookout for a graph with two vertices so that the corresponding eigenvalues are φ^2 and $\overline{\varphi}^2$. A successful candidate presents itself immediately, namely, X^2, that is, the graph whose vertices are a and b and whose edges represent walks of length 2 in X. So, X^2 will have a single loop at a, representing the single closed walk of length 2 in X based at a; a single edge between a and b, representing the single walk of length 2 in X from a to b (or reversing direction, from b to a); and *two* loops at b, representing the two closed walks of length 2 in X based at b. Not coincidentally, the adjacency matrix for X^2 is A^2. Hence the corresponding eigenvalues are φ^2 and $\overline{\varphi}^2$.

It remains to show that the number of closed walks of length n in X^2 is precisely $5F_n^2 + 2(-1)^n$. Because every edge in X^2 represents a walk of length 2 in X, it follows that closed walks of length n in X^2 are in one-to-one correspondence with closed walks of length $2n$ in X. We know from our work a few paragraphs ago that the number of such walks is precisely L_{2n}. So it suffices to show that $L_{2n} = 5F_n^2 + 2(-1)^n$. This equation is not unduly difficult to verify directly. Alternatively, it follows immediately from known results about Lucas and Fibonacci numbers; for example, combine Identities 36 and 53 in [1].

2. Overview

Now that we've seen some concrete examples of the combinatorial trace method in action, let's discuss some generalities pertaining to it.

For example: Where is the line between those problems that can be approached via the combinatorial trace method, and those that cannot? (We thank the referee for posing this question.) The answer depends on the type of problem under consideration, which in turn depends on one's point of view. In this article, we have proved identities by using Theorem 0.1. To use the combinatorial trace method in such cases, we require at a minimum that the identity can be expressed in a form with a counting function on one side and a power sum on the other. For example, consider the fact that the sum of a row in Pascal's triangle equals a power of 2, or more precisely, that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^k. \tag{4}$$

This identity has Combinatorial Trace Method written all over it. The right-hand side is a power sum with one term, so the appropriate graph will have one vertex. Indeed, take the graph with one vertex and two loops ℓ_1 and ℓ_2 at that vertex. On the one hand, the adjacency matrix is (2) with eigenvalue 2; on the other hand, we can count the number of closed walks of length n by conditioning on the number of times the walk contains ℓ_1. Equation (5) follows. By contrast, consider the identity

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}. \tag{5}$$

We do not see any way to transform this equation into the desired form. Consequently, we do not see any way to apply Theorem 0.1 to obtain equation (5).

The applications we've discussed so far do not reflect the usual way this technique is employed. The more typical standpoint is that of the spectral graph theorist, who is faced with a graph X and wishes to obtain information about its eigenvalues. Theorem 0.1 may then be helpful. The extent to which it is useful depends on how hard the counting is. In most applications, it is far too difficult to obtain a precise expression for the number of closed walks of a given length, so generally one must make do with upper or lower bounds, then apply known combinatorial estimates so as to work with an analytically tractable expression.

We will make no attempt here to comprehensively collect a list of all graph eigenvalue results that have been proved via the combinatorial trace method (see [3] for a survey), but we will briefly mention a few. Perhaps most noteworthy is the Alon-Boppana theorem, which states the following. Let (X_n) be a sequence of finite d-regular graphs, so the number of vertices goes to infinity. For each n, let λ_n be the second-largest eigenvalue of X_n. Then $\liminf \lambda_n \geq 2\sqrt{d-1}$. Serre has a stronger result than this, namely that a certain fixed proportion (of the number of vertices) of the eigenvalues of a d-regular graph must exceed an expression with asymptote $2\sqrt{d-1}$. Both of these theorems can be proved in various ways, but we are partial to the closed-walk-counting approach, and indeed, Cioabă has a proof of Serre's result that makes use of the combinatorial trace method. In [2], Cioabă also uses the combinatorial trace method to prove a Serre-type theorem for Cayley graphs on abelian groups—in this case, d takes the place of $2\sqrt{d-1}$. In a more elementary vein, Dsouza applies this technique in his Master's thesis to various Cayley graphs on cyclic groups in order to obtain various formulae. Perhaps the most appealing of these comes from cycle graphs, from which we get:

$$\sum_{\substack{0\leq i\leq k \\ n|2i-k}} \binom{k}{i} = \frac{1}{n}\sum_{j=0}^{n-1}\left[2\cos\frac{2\pi j}{n}\right]^k \tag{6}$$

The left-hand side of (6) represents the sum of all terms in a fixed row of Pascal's triangle that are "evenly spaced" apart from each other and that are symmetric about the central vertical line.

Because Theorem 0.1 applies to any graph, we now have a plethora of pre-packaged, open-and-serve undergraduate research projects at our disposal. Namely, take any small graph X, and attempt to both compute the graphs eigenvalues and also find a formula for the number of closed walks of length n. Plugging your calculations into Theorem 0.1, the result is almost sure to be a nice combinatorial formula. If you obtain a copy of [5], it will be quite some time before you run out of graphs X to explore. For a more sophisticated project, take a larger graph for which the eigenvalues are too difficult to compute explicitly, and attempt to use the number of walks to obtain information about the eigenvalues. For d-regular graphs, the largest eigenvalue is d; many computer scientists are particularly interested in the second-largest eigenvalue. (See [4] for more on the rich mathematical theory underlying their interest there.) Another, more ambitious, project would be to find formulae of this sort for an infinite family of graphs.

The authors would like to acknowledge Anthony Shaheen for his assistance and again thank the referees for their thoughtful comments, as well as the editor for suggesting that we broaden the scope of the paper beyond the Binet formulae to a more general overview. This research was conducted with support from the MORE (Minority Opportunities in Research and Education) program at California State University, Los Angeles, through NIH grant GM61331.

References

1. A. Benjamin and J. Quinn, *Proofs that really count*, The Dolciani Mathematical Expositions, vol. 27, Mathematical Association of America, Washington, DC, 2003.
2. S. Cioabă, *Closed walks and eigenvalues of abelian Cayley graphs*, Comptes Rendus Mathematique **342** (2006), no. 9, 635–638.
3. K. Dsouza and M. Krebs, *A combinatorial trace method: Counting closed walks to assay graph eigenvalues*, to appear in Rocky Mountain Journal of Mathematics.
4. M. Krebs and A. Shaheen, *Expander families and Cayley graphs: A beginner's guide*, Oxford University Press, USA, 2011.
5. R. Read and R. Wilson, *An atlas of graphs*, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1998. MR 1692656 (2000a:05001)

Department of Mathematics, California State University - Los Angeles, 5151 State University Drive, Los Angeles, California 90032

E-mail address: `mkrebs@calstatela.edu`

Department of Mathematics, California State University - Los Angeles, 5151 State University Drive, Los Angeles, California 90032

E-mail address: `nmarti20@calstatela.edu`