Lecture Notes
on
Ordinary Differential Equations

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Preface

The *Lecture Notes on Differential Equations* ("Notes") presented here are simply a collection of all the handouts I have given to my students in our elementary differential equations course (Math 215) over the past several years.

In the presence of a good text, I don't normally bother giving handouts. Occasionally I give handouts on topics that seem somewhat muddled or deal only with sheer drudgery, or solution methods that are just laborious. What started out as one handout slowly expanded into many!

1. The first time I taught the course, I found that none of the books talked about operator methods for finding particular solutions. In most cases, it is far simpler to use operator methods instead of the method of coefficients which can be extremely laborious. Ever since student days in India, I have always been captivated by the sheer beauty, elegance and power of the operator methods. So as I taught this topic it became necessary to give fairly detailed handouts on this topic. Thus the first handout was born. Somewhat to my surprise, they were really well received.

2. Most books treat the method of undetermined coefficients as if it is simply a piece of intelligent guess work. My students asked for a handout that would clarify the actual process of writing the correct form of a particular solution and the process of finding the coefficients. This was the second handout.

3. The usual process for solving second order equations using power series methods is to substitute an assumed power series, manipulate indices, collect coefficients, etc., to find the recurrence relations satisfied by the coefficients. Like the method of coefficients, however, this is also a laborious method where it is easy to make mistakes. But one can dispense with this drudgery and instead find the recurrence relations more elegantly by simply using Leibniz's theorem on derivatives of products. This gave rise to my third handout.

Thus, over the years I wrote more handouts on other topics that can be treated more simply or directly such as systems of equations, Cauchy-Euler equations, etc., where the underlying method is the operator method. Often I was impelled by my students to write on just about every other topic that was in the syllabus. By now the Notes had grown to a good size and I started using them as the basis for my lectures supplemented by reading material from other texts. Another important reason has been the sharp rise in the cost of textbooks with prices beyond the reach of many of our students.

Our department surveys have shown that the students surely read the notes more than they read their text books (my experience is that except for some very bright students, most avoid whatever book is prescribed!). To make the Notes somewhat self contained, I included handouts on practically all the topics in our syllabus including several applications. These notes have been class tested by other instructors as well. They could be used by themselves or as a supplement to a text book.

The Notes however do not form a text book. Our students (math majors, engineering and science majors) who take this course have completed the usual calculus sequence but not a linear algebra course, so there is little discussion of eigenvalues, etc. In discussing power series methods, we restrict our discussion to solutions around ordinary points. Numerical solution of differential equations is a very important topic, but it does not find a place in Notes. Nor do partial differential equations. Even so, there is possibly too much material here to be covered in a quarter or even a semester, and instructors may want to skip some material. We would urge that they certainly include some of the applications.

I am much indebted to Melisa Hendrata, my former student and protégé. Now a colleague in the department, she joins me as a coauthor. Some of the applications and the chapter on Laplace transforms are due to her. She has transformed what was once an unattractive TEXdocument into a beautiful monograph. Her keen eye has caught many errors.

It is now my pleasant duty to acknowledge my indebtedness to the many authors from whose books I learned the material or used in my teaching. The classical texts by Forsyth, Piaggio, Murray, Coddington were suggested texts for me at the University of Delhi, India. I have taught from the books by Simmons (an all time favorite), Ross, Tennenbaum & Pollard, Boyce & DiPrima and many others.

Carlos Arcos and David Beydler of the department used the notes as text when they taught the course. I am grateful to them for pointing out the typos and other minor errors.

Above all, I am grateful to the several generations of students who took my course, made teaching them all a joyful experience, and were primarily responsible for the Notes to come into existence!

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Chapter 1

Preliminaries

1.1 Introduction

A ***differential equation*** is an equation that involves a function $y = f(x)$ and one or more of its derivatives. For example,

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = \sin x \tag{1.1}$$

is a differential equation. Sometimes one uses shortened notations to write the same equation as

$$y'' + 2xy' + y = \sin x$$

or using the differential operator $D \equiv \frac{d}{dx}$,

$$D^2y + 2xDy + y = \sin x.$$

The ***order*** of a differential equation is the order of the highest derivative in the equation. For example, equation (1.1) is a second-order equation, while $y' - \sin(x + y) = 0$ is of first order.

A solution of a differential equation is a relationship that involves only y and other functions of x either *explicitly* like

$$y = 2c_1e^x + c_2 \sin x$$

or *implicitly* as

$$\frac{y}{x-y} = \tan y$$

but not any of its derivatives. Notice that often implicit solutions cannot be expressed as direct explicit solutions.

It is much *easier* to verify that a function (given either implicitly or explicitly) is a solution of a given differential equation.

Example 1.1. *Verify that* $y = c_1 \sin 2x + c_2 \cos 2x$ *is a solution of* $y'' + 4y = 0$.
Solution: Since the second derivative is involved, we differentiate y twice to get

$$y' = 2c_1 \cos 2x - 2c_2 \sin 2x, \quad y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$$

and substituting these in the given equation,

$$\{-4c_1 \sin 2x - 4c_2 \cos 2x\} + 4\{c_1 \sin 2x + c_2 \cos 2x\} \equiv 0.$$

Example 1.2. *Verify that the function* $x^2 = 2y^2 \ln y$ *solves the equation* $y' = xy/(x^2 + y^2)$.
Solution: We differentiate both sides of the given solution to get

$$\begin{aligned} 2x &= 4yy' \ln y + 2y^2(1/y)y', \\ &= 2yy'(2 \ln y + 1) \end{aligned}$$

from which we get

$$\begin{aligned} y' &= \frac{x}{y(1 + 2\ln y)} \\ &= \frac{xy}{y^2(1 + 2\ln y)} \\ &= \frac{xy}{x^2 + y^2}. \end{aligned}$$

1.2 The differential

If $y = f(x)$ is a differentiable function of x, the ***differential*** dy of $f(x)$ is defined to be

$$df = f'(x)\, dx$$

and sometimes written simply as $dy = f'(x)\, dx$. Notice that this leads to the familiar derivative or *differential coefficient* $y' = dy/dx = f'(x)$. However, the differential can also be thought of as a linear approximation. Suppose the value of $f(x_0)$ is known at some point x_0 and we wish to approximate the value of f at a nearby point $x_0 + \Delta x$.

The actual change is of course $\Delta y = f(x_0 + \Delta x) - f(x_0)$. We can rewrite this as

$$\Delta y = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \cdot \Delta x,$$

and for very small values dx of Δx, we can rewrite this as

$$dy = f'(x)\, dx.$$

It is important to observe that dy is a function of both x and dx, and for this reason some authors write the differential as $dy(x, dx) = f'(x)\, dx$.

One immediate use of the differential is that it helps us solve differential equations! If we know the differential of $df = f'(x)\, dx$, it is immediate that $f(x) = \int f'(x)\, dx + c$, where c is a constant of integration. Another observation is that the solution is not a single function but *a family* of similar curves ("parallel") in some sense. For example, if

$$dy = 2x\, dx,$$

then $y = x^2 + c$, which is a family of parabolas all opening upwards and with $y-$axis as their axis. There is one such parabola through every point on the $y-$axis. If $c = 0$, it goes through the origin. If $c = 1$, we get the parabola $y = x^2 + 1$ which goes through (0,1).

Let $y = f(t)$ be a function of t and let $t = g(x)$ be a function of x. Then the differential of y is

$$dy = f'(t)\, dt$$

and since dt is the differential of t and $t = g(x)$ we also have

$$dt = g'(x)\, dx.$$

Substituting, we finally get

$$dy = f'(t) g'(x)\, dx = f'(g(x)) g'(x)\, dx$$

or

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$$

and this is the familiar *chain rule.*

Conversely, given a differential equation say

$$y' = 2\sin x\cos x,$$

we rewrite it using differentials as

$$dy = 2\sin x\cos x\,dx$$

and solve it as

$$y = \int 2\sin x\cos x\,dx + c.$$

One way of evaluating the integral is to use the substitution $t = \sin x$ to get $dt = \cos x\,dx$. Substituting we get

$$y = 2\int t\,dt + c = t^2 + c = \sin^2 x + c.$$

If on the other hand, we make the substitution $t = \cos x$, we would get

$$y = -\cos^2 x + c = \sin^2 x + (c-1) = \sin^2 x + c_1,$$

where c_1 is a different constant. Finally, we can also use the trigonometric identity $\sin 2x = 2\sin x\cos x$ to get $dy = \sin 2x\,dx$ from which

$$y = -\frac{\cos 2x}{2} + c = -\frac{1 - 2\sin^2 x}{2} + c = \sin^2 x + (c - 1/2) = \sin^2 x + c_2$$

and we see that although there are apparently three different solutions, they are all equivalent!

1.3 Direction Fields

Often times it is not easy to solve a differential equation, even for a first-order equation. Yet we may need to know at least the *behavior* of the solution.

First note that the first-order equation

$$\frac{dy}{dx} = f(x, y)$$

gives us the slope of the solution curve y at each point (x, y). If we draw the slope at various points on the xy-plane as short line segments, we will get what is called the ***direction field*** (often also called ***slope field***). The direction field will give us

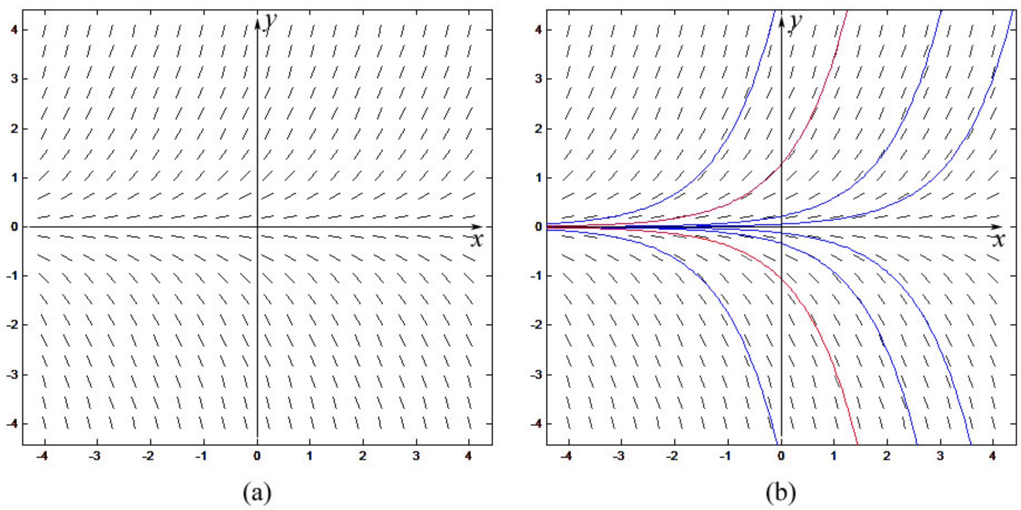


Figure 1.1: (a) Direction field of $dy/dx = y$. (b) Family of solutions of $dy/dx = y$. The red curves are solutions satisfying the initial conditions $y(0) = 1.25$ and $y(0) = -1.1$.

an idea of how the solution might look like. Consider, for example, the equation

$$\frac{dy}{dx} = y. \tag{1.2}$$

The solution curve to (1.2) has a horizontal slope at any point along the x-axis, but has the slopes of 3 and -4 at the points (2,3) and (-1,-4), respectively. The direction field of (1.2) is shown in Figure 1.1(a).

One can get much information by looking at this direction field. First, its "flow" pattern shows that as $x \to \infty$, all solutions diverge to either $+\infty$ or $-\infty$. In fact, equation (1.2) can be easily solved using the method of separation of variables (will be discussed in the next chapter) and the solution is given by $|y| = ce^x$, where c is the constant of integration. Note that the solutions are a family of exponential functions whose behavior matches the flow pattern shown by the direction field.

Secondly, given an initial condition $y(x_0) = y_0$, we can look at the point (x_0, y_0) and trace the slopes to get the solution curve that passes through (x_0, y_0). In this particular example, we see that a small change in the initial condition can cause a very different behavior of the solutions. With the initial condition $y(0) = 1.25$, the solution $y \to \infty$ as $x \to \infty$, while the initial condition $y(0) = -1.1$ causes $y \to -\infty$ as x gets larger. See Figure 1.1(b).

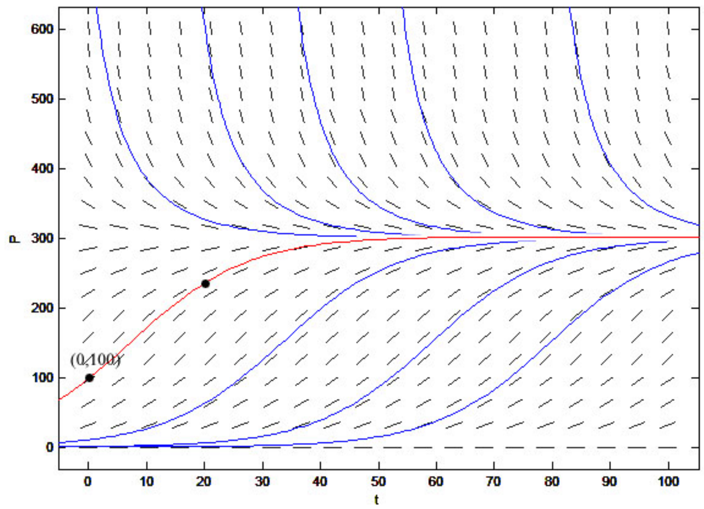


Figure 1.2: Direction field and family of solutions of $\frac{dP}{dt} = 0.1\,P(t)(1 - (P(t)/300))$. The solution curve satisfying the initial condition $P(0) = 100$ is shown in red.

Example 1.3. *The elk population in a small mountain area is given by a first-order equation*

$$\frac{dP}{dt} = 0.1\,P(t)\Big(1 - \frac{P(t)}{300}\Big),$$

where $P(t)$ is the number of elks at any time t. This equation is called the logistic equation. More details on this type of equation will be discussed in Section 2.5.2. By sketching the direction field,

(a) Estimate the limiting size of the population as $t \to \infty$.

(b) If the initial elk population is 100, estimate the size of the population in 20 years. Can the population ever reach 500?

(c) If the initial population is 600, can it decrease to 200?

Solution:

(a) Since the variables t and P represent the time and the number of elks, respectively, they cannot be negative. Thus, we only need to sketch the direction field for $t, P \geq 0$ and it is given in Figure 1.2. From the flow pattern, we can see that all solutions tend to P-value of 300 as $t \to \infty$. In Section 2.5.2 we will see that this is indeed the case.

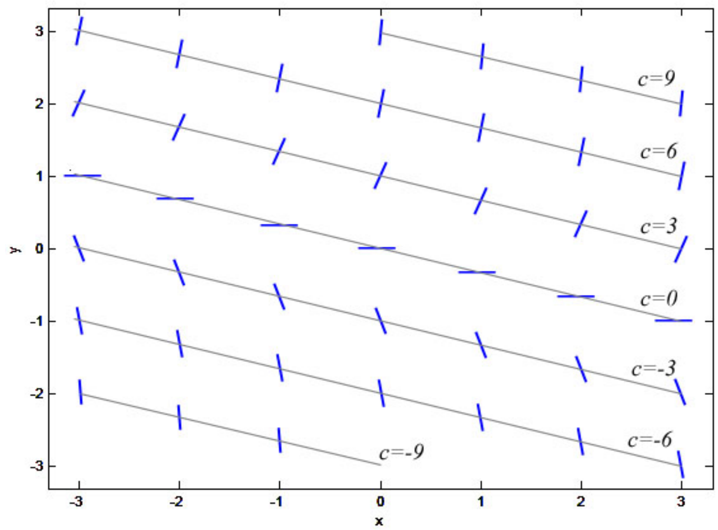


Figure 1.3: Isoclines for $y' = x + 3y$

(b) The initial condition here is when $t = 0$, $P = 100$. We trace the solution curve that passes through the point $(0, 100)$ and see that when $t = 20$, $P \approx 233$. It is clear that the population will increase to 300, but never reaches 500.

(c) The population will steadily decrease to 300, but it can never decline to 200.

Sketching direction field is straightforward, but it can be very tedious without the aid of computer software packages. If one really needs to hand sketch a direction field, the ***method of isoclines*** can help ease the job. For a first-order equation $dy/dx = f(x, y)$, an *isocline* is any member of the family of curves $f(x, y) = c$, where c is an arbitrary constant (Yes, it is very much the level curve for $f(x, y)$). At each point on an isocline, the solution curve has slope that is equal to c.

We illustrate the technique in the following example.

Example 1.4. *Sketch the direction field for* $y' = x+3y$ *using the method of isoclines.*
Solution: The isoclines for the given equation is $x + 3y = c$, which is a family of parallel lines. Pick several values for c and draw short line segments with slope c along the isocline $x + 3y = c$. Figure 1.3 shows isoclines for $c = 0, \pm 3, \pm 6, \pm 9$ and the resulting direction field. Some solution curves are given in Figure 1.4

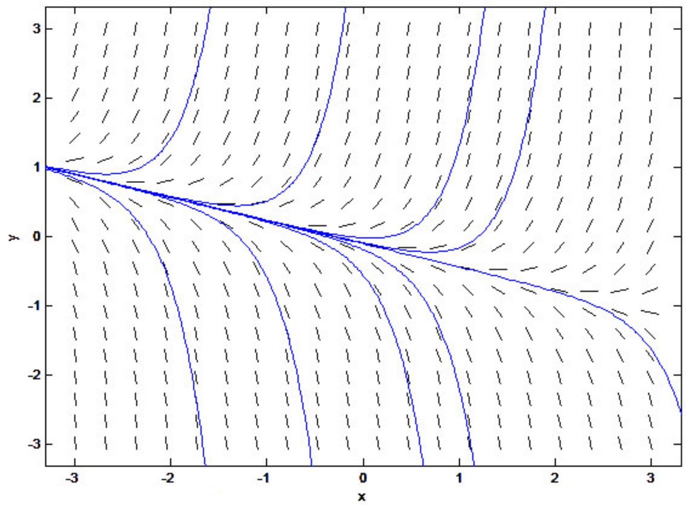


Figure 1.4: Family of solutions to $y' = x + 3y$

1.4 Exercises

1. Verify that the following functions are solutions to the corresponding differential equations.

   (a) $y = 2\sqrt{x} + c, \qquad y' = 1/\sqrt{x}$

   (b) $y^2 = e^{2x} + c, \qquad yy' = e^{2x}$

   (c) $y = \arcsin xy, \qquad xy' + y = y'\sqrt{1 - x^2y^2}$

   (d) $x + y = \arctan y, \qquad 1 + y^2 + y^2y' = 0$

   (e) $y = a \sin x + b \cos x, \qquad y'' + y = 0$

2. Draw the direction field for the following first-order equations using the method of isoclines, including the curve satisfying the given conditions. From the sketch of the direction field, what can you say about the behavior of the solution?

   (a) $\dfrac{dy}{dx} = 2y, \quad y(0) = 1$

   (b) $\dfrac{dy}{dx} = 2x - y, \quad y(0) = 0$

   (c) $\dfrac{dy}{dx} = x^2 - y, \quad y(1) = 0$

   (d) $\dfrac{dy}{dx} = 2y(3 - y), \quad y(0) = 1$

Chapter 2

First Order Linear Equations

2.1 Separable equations

A differential equation is said to be ***separable*** if it can be rewritten so that terms involving the differential of y is on one side of the equation, and those of x on the other side. One then integrates to get rid off the differentials leading to an equation that implicity or explicitly gives y.

Example 2.1. *Solve* $y^2 y' + 2x = 0$.
Solution: We rewrite this as $y^2\{\, dy/\, dx\} = -2x$ and using differentials as

$$y^2\, dy = -2x\, dx.$$

Integrating both sides we get

$$\int y^2\, dy = -\int 2x\, dx + c,$$

that is,

$$\frac{1}{3}y^3 = -x^2 + c,$$

or as is commonly written

$$\frac{1}{3}y^3 + x^2 = c.$$

This equation can be solved for y explicitly but that is not necessary. Because the given equation is of the first order, there is only one constant of integration c. The solution is said to be a *one parameter family*. The solution is unique once c is known, such as for instance $y(0)$, the value of y at the origin.

Example 2.2. *Find the particular solution of the equation*

$$y \ln y\, dx - x\, dy = 0$$

such that when $x = 1, y = 2$.
Solution: We rewrite the equation as

$$\frac{dx}{x} - \frac{dy}{y \ln y} = 0$$

from which we integrate

$$\int \frac{dx}{x} - \int \frac{dy}{y \ln y} = c.$$

The first integral is easy. To evaluate the second, we use the substitution $t = \ln y$, $dt = dy/y$ to obtain

$$\int \frac{dx}{x} - \int \frac{dt}{t} = c$$
$$\ln |x| - \ln |t| = c,$$

that is,

$$c = \ln |x| - \ln |t| = \ln |x/t| = \ln |x/ \ln y|.$$

This is equivalent to

$$\left| \frac{x}{\ln y} \right| = e^c = c_1,$$

where c_1 is another constant. This solution is again a one parameter family. To find the particular solution required, we plug in $x = 1, y = 2$ to obtain $c_1 = |1/ \ln 2|$. Hence, the particular solution is $|\ln y| = |x| \ln 2$.

Notes:

1. Simply because an equation is separable, it does not follow that it is solvable! The resulting integrals may not be expressible in terms of known functions, e.g. $dy = e^{x^2}\, dx$.

2. An equation may not be *apparently* separable but a little effort can make it so. As an example, $y' = x^2 y + x^2 e^y - y - e^y$ does not seem separable but if you look carefully, the right side is factorable: $y' = (x^2 - 1)(y + e^y)$ which is now readily separable.

2.2 First order linear equations

A typical such equation is of the form

$$\frac{dy}{dx} + P(x)y = Q(x). \tag{2.1}$$

Here P and Q are functions of x. This equation is not separable. But we can employ a trick that occurs many times in differential equations. We rewrite the equation as $dy + P(x)y\,dx = Q(x)\,dx$ and multiply this by (as yet) an unknown function $H(x)$ so that the resulting equation

$$H(x)\,dy + H(x)P(x)y\,dx = H(x)Q(x)\,dx \tag{2.2}$$

consists of differentials only! The right side is obviously so. If the left side is to be a differential, then (by looking at the first term) it must be of the form $d\big(H(x)\cdot y\big)$. But by the product rule of derivatives,

$$d\big(H(x)\cdot y\big) = H\,dy + y\,dH.$$

Comparing this with the left hand side of equation (2.2), the second terms must agree, that is, $y\,dH = HPy\,dx$, and since $y \not\equiv 0$, we must have

$$dH = HP\,dx, \qquad \frac{dH}{H} = P\,dx.$$

Integrating the second equation, we immediately get

$$\ln H(x) = \int P(x)\,dx, \qquad H(x) = e^{\left(\int P\,dx\right)}. \tag{2.3}$$

Equation (2.2) now becomes $d(H(x)\cdot y) = H(x)Q(x)\,dx$, whose solution is therefore

$$H(x)y = \int H(x)Q(x)\,dx + c.$$

From (2.3) we finally get

$$ye^{\left(\int P\,dx\right)} = \int e^{\left(\int P\,dx\right)} \cdot Q(x)\,dx + c. \tag{2.4}$$

The function $H(x)$ is called an ***integrating factor***. Multiplication by an integrating factor produces exact differentials. We will have more occasions to deal with integrating factors. Note that using equation (2.4) we can write down the solution

$$y = e^{-\int P\,dx}\left[\int e^{\int P\,dx}\cdot Q(x)\,dx + c\right].$$

Example 2.3. *Solve* $y' + y = 1/(1+e^{2x})$.
Solution: Here $P = 1$ and $Q = 1/(1+e^{2x})$. Hence, $e^{\int P\,dx} = e^x$ and the solution is

$$ye^x = \int \frac{e^x}{1+(e^x)^2}\,dx + c.$$

The integral is evaluated by the substitution $t = e^x$ so that $ye^x = \int dt/(1+t^2) + c$. That is, $ye^x = \arctan t + c$ and finally we get $y = e^{-x}(\arctan e^x + c)$.

Remark 1. You should note that in equation (2.1), x is the independent variable and y the dependent variable. Sometimes a linear equation may not appear in this form! In some of those cases if we rewrite the equation with y as the independent variable and x as the dependent variable (reversing the roles of x and y), the equation will look very similar to (2.1) and may appear in the form

$$\frac{dx}{dy} + P(y)x = Q(y).$$

In this case the integrating factor is $e^{\int P\,dy}$ and the solution would be

$$x = e^{-\int P\,dy}\left[\int e^{\int P\,dy}\cdot Q(y)\,dy + c\right].$$

2.3 Exact equations

A first order equation can often be written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \qquad (2.5)$$

This equation is said to be ***exact*** if the left hand side is the differential of a function $F(x,y)$ so that

$$dF(x,y) = M(x,y)\,dx + N(x,y)\,dy = 0 \qquad (2.6)$$

and the solution is clearly $F(x,y) = c$ for some constant c. The problem then is to find F!

Given $y = f(x)$, we know its differential is defined as $df = f'(x)\,dx$. For a function $F(x,y)$ of two (or several) variables one needs to be careful in writing its differential. Suppose for a moment $x = g(t), y = h(t)$ are both themselves functions of another variable t. Then from Calculus we know $F'(t)$ exists and is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt}.$$

In analogy with the one variable case, we can now write the differential dF of $F(x,y)$:

$$dF = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy.$$

Hence, if equation (2.5) is exact and F exists as required, then we must have

$$dF = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy = M(x,y)\ dx + N(x,y)\ dy = 0.$$

It follows by comparing the coefficients of dx and dy that we must have

$$\frac{\partial F}{\partial x} = M(x,y) \text{ and } \frac{\partial F}{\partial y} = N(x,y). \tag{2.7}$$

Since F is necessarily continuous, its mixed partials with respect to x and y must necessarily be equal:

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

Hence, from equation (2.7) we finally have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{2.8}$$

which gives us *a necessary condition* for equation (2.5) to be exact. But let us first consider an example.

Example 2.4. *Solve* $2x \sin y\ dx + x^2 \cos y\ dy = 0.$

Solution: Here $M = 2x \sin y$ and $N = x^2 \cos y$. It is easy to verify that $\partial M/\partial y = 2x \cos y = \partial N/\partial x$, showing equation (2.8) is satisfied and the equation is indeed exact. Hence, there exists some function $F(x,y)$ whose differential $dF = 2x \sin y\ dx + x^2 \cos y\ dy$. But how do we find F?

Since $\partial F/\partial x = M(x,y) = 2x \sin y$ we can integrate this equation partially with respect to x (that is, treating y as a constant just here) and write

$$F(x,y) = \int M\,\partial x = \int 2x\sin y\,\partial x, \quad \text{(Note: This notation is } \textbf{\textit{not}} \text{ standard!)}$$

to obtain $F(x,y) = x^2\sin y + h(y)$, where $h(y)$ is a function *independent of* x (think about this!). Taking partial derivative with respect to y gives us

$$\frac{\partial F}{\partial y} = x^2\cos y + h'(y).$$

But from equation (2.7)

$$\frac{\partial F}{\partial y} = N(x,y) = x^2\cos y.$$

It follows that $h'(y) = 0$ and $h(y) = c$, where c is a pure constant. The final solution is $F(x,y) = x^2\sin y + c$.

Remark 2. This is a brute force method! It involves a partial integration followed by a partial derivative. The method always works but can be trying in complicated problems. Often terms can be rearranged to form an exact differential.

Grouping

Grouping involves a clever combining of the terms of the equation in such a way that it is evident it is an exact differential. After verifying an equation is exact, if the expression consists of addition of terms, it is invariably of the form $(.)\,dx + (.)\,dy$ and this is clearly the differential of a product. In Example 2.4, no grouping is necessary and a moment's reflection shows that

$$(2x\sin y)\ dx + (x^2\cos y)\ dy \equiv d(x^2\sin y) = 0$$

and the solution is as before. With practice this becomes very easy.

Example 2.5. *Solve* $(2y^2 - 4x + 5)\ dx + (2y - 4 + 4xy)\ dy = 0.$
Solution: The equation is exact since $\partial M/\partial y = 4y = \partial N/\partial x$. Now group the terms as follows

$$\underbrace{(-4x+5)\ dx}_{d(-2x^2+5x)} + \underbrace{(2y-4)\ dy}_{d(y^2-4y)} + \underbrace{(2y^2\ dx + 4xy\ dy)}_{d(2xy^2)} = 0$$

and clearly the solution is $-2x^2 + 5x + y^2 - 4y + 2xy^2 = c$.

Some important exact differentials

The following list of exact differentials would be quite useful in solving exact differential equations by grouping:

1. $d(xy) = y\ dx + x\ dy$
2. $d\Big(\dfrac{x}{y}\Big) = \dfrac{y\ dx - x\ dy}{y^2}$
3. $\Big(\dfrac{y\ dx - x\ dy}{x^2 + y^2}\Big) = \dfrac{d\big(\frac{x}{y}\big)}{1 + \big(\frac{x}{y}\big)^2} = d\Big\{\ \arctan\Big(\dfrac{x}{y}\Big)\Big\}.$
4. $d\Big(\dfrac{y}{x}\Big) = \dfrac{x\ dy - y\ dx}{x^2}$

Example 2.6. *Solve* $y\ dx - x\ dy + \ln x\ dx = 0.$
Solution: It is easy to see that this equation is not exact. But we rewrite it as

$$x\,dy - y\,dx - \ln x\,dx = 0,$$

and if we divide by x^2 it becomes

$$\underbrace{\frac{x\,dy - y\,dx}{x^2}}_{d(y/x)} - \underbrace{\frac{\ln x}{x^2}\,dx}_{d\left(\int x^{-2}\ln x\,dx\right)} = 0$$

The integral is evaluated by parts as follows:

$$\int x^{-2}\ln x\,dx = -x^{-1}\ln x + \int x^{-2}\,dx = -x^{-1}\ln x - 1/x + c.$$

Hence, the solution of the equation is

$$\frac{y}{x} = -\frac{\ln x}{x} - \frac{1}{x} + c \quad \text{or} \quad y = -\ln x + cx - 1.$$

Integrating factors

In the last example, a non-exact equation became exact after multiplication by $\mu(x) = 1/x^2$. This is another instance of an *integrating factor* which we first encountered in first order linear equations. There is a vast theory on integrating factors. This method is often cumbersome as involves e^f where f may be either a function of

x alone or of y alone. Often simpler methods may be more useful. Here we provide a couple of simple rules one could use in some cases:

Suppose the equation (2.8) *is not satisfied.*

Rule 1. *If*

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = h(x) \tag{2.9}$$

is a function of x alone, then $\mu = e^{\int h(x)\,dx}$ is an integrating factor.

Rule 2. *If*

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y) \tag{2.10}$$

is a function of y alone, then $\mu = e^{\int g(y)\,dy}$ is an integrating factor.

Example 2.7. *Solve* $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0.$
Solution: Here $\partial M/\partial y = 2y$ and $\partial N/\partial x = 0$, so that

$$h(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = 1$$

is a function of x alone. Hence, $\mu = e^x$ is an integrating factor and we obtain the equation

$$\underbrace{(e^x y^2\,dx + 2e^x y\,dy)}_{d(e^x y^2)} + \underbrace{e^x(x^2 + 2x)\,dx}_{d\left(\int (x^2+2x)e^x\,dx\right)} = 0.$$

Using integration by parts, then solution then becomes $e^x y^2 + x^2 e^x = c$.

2.4 Homogeneous equations

We say that a function $z = f(x, y)$ is ***homogeneous of degree*** n if it can be written in the form

$$f(x, y) = x^n g(v), \quad v = \frac{y}{x}.$$

For instance, $f(x, y) = x^3y^2 + x^2y^3 - xy^4$ is homogeneous of degree 5 since we can write

$$f(x, y) = x^5\left\{\frac{y^2}{x^2} + \frac{y^3}{x^3} - \frac{y^4}{x^4}\right\} = x^5 g(v)$$

where $g(v) = v^2 + v^3 - v^4$, $v = y/x$.

The differential equation (2.5) is said to be **homogeneous** if *both* $M(x, y)$ and $N(x, y)$ are homogeneous and of the same degree. This definition applies right now only to equations of type (2.5). Later we use the same term to call higher order equations as homogeneous if the right hand side is zero (they may not always be so!). This would cause no confusion as you will see. For example, in equation $(x+y)\,dx+(x-y)\,dy=0$, both $M=(x+y)$ and $N=(x-y)$ are of the same degree 1 since $M=x(1+\frac{y}{x})$ and similarly for N. Hence, the differential equation is homogeneous of degree 1. The equation $(x^3+y^3)\,dx-xy^2\,dy=0$, is homogeneous of degree 3. But $x^3\,dy=(x^2y-y^2)\,dx$, is not homogeneous.

The substitution $y = vx$

When an equation is homogeneous, the substitution $y=vx$ is very useful. With this substitution, and assuming $M(x,y)$ and $N(x,y)$ are both of degree α, equation (2.5) can be written as

$$\frac{dy}{dx}=-\frac{M(x,y)}{N(x,y)}=-\frac{x^\alpha M(v)}{x^\alpha N(v)}=-g(v). \tag{2.11}$$

for some function g. Further, by product rule,

$$y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}. \tag{2.12}$$

By (2.11) and (2.12), equation (2.5) can be transformed into

$$x\frac{dv}{dx}=-g(v)-v,$$

which is separable with x and v as variables. Its solution is then given by

$$\int\frac{dv}{-g(v)-v}=\int\frac{dx}{x}.$$

Even though this method works, the integration after the variables have been separated may be sometimes daunting and often can be solved by other methods.

Example 2.8. *Solve* $(x^3+y^3)\,dx-xy^2\,dy=0$.
Solution: We rewrite the given equation as

$$\frac{dy}{dx}=\frac{x^3+y^3}{xy^2}$$

Using the substitutions from equation (2.12),

$$v+x\frac{dv}{dx}=\frac{1+v^3}{v^2}$$

and

$$x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v = \frac{1}{v^2}.$$

This is a separable equation and leads to

$$\frac{dx}{x} = v^2 dv.$$

Its solution is $\ln|x| = \{v^3/3\} + c$, which is equivalent to $3x^3 \ln|x| = y^3 + c_1 x^3$.

Integrating factor for Homogeneous Equations

The following rule applies only to homogeneous equations.

Rule 3. *If $Mx + Ny \neq 0$ and the equation is homogeneous, then*

$$\frac{1}{Mx+Ny} \text{ is an integrating factor of } M\,dx + N\,dy = 0.$$

Notice that unlike other integrating factors, there is no exponential involved in this integrating factor! This may often trump the usual method for homogeneous equations.

Example 2.9. *Solve* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.
Solution: Simplify by removing the common factor x to get

$$(xy - 2y^2)\,dx - (x^2 - 3xy)\,dy = 0.$$

This equation is homogeneous of degree 2 and

$$Mx + Ny = x^2y - 2xy^2 - x^2y + 3xy^2 = xy^2 \neq 0$$

and the rule applies. Hence, $1/(Mx+Ny) = 1/xy^2$ is an integrating factor. Multiply the last equation by $1/xy^2$ and rewrite to get

$$\begin{aligned} \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy &= 0 \\ \frac{dx}{y} - \frac{x\,dy}{y^2} - \frac{2\,dx}{x} + \frac{3\,dy}{y} &= 0 \\ \underbrace{\frac{y\,dx - x\,dy}{y^2}}_{d(x/y)} - \underbrace{\frac{2\,dx}{x}}_{2\,d(\ln x)} + \underbrace{\frac{3\,dy}{y}}_{3\,d(\ln y)} &= 0, \end{aligned}$$

which implies that

$$\frac{x}{y} + \ln \frac{y^3}{x^2} = c.$$

Example 2.10. *Solve* $(x - y)\,dx = (x + y)\,dy$.

Solution: This is homogeneous (M and N are of degree 1). We rewrite the equation as

$$\frac{dy}{dx} = \frac{x - y}{x + y}. \tag{2.13}$$

Substituting $y = vx$, and using equation (2.12) we get

$$\begin{aligned}
v + x\frac{dv}{dx} &= \frac{1 - v}{1 + v} \\
x\frac{dv}{dx} &= \frac{1 - v}{1 + v} - v = \frac{1 - 2v - v^2}{1 + v} \\
\frac{dx}{x} &= \frac{1 + v}{1 - 2v - v^2}\,dv \\
\ln|x| &= (-1/2)\ln|1 - 2v - v^2| + c \\
2\ln|x| + \ln|1 - 2v - v^2| &= 2\ln c_1 \\
x^2(1 - 2v - v^2) &= c_1^2 = c_2.
\end{aligned}$$

If you observe carefully, the given equation is in fact *exact*! It can be solved very easily by grouping:

$$x\,dx - y\,dy = x\,dy + y\,dx$$

leading to $x^2 - y^2 = 2xy + c$, same as before! Also notice that the last rule for finding integrating factor does apply but the factor itself is not particularly simple! It pays to study the equation carefully before attempting a solution!

2.5 Applications

In this section we consider two models in which the central theme is the rate of change of a physical quantity. Examples of this type abound. The rate at which a given quantity of a substance cools when placed in an environment which is cooler than itself, increase of population in a demographic area, radioactive decay, etc. Since the rate of change refers to the derivative, most such situations are modeled by a first order linear equation which is solved easily. Because the rate of change is usually proportional to the quantity left, the solution is invariably an exponential function.

2.5.1 Newton's law of cooling

Newton's law of cooling says simply that:

> *The rate of change of the temperature of a cooling body is proportional to the difference between the temperature of the body and the constant temperature of the medium surrounding the body.*

An immediate consequence of the law is that the cooling body loses its heat more rapidly initially and this rate slows down with time.

Let us suppose that we have an object, say a cup of coffee, whose initial temperature is T_0 and it is placed in a room which is maintained at a temperature M_0 where $M_0 < T_0$. Let T be the current temperature of the body. Then by Newton's law of cooling,

$$\frac{dT}{dt} \propto (T - M_0),$$

that is,

$$\frac{dT}{dt} = k(T - M_0), \tag{2.14}$$

where k is a constant of proportionality. This is a first order equation which is separable and we can rewrite as

$$\frac{dT}{(T - M_0)} = kdt$$

and whose solution is

$$\ln|T - M_0| = kt + c,$$

with c being the constant of integration. At $t = 0,\ T = T_0$ so that $c = \ln|T_0 - M_0|$. Hence, the solution can be written as

$$\ln|T - M_0| = \ln|T_0 - M_0| + kt.$$

Also, since $T_0 > T > M_0$, we can drop the absolute values and write the solution to (2.14) as

$$T - M_0 = (T_0 - M_0)e^{kt}. \tag{2.15}$$

A moment's reflection shows that in most cases we really need e^k, not the actual value of k.

Example 2.11. *The temperature of a cup of coffee initially is* $185°F$ *and it is in a room maintained at* $72°F$*. Fifteen minutes later, the temperature is* $150°F$*. When* *will the coffee be at* $100^\circ F$*?*

Solution: Let the temperature of the coffee at time t be T°F. We will measure t in hours. Using equation (2.15),

$$T - 72 = (185 - 72)e^{kt}, \tag{2.16}$$

where we need to determine k (or e^k). Since $T = 150^\circ$ when $t = 0.25$, we get from (2.16)

$$150 - 72 = 113\ e^{0.25k},$$

that is, $(e^k)^{0.25} = 78/113 \approx 0.690$. Hence, $e^k = (0.690)^4 \approx 0.227$.

To find when the temperature of the coffee would be 100°, from (2.16) we have

$$100 - 72 = 113\ e^{kt},$$

that is, $0.227^t = 28/113 \approx 0.247$. Hence,

$$t = \ln 0.247 / \ln 0.227 = 0.943 \text{ hours } \approx 57 \text{ minutes}.$$

Example 2.12. *The police discover the body of a murdered young man at 10:30 a.m. in a condo kept at* $72^\circ F$*. The coroner records the body temperature at* $84^\circ F$ *and an hour later at* $80^\circ F$*. When did the murder take place?*

Solution: Let us measure time in hours and count it from the moment the body is discovered. Since the initial temperature is 84°F, by Newton's law of cooling, using (2.15)

$$(T - 72) = (84 - 72)e^{kt} = 12e^{kt}.$$

The body temperature at the end of one hour is 80° so that

$$(80 - 72) = 12e^k$$

from which $e^k = 8/12 = 2/3$. The normal temperature of a healthy human body is 98.6°F. This must have been the temperature just prior to death. Hence

$$(98.6 - 72) = 12(e^k)^t.$$

Since $e^k = 2/3$ this gives us $26.6/12 = (2/3)^t$ from which

$$t = \frac{\ln(26.6/12)}{\ln(2/3)} = -1.963 \text{ hours},$$

that is 1 hour and 57 minutes before we started the clock. Hence, the murder took place around 8:33 a.m.

2.5.2 Malthus' law of population dynamics

We all know that population grows in general, but how fast does the population of a city grow? What will the population of the US be in 2025? Questions of this kind have been of interest to many scientists and the first person to propose a mathematical law was Rev. Thomas Robert Malthus, an English clergy man who laid out his findings in his 1798 writings.

Malthusian law says:

> *The rate of change of population is proportional to the actual population at any given time.*

Notice that this law is very similar in nature to Newton's law. Both laws predict that the rate of change of a quantity is in some sense proportional to the quantity remaining at a given time. The only difference is that Newton's law has to do with the *decrease* in heat while Malthus law has to do with *increase* in population.

Let the population at a given time t_0 be P_0 and at any time t later be $P(t)$. Then according to Malthus law,

$$\frac{dP}{dt} \propto P. \tag{2.17}$$

If k is the constant of proportionality, equation (2.17) becomes

$$\frac{dP}{dt} = kP. \tag{2.18}$$

This is a very simple equation whose solution is $\ln P = kt + c$, where c is the constant of integration. Using the initial condition that at $t = t_0, P = P_0$, we get

$$\ln P_0 = k\,t_0 + c.$$

We can then write the solution as

$$\begin{aligned}\ln P &= kt + \ln P_0 - kt_0 \\ &= \ln P_0 + k(t - t_0),\end{aligned}$$

that is,

$$P = P_0 e^{k(t-t_0)}. \tag{2.19}$$

Before we proceed further we should mention that Malthus law has proved remarkably accurate in predicting world population for certain periods and also for some species of mammals.

Example 2.13. *The world population in 1965 and in 1970 was 3.345 billions and 3.706 billions, respectively. What was the population in 1973?*

Solution: From equation (2.19), using the data for 1965 and 1970,

$$3.345 = 3.706\, e^{k(1965-1970)},$$

that is,

$$\begin{aligned} 3.345 &= 3.706\, e^{-5k} \\ e^k &= (3.706/3.345)^{0.2} = 1.020. \end{aligned}$$

To find the population in 1973, we have (again from equation (2.19)) that

$$P = 3.706\, e^{k(1973-1970)} = 3.706\,(e^k)^3 = 3.706(1.020)^3 = 3.932 \text{ billions.}$$

The actual population of the world in 1973 was 3.937 billions!

If the population of a city at time t_0 is P_0, it is apparent from equation (2.19) that the Malthus law predicts that the population of the city will grow exponentially without any limit. This is obviously not very realistic. In most cases, population growth is limited by available living space, decrease in food supply, etc. A different model has been proposed by the *logistic model*, also known as the *Verhulst-Pearl model*. According to this model, equation (2.18) is to be modified to:

$$\frac{dP}{dt} = kP\Big(1 - \frac{P}{M}\Big). \tag{2.20}$$

The only difference between this and the Malthus equation is the additional term $-kP^2/M$. This can be explained in the following way. The denominator M denotes a limiting factor. If the *decrease* in population per person is proportional to $-P/M$, then for the entire population it is proportional to $P \cdot (-P/M) = -P^2/M$. When $P \ll M$, that is, P is very small in comparison to M, the logistic model coincides with the Malthus model. In fact, it has been observed that the Malthus model is fairly accurate for world population for years past but not so for the distant future.

The solution of (2.20) is not as easy as that of (2.18), but not very difficult either. The variables are separable and we have

$$\frac{dP}{P\left(1-\frac{P}{M}\right)} = kdt.$$

Using partial fractions the last equation can be rewritten as

$$\left\{\frac{1}{P} + \frac{1/M}{1-(P/M)}\right\} dP = kdt,$$

which on integration yields

$$\ln P - \ln(1 - P/M) = kt + \ln c,$$

$\ln c$ being the constant of integration written in this form for easier simplification. We rewrite the last equation in the form

$$\frac{P}{1-(P/M)} = ce^{kt}. \tag{2.21}$$

If we use the initial conditions that at time $t = 0$, the starting population is P_0, we have from the last equation

$$c = \frac{P_0}{1-(P_0/M)} = \frac{MP_0}{M-P_0}. \tag{2.22}$$

From (2.21), solving for P we get

$$\begin{aligned} P &= (1 - P/M)ce^{kt} \\ \Rightarrow P &= \frac{Mce^{kt}}{M + ce^{kt}}, \end{aligned} \tag{2.23}$$

as is easily verified. In equation (2.23), we substitute for c from (2.22) to get

$$\begin{aligned} P &= \frac{\frac{MP_0}{M-P_0}\, Me^{kt}}{M + \frac{MP_0}{M-P_0}\, e^{kt}} \\ &= \frac{e^{kt}MP_0}{(M-P_0) + P_0e^{kt}} \\ &= \frac{MP_0}{P_0 + (M-P_0)e^{-kt}}, \end{aligned} \tag{2.24}$$

which is the solution of (2.20). A little laborious perhaps, but not hard.

If we let $t \to \infty$ in (2.24) we get $P = M$, the limiting size of the population.

Example 2.14. *The elk population in a small mountain area is given by the logistic equation*

$$\frac{dP}{dt} = 0.1\,P(t)\Big(1 - \frac{P(t)}{300}\Big).$$

If the initial elk population is 120, find P *as a function of* t. *What is the population in ten years? When does the population double?*

Solution: In this problem $M = 300, P_0 = 120, k = 0.1$. From equation (2.24),

$$\begin{aligned} P(t) &= \frac{MP_0}{P_0 + (M - P_0)e^{-kt}} \\ &= \frac{300 \cdot 120}{120 + 180\,e^{-(0.1 \cdot 10)}} \\ &= \frac{600}{2 + 3e^{-1}} \\ &\approx 193 \end{aligned}$$

which is the population of the elk in 10 years. To find the time when the population doubles, we need t when $P = 2P_0 = 240$. From equation (2.24) we have

$$240 = \frac{300 \cdot 120}{(120 + 180e^{-0.1t})}$$

from which $e^{0.1t} = 6$ or $t = 10\ln 6 = 17.9$ years.

2.6 Exercises

1. Use separation of variables to solve the following equations:

   (a) $x^5 y' + y^5 = 0$.

   (b) $y' - y\tan x = 0$.

   (c) $(y + yx^2 + 2 + 2x^2)dy = dx$.

   (d) $y'/(1 + x^2) = x/y$ and $y = 3$ when $x = 1$.

   (e) $y' = x^2 y^2$ and the curve passes through (-1,2).

2. Solve the following linear equations:

   (a) $\frac{dy}{dx} + \frac{4y}{x} = 6x^2$.

   (b) $\frac{dy}{dx} + 3y = 4x^3 e^{-3x}$.

(c) $x\dfrac{dy}{dx} + \dfrac{2x+1}{x+1}y = x - 1.$

(d) $xdy + (xy + y - 1)dx = 0.$

(e) $ydx + (xy^2 + x - y)dy = 0.$

(f) $(\cos^2 x - y\cos x)dx - (1 + \sin x)dy = 0.$

(g) $\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{-y^2}{x}.$

*Note: This type of equation is called **Bernoulli's equation** and it can be rewritten so that it is linear. Divide throughout by $-y^2$ and make a change of variables by substituting $u = 1/y$. The resulting equation will be linear in u and solve this as usual. Substitute back for u in terms of y.*

3. Verify that the following are exact equations. If they are, you may use the l..o..n..g method(!) or look for clever groupings where you can. *You must do the last two problems by long method only.*

(a) $(3x + x^2 + 2y)dx + (2x + y - \frac{1}{y^2})dy = 0.$

(b) $(2xy + x)dx + (x^2 + 3y^2)dy = 0.$

(c) $(6xy + 2y^2)dx + (3x^2 + 4xy)dy = 0.$

(d) $(y\sec^2 x + \sec x\tan x)dx + (\tan x + \sec^2 y)dy = 0.$

(e) $\Big(\dfrac{3s^2 - 2}{t}\Big)ds + \Big(\dfrac{2s - s^3}{t^2}\Big)dt = 0.$

(f) $2xydx + x^2dy - 3dx + 4ydy = 0.$

(g) $(y\sin 2x\cos x + y^3\sin x)dx + (\sin^2 x - 3y^2\cos x)dy = 0.$

(h) This is a toughie!

$$\Big(-\frac{1+y}{x^2}\Big)dx + \Big(\frac{y^2 - x}{xy^2}\Big)dy = 0.$$

(i) $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0.$

4. Solve the following homogeneous equations:

(a) $(x^3 - 3xy^2)dx + 2x^2ydy = 0.$

(b) $(xy + y^2)dx - x^2dy = 0.$

(c) $\dfrac{dy}{dx} = \dfrac{y(\ln y - \ln x + 1)}{x}.$

(d) Example 8 is solved in the text as a homogeneous equation. Show that it is exact and solve the equation.

5. A frozen pizza is brought home from a grocery store and placed in a freezer maintained at zero degrees. After 15 minutes, its temperature is 15 degrees and after 30 minutes it is 7.5 degrees. What was the original temperature of the pizza when it was placed in the freezer? Can you first guess the answer?!

Chapter 3

Higher Order Homogeneous Linear Equations

3.1 Introduction

We now proceed to the study of differential equations that are still linear but possibly of higher order. A typical equation is of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = Q(x), \tag{3.1}$$

where the coefficients $a_0(x), a_1(x), a_2(x), \ldots, a_n(x)$, and $Q(x)$ are continuous functions of x. When $Q(x) = 0$, equation (3.1) is said to be *homogeneous*. Notice that the word *homogeneous* has a different meaning in the context of first order linear equations.

The notation used in equation (3.1) is laborious to write (and to typeset!) and we will use the following equivalent forms often:

$$\begin{aligned} a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y &= Q(x), \\ a_0(x)D^n y + a_1(x)D^{n-1} y + \ldots + a_{n-1}(x)Dy + a_n(x)y &= Q(x) \end{aligned} \tag{3.2}$$

In the second form (3.2) we have used the ***differential operator*** notation D:

$$D \equiv \frac{d}{dx}, \quad D^2 \equiv \frac{d^2}{dx^2}, \quad \text{etc.}$$

We will find the differential operator notation to be particularly useful when dealing with *Operator methods* in Chapter 4.

Although we will be concerned in this chapter with homogeneous equations and where the coefficients $a_0, a_1, \ldots, a_n$ are constants, the following theorem is important for all our work from now on.

Theorem 3.1. *In equation* (3.1)*, assume that the coefficients* $a_0(x), a_1(x), \ldots, a_n(x)$ *and* $Q(x)$ *are continuous functions of* x *on some interval* $a \leq x \leq b$*. Let* $k_0, k_1, \ldots, k_{n-1}$ *be any* n *arbitrary constants. Let* x_0 *be any point in* $[a, b]$*. Then there exists a unique solution of* (3.1) *such that*

$$f(x_0) = k_0, f'(x_0) = k_1, \ldots, f^{(n-1)}(x_0) = k_{n-1}.$$

Further, if $Q(x) = 0$*, then the set of solutions to* (3.1) *is a vector space* $\mathcal{V}$ *over the real numbers* $\mathbb{R}$*. The dimension of* $\mathcal{V}$ *is* n*, the order of the equation. In particular any solution of* (3.1) *is of the form*

$$y = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x), \tag{3.3}$$

where $c_1, c_2, \ldots, c_n$ *are arbitrary constants and* $f_1(x), f_2(x), \ldots, f_n(x)$ *are linearly independent functions that are solutions of* (3.1)*.*

The proof of Theorem 3.1 is beyond the scope of these notes. But it is easy enough to see why the set of solutions $\mathcal{V}$ to the homogeneous equation is a vector space. So assume $Q(x) = 0$. We need to show that if f and g are any two solutions of (3.1), then so are $f + g$ and kf for any real number k. Since f and g are solutions of (3.1) we have

$$a_0(x)D^n(f) + a_1(x)D^{n-1}(f) + \ldots + a_n(x)(f) = 0 \tag{3.4}$$

and

$$a_0(x)D^n(g) + a_1(x)D^{n-1}(g) + \ldots + a_n(x)(g) = 0. \tag{3.5}$$

Adding equations (3.4) and (3.5) and using the fact $D(f+g) = Df+Dg$, $D^2(f+g) = D^2 f + D^2 g$, etc., we get

$$a_0(x)D^n(f+g) + a_1(x)D^{n-1}(f+g) + \ldots + a_n(x)(f+g) = 0$$

and this shows that $f + g$ is also a solution of (3.1). Multiplying both sides of (3.4) by any real number k, we get

$$ka_0(x)D^n(f) + ka_1(x)D^{n-1}(f) + \ldots + ka_n(x)(f) = 0$$

and again by the properties of derivatives, this is the same as

$$a_0(x)D^n(kf) + a_1(x)D^{n-1}(kf) + \ldots + a_n(x)(kf) = 0,$$

showing that kf is also a solution of (3.1).

Remark 3. Theorem 3.1 says that for the homogeneous case of (3.1), $\mathcal{V}$ has n degrees of freedom. Any solution in $\mathcal{V}$ is determined by n arbitrary constants, that is $\mathcal{V}$ has dimension n (see Appendix A: Review of Basic Linear Algebra). In particular, in equation (3.3), the functions $f_1(x), f_2(x), \ldots, f_n(x)$ form a *basis* for $\mathcal{V}$.

3.2 Solution of the homogeneous linear equations

In this chapter we will be concerned with the case when (3.1) is homogeneous and the functions $a_0(x), a_1(x), \ldots, a_n(x)$ are constants. In light of Remark 3, to solve such an equation we need to find n linearly independent solutions of (3.1) to find its general solution. While we know what linear independence of vectors means (see Appendix A: Review of Basic Linear Algebra), what do we mean by linear independence of functions?

Definition 3.1. *The functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be linearly dependent if there exist constants $c_1, c_2, \ldots, c_n$, not all zero such that*

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0.$$

Otherwise they are called linearly independent.

This definition is however not particularly useful in testing for linear independence. What we need is the notion of the *Wronskian.*

Definition 3.2. *The Wronskian of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ is the determinant*

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & f_3^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

We omit the proof of the following important theorem.

Theorem 3.2. *The functions $f_1(x), \ldots, f_n(x)$ are linearly independent if their Wronskian $W(f_1, \ldots, f_n) \neq 0$.*

Remark 4. Theorem 3.2 is useful if we have three or more functions. If we only have two functions, say f_1, f_2, we can check their linear independence much more quickly. The two functions are linearly *dependent* if there exist constants c_1, c_2, not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0.$$

Let us assume that $c_1 \neq 0$. Then the last equation can be rewritten as

$$f_1(x) = -\frac{c_2}{c_1} f_2(x), \text{ or equivalently } \frac{f_1(x)}{f_2(x)} = c,$$

for some constant c. This says their ratio must be a constant function. If this is not the case, f_1 and f_2 are linearly independent. This is often much simpler to verify.

Example 3.1. *Show that e^{ax} and e^{bx} are linearly independent unless $a = b$.*
Solution: It is easy to see that $e^{ax}/e^{bx} = e^{(a-b)x}$ is a constant only if $a - b = 0$, that is, $a = b$.

We now have the tools to solve homogeneous linear equations with constant coefficients. Consider the following example.

Example 3.2. *Solve*

$$y'' - 5y' + 6y = 0. \tag{3.6}$$

Solution: It is easy to see that y'' is a linear combination of y' and y. But $y = e^{mx}$ has the property that

$$D(e^{mx}) = me^{mx}, \quad D^2(e^{mx}) = m^2 e^{mx}, \ldots, D^k(e^{mx}) = m^k e^{mx},$$

that is, the derivatives of y are constant multiples of y. If we write (3.6) using differential operator and substitute $y = e^{mx}$, we get

$$\begin{aligned}(D^2 - 5D + 6)y &= 0\\ (m^2 - 5m + 6)e^{mx} &= 0,\end{aligned}$$

and since e^{mx} is not identically zero,

$$m^2 - 5m + 6 = 0,$$

which is a quadratic equation and referred to as the ***auxiliary equation***. Its roots are $m = 2, 3$. Hence e^{2x} and e^{3x} are both solutions of (3.6). It is obvious that they are linearly independent (Remark 4). The solutions to the given equation form a vector space of dimension 2. Hence, the general solution to the given equation is $y = c_1e^{2x} + c_2e^{3x}$.

Example 3.3. *Solve* $(D^2 - 1)y = 0$.
Solution: The equation is same as $y'' - y = 0$. The auxiliary equation is $m^2 - 1 = 0$, and its roots are $m = \pm 1$. Hence, the solutions are e^x and e^{-x}, which are linearly independent. The general solution is given by $y = c_1e^x + c_2e^{-x}$.

It is clear from the last example that the solution of a homogeneous linear equation very much depends on the roots of the auxiliary equation. In the two examples above, the roots were real and distinct giving rise to two linearly independent solutions leading to the general solution. In those examples, the auxiliary equation were quadratic. But the roots of a quadratic equation need not be distinct, not even real! We consider these cases next.

3.2.1 Special case: Auxiliary equation has repeated roots

If the auxiliary equation has a pair of equal roots, the corresponding solutions are identical and hence *linearly dependent* by Remark 4. What do we do? This is best explained by an example.

Example 3.4. *Solve the equation* $y'' - 4y' + 4y = 0$.
Solution: The auxiliary equation is $m^2 - 4m + 4 = 0$ with roots $m = 2, 2$. The corresponding solutions are e^{2x} and e^{2x}. We rewrite the given equation using the operator D as

$$(D^2 - 4D + 4)y = (D - 2)(D - 2)y = 0. \tag{3.7}$$

It may appear that we have treated $(D^2 - 4D + 4)$ as an algebraic polynomial (which is true!) but it makes sense as a differential operator as well:

$$\begin{aligned}(D^2 - 4D + 4)y &= \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y \\ &= \frac{d}{dx}\Big\{\frac{dy}{dx} - 2y\Big\} - 2\Big\{\frac{dy}{dx} - 2y\Big\} \\ &= \Big\{\frac{d}{dx} - 2\Big\}\Big\{\frac{dy}{dx} - 2y\Big\} \\ &= \Big\{\frac{d}{dx} - 2\Big\}\Big\{\frac{d}{dx} - 2\Big\}y \\ &= (D - 2)(D - 2)y.\end{aligned}$$

From (3.7), we make a change of variables by letting $(D-2)y = u$

$$(D-2)\underbrace{(D-2)y}_{u} = 0 \Rightarrow \begin{cases} (D-2)u = 0 \\ (D-2)y = u \end{cases}$$

Note that the first equation $(D-2)u = 0$ is homogeneous with auxiliary equation $m-2=0$. Hence, $u = e^{2x}$. Now we solve the second equation

$$(D-2)y = u = e^{2x} \text{ or what is the same thing } \frac{dy}{dx} - 2y = e^{2x},$$

which is linear of first order, which we discussed in the last chapter. Its solution is clearly

$$ye^{-2x} = \int e^{-2x} \cdot e^{2x} dx = \int dx = x. \tag{3.8}$$

It follows that $y = xe^{2x}$ is another solution of the given equation (verify!) and the two solutions e^{2x}, xe^{2x} are linearly independent. Hence, the general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x}. \tag{3.9}$$

Remark 5. You may be wondering why we did not include a constant of integration in equation (3.8). The reason is that including constants of integration leads to the same solution as in (3.9). To see this, let us solve the given equation including constants of integration. First observe that the solution of $(D-2)u = 0$ leads to $du/dx = 2u$ from which $u = c_1 e^{2x}$. Next substituting for u, we get the linear equation $y' - 2y = c_1 e^{2x}$ whose solution is $ye^{-2x} = \int e^{-2x} c_1 e^{2x} dx = c_1 x + c_2$, which leads to $y = c_1 x e^{2x} + c_2 e^{2x}$, which is no different from (3.9).

Remark 6. It is worth noting from the previous example that when the auxiliary equation has a root $m = \alpha$ that repeats, the corresponding (linearly independent) solutions are $e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots$, and so on. This fact will be of much use to us in Chapter 4 when dealing with the *Method of Undetermined Coefficients*.

3.2.2 Special case: Auxiliary equation has complex roots

Consider the following equation: $(D^2 - 4D + 5)y = 0$. The auxiliary equation is $m^2 - 4m + 5 = 0$, whose roots are $m = 2 \pm i$ with corresponding solutions $y_1 = e^{(2+i)x} = e^{2x}e^{ix}$, and $y_2 = e^{(2-i)x} = e^{2x}e^{-ix}$. But these are complex solutions. How do we find linearly independent real solutions?

It is convenient to think of the solution space $\mathcal{V}$ of the given equation as a vector space over the complex numbers $\mathbb{C}$. Since sums and differences of solutions are also solutions, we use Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$. Then we can write y_1, y_2 as

$$\begin{aligned} y_1 &= e^{2x}(\cos x + i\sin x) \\ y_2 &= e^{2x}(\cos x - i\sin x). \end{aligned}$$

Adding and subtracting these equations we get the solutions,

$$\begin{aligned} u_1 &= y_1 + y_2 = 2e^{2x}\cos x \\ u_2 &= y_1 - y_2 = 2ie^{2x}\sin x. \end{aligned}$$

Since a constant multiple of a solution is also a solution (and $\mathcal{V}$ is now a vector space over the complex numbers) we get

$$v_1 = (1/2)u_1 = e^{2x}\cos x, \quad v_2 = (1/2i)u_2 = e^{2x}\sin x$$

as solutions as well. However, v_1, v_2 are real solutions. They are also linearly independent since $v_1/v_2 = \cot x$ is not a constant (Remark 4) and hence the general solution of $(D^2 - 4D + 5)y = 0$ is $y = e^{2x}(c_1\cos x + c_2\sin x)$. You should verify this satisfies the given equation! With some practice you should be able to write down the general solution of such equations with ease.

Example 3.5. *Solve the equation* $y'''' + 3y''' + 4y'' + 3y' + y = 0$.
Solution: The auxiliary equation is $m^4 + 3m^3 + 4m^2 + 3m + 1 = 0$. By inspection (the sum of the coefficients of odd powers of m is equal to the sum of the coefficients of the even powers of m), we see that $m = -1$ is a root of the equation. Hence, $(m+1)$ is a factor of the left side of the auxiliary equation. We can divide by $(m+1)$ and write

$$m^4 + 3m^3 + 4m^2 + 3m + 1 = (m+1)(m^3 + 2m^2 + 2m + 1).$$

We see once again that $(m+1)$ is a factor of $m^3 + 2m^2 + 2m + 1$. We divide the latter by $(m+1)$ and can finally write

$$m^4 + 3m^3 + 4m^2 + 3m + 1 = (m+1)^2(m^2 + m + 1).$$

Thus, the roots of the auxiliary equation are $m = -1, -1, \frac{(-1 \pm i\sqrt{3})}{2}$, where the complex roots being the roots of $m^2 + m + 1 = 0$. The general solution of the given equation is

$$y = c_1 e^{-x} + c_2 x e^{-x} + e^{-\frac{1}{2}x}\left\{c_3 \cos \frac{\sqrt{3}}{2}x + c_4 \sin \frac{\sqrt{3}}{2}x\right\}.$$

Remark 7. When the given differential equation is of order 2, the auxiliary equation is simply a quadratic equation which can be solved by either factoring or using the quadratic formula. For higher order equations, one still attempts to factor the auxiliary equation. We suggest you try $m = 1$ (the sum of the coefficients should be zero) or $m = -1$ as in the example above. If these don't work, try if $m = \pm 2$ are solutions by actual substitution.

Remark 8. If you are wondering what would happen if the complex roots, say $m = \alpha \pm i\beta$ repeat, such as in fourth order equation $(D^4 + 2D^2 + 1)y = 0$, Remark 4 should enable you to guess that the general solution would be of the form

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + x e^{\alpha x}(c_3 \cos \beta x + c_4 \sin \beta x).$$

We shall prove this is indeed the case when we discuss the *Operator Methods*. But most problems one faces are usually of the second degree and this complicated situation rarely occurs!

3.3 Applications

In this section we consider several applications of the material we have studied so far. In the first two applications (Sections 3.3.1-3.3.2), our object of interest is a curve that often occurs in daily life. The goal is to find the cartesian equation of the curve when we model it using mathematics. The modeling procedure is pretty similar in both applications. At an arbitrary point P on the curve, we use the fact that if the tangent at P makes an angle ψ with the horizontal, then $\tan \psi = dy/dx$, the derivative at the point. In each case, our model leads to a second order differential equation satisfied by the equation of the curve.

The rest of the applications in this section deal with objects that have natural tendency to vibrate around an equilibrium position, oscillating back and forth about the point of equilibrium. Well known examples are the pendulum of a clock, a spring balance, and the string of a musical instrument when plucked. Of course in most cases the object comes to its position of rest due to external forces. When the external forces are negligible, the motion is called *free undamped motion.* Otherwise it is called *damped motion.* In this section we shall see that the mathematical model that describes this motion leads to a second order linear differential equation as well.

3.3.1 Catenary

Can you guess the shape assumed by a rope or cable, of uniform density, suspended between two points, and hanging under its own weight? If you guessed it would be a parabola, you would be in good company - Galileo thought as much. But then you would be wrong as was he! The curve is called a *catenary* (the Latin word for chain is *catena*).

The catenary occurs freely in nature. In a spider's web, the woven silk is several parallel elastic catenaries. In many parks, a simple chain fence consists of a chain hanging from two identical posts. Many suspension bridges are of this type. The correct equation for the catenary was obtained by Gottfried Leibniz, Christiaan Huygens, and Johann Bernoulli in 1691.

Consider a chain hanging from two identical posts. To derive the cartesian equation for the catenary, let us choose a vertical through the lowest point O of the chain as the y-axis. The position of the x-axis will be determined later. We will assume that the curve has uniform density, that is, the weight of a unit length of the chain is a constant, say μ.

Consider a point $P(x, y)$ on the curve. Let s be the length of the arc OP. The portion OP of the chain is in equilibrium under the action of three forces: The horizontal tension of the chain T_0, the weight of the chain μs and the variable tension of the chain at the point P which acts along the tangent PT (due to the flexibility of the chain). See Figure 3.1.

Let ψ be the angle PT makes with the x-axis. Then because of the equilibrium of the chain, we have

$$T_0 = T\cos\psi, \quad \mu s = T\sin\psi.$$

Recall from calculus that $\tan\psi = y'$. It follows from these equations that

$$\frac{dy}{dx} = y' = \tan\psi = \frac{\mu s}{T_0}.$$

Let us write $c = T_o/\mu$ so that the last equation can be written as

$$cy' = s.$$

Differentiating with respect to x,

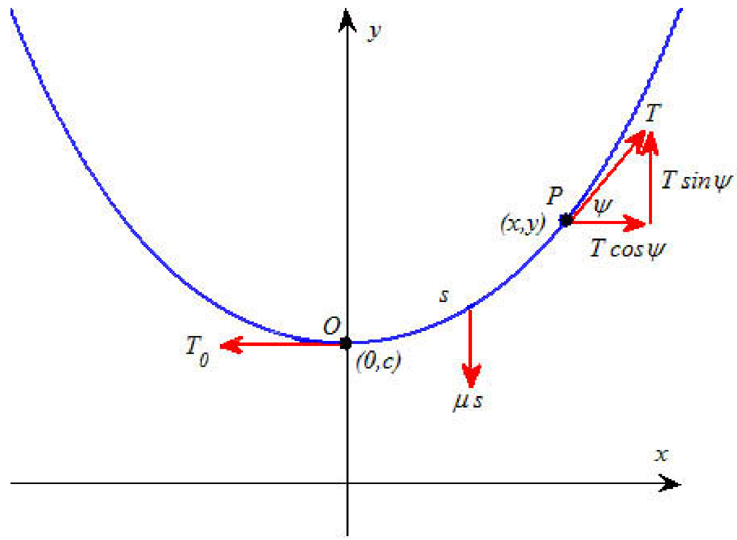


Figure 3.1: Catenary

$$cy'' = \frac{ds}{dx} = \sqrt{1 + (y')^2}.$$

This is a second order equation but the y term is absent! We can integrate it as follows. Let $z = y'$. Then the last equation can be written as

$$c\frac{dz}{dx} = \sqrt{1 + z^2},$$

that is,

$$\frac{dz}{\sqrt{1 + z^2}} = \frac{dx}{c}. \tag{3.10}$$

Integrating equation (3.10) gives us

$$\frac{x}{c} = \int \frac{dz}{\sqrt{1 + z^2}} + c_1, \tag{3.11}$$

where c_1 is a constant of integration. The integral in (3.11) is a standard integral. To evaluate it, substitute $z = \tan\theta$ to get

$$\int \frac{dz}{\sqrt{1 + z^2}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{1 + \tan^2\theta}} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| = \ln|z + \sqrt{1 + z^2}|.$$

Hence, from equation (3.11),

$$\frac{x}{c} = \ln|z + \sqrt{1+z^2}| + c_1.$$

When $x = 0$, $\psi = 0$, so that $\tan\psi = y' = z = 0$. Hence, $c_1 = 0$. Thus,

$$\frac{x}{c} = \ln|z + \sqrt{1+z^2}|. \tag{3.12}$$

From (3.12) we get

$$e^{x/c} = |z + \sqrt{1+z^2}| = z + \sqrt{1+z^2}$$

$$\text{and} \qquad e^{-x/c} = \frac{1}{e^{x/c}} = |z - \sqrt{1+z^2}| = -(z - \sqrt{1+z^2}).$$

It follows that,

$$y' = z = \frac{e^{x/c} - e^{-x/c}}{2}.$$

Integrating once more,

$$y = c\,\frac{e^{x/c} + e^{-x/c}}{2} + c_2,$$

where c_2 is a constant of integration. Remember that the position of $x - axis$ is still not defined. We now choose the x-axis to be c units below O so that $x = 0 \Rightarrow y = c$. It follows that $c_2 = 0$ and the equation of the catenary is

$$y = c\,\frac{e^{x/c} + e^{-x/c}}{2}.$$

Recall from calculus that the *hyperbolic cosine*, $\cosh x$ is defined as

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Using this the equation of the catenary can written more succinctly as

$$y = c\,\cosh\left(\frac{x}{c}\right).$$

3.3.2 Curve of pursuit

Suppose that a mouse is running away from a cat in a straight line and the cat is chasing him. What path does the cat pursue? Will the cat catch the mouse or would the mouse get away? Problems of this kind are usually called *curves of pursuit*.

Assume the usual coordinate axes and assume that the mouse is at the point $(x_0, 0)$

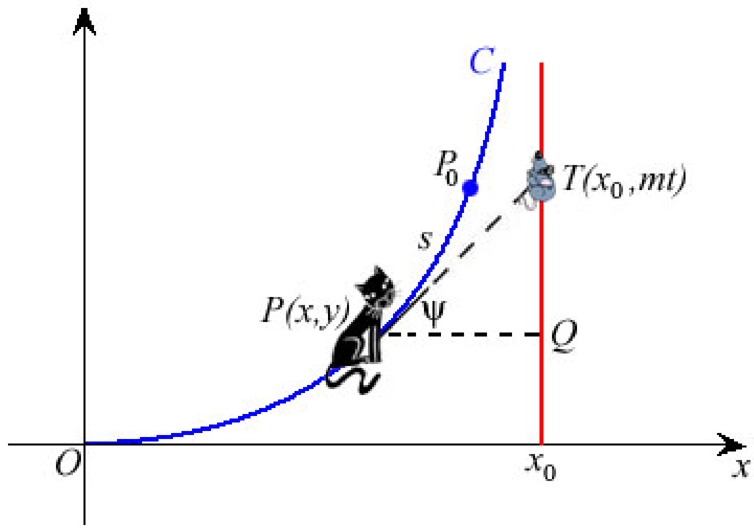


Figure 3.2: Curve of pursuit

with $x_0 > 0$, and that the cat is at the origin when it first spots the mouse. We will assume that the mouse runs along vertically along the line $x = x_0$. Let us call the curve followed by the cat as C. See figure 3.2.

We will assume both the cat and the mouse run at constant speeds c and m respectively. At time t, the cat is at some point $P(x, y)$ on C while the mouse is at the point $T(x_0, mt)$. Since the cat always keeps the mouse in his sight, PT is the tangent to C at point P. Let PQ be perpendicular to the y-axis. In $\triangle QPT$,

$$\angle QPT = \psi, \quad \tan\psi = \frac{TQ}{PQ} = \frac{mt - y}{x_0 - x}.$$

But $\tan\psi = y'$ so that

$$\frac{dy}{dx} = y' = \frac{mt - y}{x_0 - x},$$

from which we get

$$(x_0 - x)y' = mt - y. \tag{3.13}$$

The cat is not running along a straight line but along the curve C. If the length of the arc $P_0P = s$, then $s = ct$. It follows that

$$\frac{ds}{dt} = c, \tag{3.14}$$

the speed of the cat. We have three variables x, y, t and since we need the Cartesian equation of C, let us eliminate t. We differentiate equation (3.13) *with respect to* x to get

$$-y' + (x_0 - x)y'' = m\frac{dt}{dx} - y'$$
$$(x_0 - x)y'' = m\frac{dt}{dx} \tag{3.15}$$

Now $dx/dt = (dx/ds) \cdot (ds/dt)$ by chain rule. From calculus,

$$\frac{ds}{dx} = \pm\sqrt{1 + (y')^2}.$$

From (3.14) and (3.15) we now get

$$(x_0 - x)y'' = \frac{m}{c}\sqrt{1 + (y')^2}. \tag{3.16}$$

To avoid cumbersome notation, let $r = m/c$. As we have done before, we set $z = y'$ in (3.16) to get $(x_0 - x)z' = r\sqrt{1 + z^2}$, which we rewrite as

$$\frac{dz}{\sqrt{1 + z^2}} = \frac{r\,dx}{(x_0 - x)}. \tag{3.17}$$

This is the differential equation satisfied by C and is very similar to the one we saw in the case of the catenary. We can evaluate the integral on the left as usual by substituting $z = \tan\theta$, etc., (the details are left as an exercise) to get

$$\ln|y' + \sqrt{1 + (y')^2}| = -r\ln|x_0 - x| + c_1. \tag{3.18}$$

To determine the constant of integration c_1, we note that when $x = 0$, we have $y' = z = 0$. Hence, $c_1 = r\ln x_0$. Substituting this in (3.18) and dropping the natural log, we can now rewrite (3.18) as

$$y' + \sqrt{1 + (y')^2} = \left(\frac{x_0}{x_0 - x}\right)^r. \tag{3.19}$$

To solve for y' we use the same trick as in the case of the catenary by noticing that

$$\frac{1}{y' + \sqrt{1 + (y')^2}} = -\left(y' - \sqrt{1 + (y')^2}\right) = \left(\frac{x_0}{x_0 - x}\right)^{-r}. \tag{3.20}$$

Subtracting (3.20) from (3.19) we have,

$$2y' = \left(\frac{x_0}{x_0 - x}\right)^r - \left(\frac{x_0}{x_0 - x}\right)^{-r},$$
$$= \left(1 - \frac{x}{x_0}\right)^{-r} - \left(1 - \frac{x}{x_0}\right)^r. \tag{3.21}$$

Case 1: $r \neq 1$

We integrate both sides of equation (3.21) to get

$$2y = \frac{x_0}{1+r}\left(1 - \frac{x}{x_0}\right)^{1+r} - \frac{x_0}{1-r}\left(1 - \frac{x}{x_0}\right)^{1-r} + c_2, \tag{3.22}$$

where c_2 is the constant of integration. Using the fact that $x = 0 \Rightarrow y = 0$, we get

$$0 = \frac{x_0}{1+r} - \frac{x_0}{1-r} + c_2$$

from which $c_2 = 2rx_0/(1 - r^2)$. We can therefore rewrite (3.22) as

$$y = \frac{1}{2}\left\{\frac{x_0}{1+r}\left(1 - \frac{x}{x_0}\right)^{1+r} - \frac{x_0}{1-r}\left(1 - \frac{x}{x_0}\right)^{1-r}\right\} + \frac{rx_0}{1-r^2}, \tag{3.23}$$

and this gives the Cartesian coordinates of C when $r \neq 1$.

Case 2: $r = 1$

In this case equation (3.21) reduces to

$$2y' = \frac{x_0}{x_0 - x} + \frac{x}{x_0} - 1.$$

Integrating the last equation,

$$2y = \left\{ -x_0 \ln|x_0 - x| + \frac{x^2}{2x_0} - x\right\} + c_3.$$

By using the fact that $x = 0 \Rightarrow y = 0$ once more, we get $c_3 = x_0 \ln x_0$. Substituting this and simplifying the last equation becomes

$$y = \frac{1}{2}\left\{x_0 \ln\frac{x_0}{(x_0 - x)} + \frac{(x_0 - x)^2}{2x_0} - \frac{x_0}{2}\right\}, \tag{3.24}$$

and this gives the equation of C when $r = 1$.

Now that we have found the Cartesian equation for C, let us answer some of the questions we asked in the opening paragraph.

1. *Will the cat catch the mouse?*

   If $r < 1$, then $m < c$. In this case, the cat is faster. For the cat to catch the mouse, we must have $x = x_0$. Substituting this in equation (3.23), we get

   $$y = \frac{rx_0}{1 - r^2} = \frac{mcx_0}{c^2 - m^2}.$$

   This shows that the cat would indeed catch the mouse at the point $R = (x_0, mcx_0/(c^2 - m^2))$. What if the cat is extremely fast in comparison to the mouse? In this case $r = m/c$ is negligible, that is, $r = 0$. In this case, equation (3.23) gives $y = 0$, that is, the cat will catch the mouse the moment he sees him!

2. *Can the mouse ever escape?*

   If $r = 1$, that is $m = c$, the speeds of the cat and mouse are equal. As the cat tries to approach the mouse $x \to x_0$. If we take the limit of the right side of equation (3.24) as $x \to x_0$, we notice that $y \to \infty$, that is, the mouse will leave the cat in the dust behind him! The cat cannot catch the mouse in this case.

3.3.3 Simple harmonic motion

Undamped free motion is sometimes also called *oscillator motion.* The following example best describes the principles involved.

Suppose that a point P moves along the circumference of a circle of radius a centered at the origin in *counter-clockwise* direction with constant angular velocity ω radians per second. Let Q be the foot of the perpendicular from P on the x-axis. Let $P(x, y)$ be the position of P at any time t, $P_0(x_0, y_0)$ its initial position, and $Q_0(x_0, 0)$ the corresponding initial position of Q. Let δ be the angle OP_0 makes with the x-axis. See Figure 3.3.

It is obvious that OP makes the angle $\omega t + \delta$ with the x-axis. Hence, $x = a\cos(\omega t + \delta)$.

If we measure the angles with respect to the y-axis instead, that is, OP_0 makes the angle δ with the positive y-axis, then OP makes the angle $\omega t + \delta$ with the y-axis. In this case, $x = a\sin(\omega t + \delta)$. See Figure 3.4.

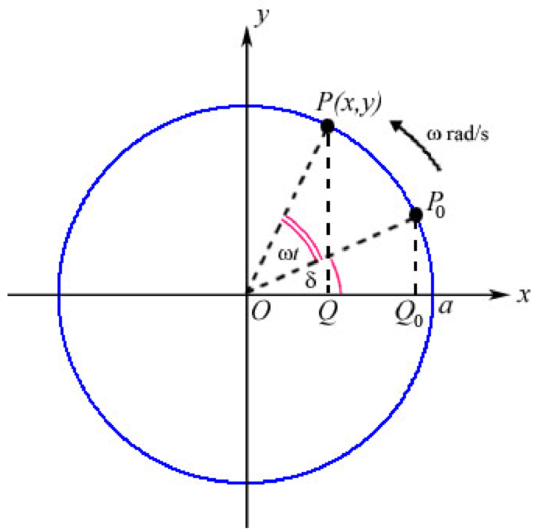


Figure 3.3: A diagram illustrating the principle of simple harmonic motion. The phase angle δ is measured with respect to the x-axis.

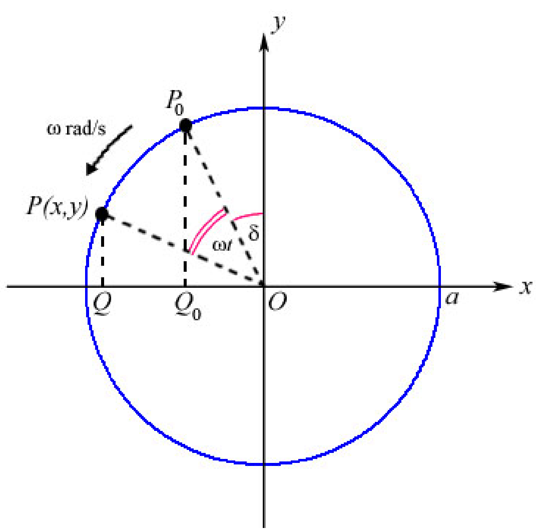


Figure 3.4: A diagram illustrating the principle of simple harmonic motion. The phase angle δ is measured with respect to the y-axis.

Thus, the position of Q at any time t, is described by either of the equations

$$x = a\cos(\omega t + \delta),$$
$$x = a\sin(\omega t + \delta). \tag{3.25}$$

As P moves, Q starting from Q_0, moves to the left. As P and Q continue to move, their positions coincide at $(-a, 0)$. As P continues to move, Q reverses direction and moves to the right towards the origin, the positions of P and Q coinciding again, this time at $(a, 0)$. Then Q reverses direction and starts moving to the left. As P ends at P_0, Q returns to Q_0.

As P revolves around the circle, Q moves back and forth along the x-axis, changing directions at the endpoints. The motion of Q is said to be an *undamped simple harmonic motion*.

Since the maximum value of cosine (or sine) is 1, it follows from (3.25) that $|x| \le a$. We call O the *equilibrium position* of Q and a its *amplitude*. Note that the amplitude is always considered positive. The angle δ is often called the *phase angle*. In equations (3.25) if $t = 2n\pi/\omega$, where n is an integer, then P has completed n revolutions or equivalently Q has completed n oscillations. The *time for one oscillation* of Q is called its *period* and denoted by T. Thus,

$$T = \frac{2\pi}{\omega}.$$

The reciprocal f of T, which is the *number of oscillations or cycles per unit of time* is called the *frequency* of Q. It is important to note that if T is measured in seconds, f is measured as cycles/sec. Sometimes f may be specified as *radians/sec* and in this case we can convert it to cycles/sec by using the fact that one cycle equals 2π radians.

It is easy to verify (exercise!) that the function $x(t)$ given by either of the equations (3.25) satisfies the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{3.26}$$

which is the differential equation satisfied by Q. Notice the special form of the equation: *It is a second order homogeneous equation where the first derivative term is missing and the coefficient of the dependent variable is positive.*

Conversely, if the position $x(t)$ of a moving object satisfies an equation of the form of (3.26), we can show it is executing a simple harmonic motion, that is, $x(t)$ must be of the form given by equations (3.25). To see this, consider the equation

$$\frac{d^2x}{dt^2} + sx = 0, \quad s > 0. \tag{3.27}$$

Using the methods of this chapter, we can write the solution to (3.27) as

$$x = c_1 \cos \sqrt{s}t + c_2 \sin \sqrt{s}t, \tag{3.28}$$

where c_1, c_2 are constants.

Let $c = \sqrt{c_1^2 + c_2^2}$. Then (3.28) becomes

$$x = c\left(\frac{c_1}{c} \cos \sqrt{s}t + \frac{c_2}{c} \sin \sqrt{s}t\right). \tag{3.29}$$

Let μ be an angle such that $\tan \mu = c_1/c_2$, so that $\sin \mu = c_1/c,\ \cos \mu = c_2/c$. Then (3.29) can be written as

$$x = c\left(\sin \mu \cos \sqrt{s}t + \cos \mu \sin \sqrt{s}t\right), \tag{3.30}$$

or

$$x = c \sin(\sqrt{s}t + \mu). \tag{3.31}$$

You should verify that had we interchanged c_1 and c_2 in (3.28) and replaced μ by $-\mu$, then the solution of (3.27) could be written as

$$x = c \cos(\sqrt{s}t + \mu). \tag{3.32}$$

Both (3.31) and (3.32) are similar to equations (3.25) showing that the motion is indeed a simple harmonic motion with amplitude c and period $2\pi/\sqrt{s}$.

To summarize, a simple harmonic motion can be described by either of the equations (3.25)*, or equivalently by the differential equation* (3.26)*. The period is readily readable from any of these equations.*

Equation (3.26) provides the period ω, but not the amplitude c. To uniquely determine the solution given by equations (3.25), from Theorem 3.1 of this chapter, we need two initial conditions that specify $x(t_0)$ and $x'(t_0)$. Often $t_0 = 0$.

Example 3.6. *An object executes a simple harmonic motion with frequency 6 radians/sec. It starts at the origin (its equilibrium position) with initial velocity 5 ft/sec. Find (a) its period, (b) the differential equation satisfied by the object, (c) the equation of motion, (d) its amplitude, and (e) the phase angle.*

Solution: Here $f = 6$ radians/sec $= 6/2\pi = 3/\pi$ cycles/sec, hence the period $T = 2\pi/\omega = \pi/3$ sec. Clearly, $\omega = 6$ radians/sec, and the differential equation satisfied by the object is

$$\frac{d^2x}{dt^2} + 36x = 0.$$

For the equation of motion, let us pick the first equation in (3.25). Then

$$x = c\cos(6t + \delta), \quad v = \frac{dx}{dt} = -6c\sin(6t + \delta).$$

At $t = 0$ we have

$$x = 0 = c\cos\delta, \quad v = 5 = -6c\sin\delta.$$

Since $c > 0$, from the first equation we have $\delta = \pm\pi/2 \Rightarrow \sin\delta = \pm 1$. From the second equation, $\sin\delta < 0$, that is $\delta = -\pi/2$. It follows that the amplitude $c = 5/6$ ft, the phase angle $\delta = -\pi/2$ radians, and the equation of motion is

$$x = \frac{5}{6}\cos\left(6t - \frac{\pi}{2}\right).$$

1. Hooke's law

Assume that a toy cart of mass m is attached to a wall by means of a spring. When the cart is at rest at the point O, there are no forces acting on it except its weight. We choose O for the origin, and initially $x = 0$. When the cart is displaced by a distance x from O, away from the wall, then the spring's tension exerts a restoring force F_x (see Figure 3.5). According to Hooke's law, this force is proportional to x, that is,

$$F_x \propto x, \text{ or } F_x(x) = -kx. \tag{3.33}$$

The negative sign shows that the force acts in the negative direction; in fact F_x always acts against the motion and hence has a sign opposite to that of x. The constant of proportionality $k > 0$ is a measure of the stiffness of the spring. If the spring is very stiff, i.e. k is large, common sense tells us that there will be little or no vibration.

The only force that influences the motion of the cart is the restorative force of the spring. By Newton's second law of motion, the force F_x acting on the cart equals the product of its mass and acceleration. Hence, from (3.33),

$$F_x = m\frac{d^2x}{dt^2} = -kx,$$

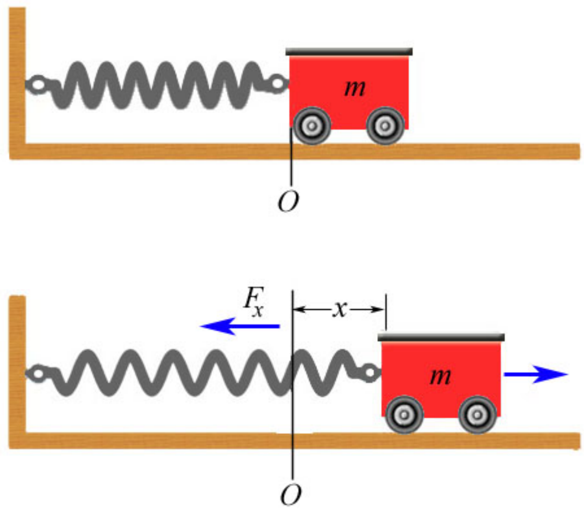


Figure 3.5: Mass-spring oscillator

from which

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

It is immediate that the cart executes a simple harmonic motion whose equation of motion is

$$x = c\cos(\sqrt{k/m}\; t + \delta). \tag{3.34}$$

It is clear that the cart has an amplitude c, period T and frequency f, which by substituting for ω, we have

$$T = 2\pi\sqrt{\frac{m}{k}}, \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

The expression for T shows that if the mass of the cart is large or the stiffness of the spring is light, then T is larger, that is it takes more time to complete one period (as we noticed earlier) and the frequency f is correspondingly smaller.

From (3.34), since $|x| \le c$, the cart oscillates around O with maximum displacement $\pm c$. If we write $\omega = \sqrt{k/m}$, it follows from (3.34) that the velocity v of the cart is given by

$$v = \frac{dx}{dt} = -c\,\omega\sin(\omega t + \delta) \tag{3.35}$$

and when $|x| = c$, that is at the end points, $|\cos(\omega t+\delta)| = 1$, that is, $|\sin(\omega t+\delta)| = 0$. At the end points, where $x = \pm c$, the speed of the cart is $s = |dx/dt| = 0$.

When the cart is at the origin, from (3.34), $\cos(\omega t+\delta)=0$ and $\sin(\omega t+\delta)=\pm 1$. Hence, at the origin the cart has its maximum speed $s=c\omega$.

The motion of the cart is thus clear. Once it is pulled to the right and let go (with no initial velocity), it moves towards the wall with increasing speed reaching the maximum at the origin. The speed then reduces becoming zero at the left end point. The cart then moves to the right, again reaching its maximum speed at the origin and moving to the right with reducing speed.

We can simplify (3.34) in specific cases if we know $x(t)$ and $v(t)$ for some $t=t_0$. Suppose for instance that at $t=0$, the cart is pulled through a distance x_0 and let go with no initial velocity $v=0$. Substituting these values in (3.34) and (3.35) we get,

$$x_0 = c\cos\delta, \quad 0 = -c\omega\sin\delta,$$

from which $\delta=0,\ c=x_0$. Hence in this particular case, the equation of motion is

$$x = x_0\cos\omega t = x_0\cos\sqrt{k/m}\,t.$$

2. Simple pendulum

We now consider the motion of simple pendulum, consisting of a "pendulum bob" attached to one end of a rod. The other end of the rod is attached to a fixed point C in such a way that the pendulum is free to move in an oscillatory motion.

Let the length of the rod be ℓ and the mass of the bob be m. Assume that the bob is pulled to the right through a small angle and let go. Let $P(x,y)$ be the coordinates of the bob at any time t, and θ the angle the pendulum makes with the vertical. The bob moves along the arc of a circle centered at C with radius ℓ. Let CO be the vertical line through C such that $CO=\ell$. The length of the arc PO is $s=\ell\theta$. The only force acting on the bob is its weight $w=mg$ and it acts downwards. We resolve this force into two forces: One component along CP and another perpendicular to it. The latter is tangential to the circular arc PO. The force along CP is $mg\cos\theta$ and the one at right angle to it is the force $F=mg\sin\theta$. It is clear that F acts along the tangent to the arc PO and is the only *restorative force* trying to move the pendulum to the equilibrium position CO.

As we saw in the previous case (of the cart attached to the wall), F and θ have opposite signs. When the pendulum moves counterclockwise, F moves clockwise and vice versa. The velocity v of the pendulum acts tangentially to the arc PO. By

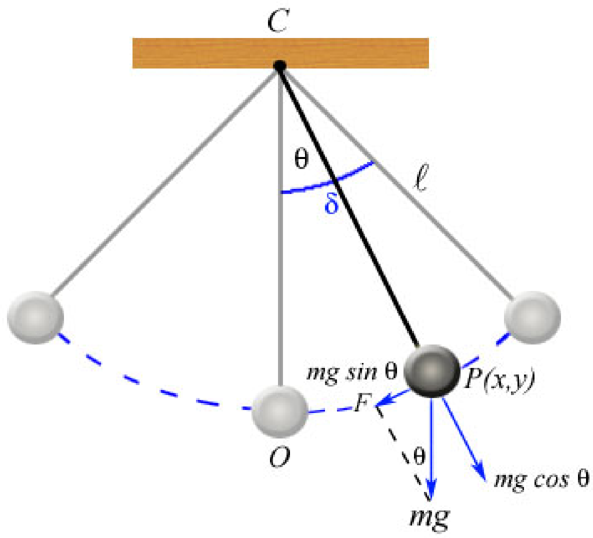


Figure 3.6: Simple pendulum

Newton's second law,

$$F = m\frac{dv}{dt} = -mg\sin\theta, \tag{3.36}$$

the negative emphasizing F and θ (and hence $\sin\theta$) have opposite signs. We have

$$v = \frac{ds}{dt}, \quad s = \ell\theta, \quad \frac{dv}{dt} = \ell\frac{d^2\theta}{dt^2}.$$

It follows from equation (3.36) that

$$\ell\frac{d^2\theta}{dt^2} + g\sin\theta = 0, \tag{3.37}$$

which is the equation satisfied by the pendulum. Equation (3.37) does not define a simple harmonic motion! But we know that

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots,$$

and for small values of θ, $\sin\theta \approx \theta$. Hence, we can rewrite (3.37) as

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0, \tag{3.38}$$

which now describes a simple harmonic motion. If we write the solution to (3.38) as

$$\theta = c\cos(\sqrt{g/\ell}\; t + \delta), \tag{3.39}$$

its amplitude is c with phase angle δ. Its period T and its frequency f are given by

$$T = 2\pi\sqrt{\ell/g}, \quad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{g/\ell}. \tag{3.40}$$

3. Simple pendulum (continued)

Assume that the pendulum is pulled through an angle α at $t = 0$. If α is not small, θ need not be small enough for us to approximate $\sin\theta$ by θ.

Let $\omega = d\theta/dt$ be the angular velocity of the bob. If we multiply both sides of equation (3.37) by $d\theta/dt$ we get

$$\ell\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + g\sin\theta\frac{d\theta}{dt} = 0,$$

and integrating the last equation it follows that

$$\ell\Big(\frac{d\theta}{dt}\Big)^2 - 2g\cos\theta = c_1,$$

for some constant c_1. At $t = 0,\ \theta = \alpha, \omega = d\theta/dt = 0$. It follows that $c_1 = -2g\cos\alpha$. Substituting this in the last equation and taking square roots we obtain

$$\Big(\frac{d\theta}{dt}\Big)^2 = \frac{2g}{\ell}(\cos\theta - \cos\alpha),$$

that is,

$$\omega = \frac{d\theta}{dt} = \pm\big(\sqrt{2g/\ell}\big)\sqrt{\cos\theta - \cos\alpha}.$$

Hence,

$$dt = \pm\big(\sqrt{\ell/2g}\big)\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

We integrate the last equation to finally obtain

$$t(\theta) = \pm\big(\sqrt{\ell/2g}\big)\int_\alpha^\theta \frac{d\eta}{\sqrt{\cos\eta - \cos\alpha}}.$$

To find the period T of the pendulum, we notice that as the pendulum swings from $\theta = -\alpha$ to $\theta = \alpha$, it covers one half of the period. Thus,

$$\frac{T}{2} = \left(\sqrt{\ell/2g}\right) \int_{-\alpha}^{\alpha} \frac{d\eta}{\sqrt{\cos\eta - \cos\alpha}}. \tag{3.41}$$

Further, $\cos\eta$ is an even function so it follows that

$$T = 2\sqrt{2}\left(\sqrt{\ell/g}\right) \int_{0}^{\alpha} \frac{d\eta}{\sqrt{\cos\eta - \cos\alpha}}. \tag{3.42}$$

Alas, as elegant as formula (3.42) is, the integral is not easy to evaluate. The rest of what follows should give you an idea of how mathematicians tackle problem of this kind. We use the double angle formula from trigonometry, $\cos\theta = 1 - 2\sin^2(\theta/2)$, etc., to rewrite (3.42) as

$$\begin{aligned} T &= 2\left(\sqrt{\ell/g}\right) \int_{0}^{\alpha} \frac{d\eta}{\sqrt{\sin^2(\alpha/2) - \sin^2(\eta/2)}} \\ &= 2\left(\sqrt{\ell/g}\right) \int_{0}^{\alpha} \frac{d\eta}{\sqrt{u^2 - \sin^2(\eta/2)}}, \end{aligned} \tag{3.43}$$

where $u = \sin(\alpha/2)$. If we make the substitution

$$\sin\frac{\eta}{2} = u\sin\phi,$$

we note that as η goes from 0 to α, then ϕ goes from 0 to $\pi/2$ and

$$\frac{1}{2}\cos\frac{\eta}{2}\,d\eta = u\cos\phi\,d\phi.$$

Substituting this in (3.43) gives us

$$T = 4\left(\sqrt{\ell/g}\right) \int_{0}^{\pi/2} \frac{u\cos\phi d\phi}{\cos(\eta/2)\sqrt{u^2 - u^2\sin^2\phi}},$$

which finally simplifies to

$$T = 4\left(\sqrt{\ell/g}\right) \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - u^2\sin^2\phi}}. \tag{3.44}$$

The integral in (3.44) cannot be evaluated in terms of elementary functions. It is called an ***elliptic integral of the first kind*** and occurs frequently in physics and in evaluating the circumference of an ellipse. Tables of elliptic integrals have been computed for values of u.

It can be shown (the details are beyond the scope of these notes) that

$$T = 2\pi\sqrt{\ell/g}\left\{1 + \frac{u^2}{2^2} + \frac{(1\cdot 3)^2}{(2\cdot 4)^2}u^4 + \cdots\right\} \tag{3.45}$$

and this gives the ***actual period of the pendulum***.

For example, if $\alpha = 6^\circ$, then $u = \sin 3^\circ = 0.0523, k^2 = 0.002739$ and from (3.45)

$$T(\text{true value}) = 2\pi\sqrt{\ell/g}\left\{1 + \frac{0.002739}{4} + \frac{9}{64}\cdot(0.002739)^2 + \ldots\right\}$$

from which

$$T \approx 2\pi\sqrt{\ell/g}\,(1.0006858).$$

The value given by equation (3.45) of course is $T \approx 2\pi\sqrt{\ell/g}$. You should note that as the pendulum completes several periods, the error is no longer negligible.

3.4 Exercises

Find the general solution to each of the following equations except where noted.

1. $y'' - 3y' + 2y = 0.$
2. $4y'' - 7y' + 3y = 0.$
3. $2y'' + y' - 3y = 0.$
4. $y'' - 2y' + y = 0.$
5. $y'' + 4y' + 4y = 0.$
6. $y''' + 3y'' + y' + 3y = 0.$
7. Solve the initial value problem: $(D^2 - 6D + 25)y = 0$ given that $y(0) = -3, y'(0) = -1.$
8. $(D^3 - 6D^2 + 5D + 12)y = 0.$
9. $16y'' + 32y' + 25y = 0.$
10. $(8D^3 - 12D^2 + 6D + 1)y = 0.$
11. A simple pendulum of length 5 feet is released from a position of 1 radian. Find

(a) the position of the pendulum as a function of time

(b) the amplitude, period and frequency

(c) the velocity (both linear and angular) with which the pendulum crosses the equilibrium position.

12. A clock pendulum is regulated so that it crosses the equilibrium position every 2 seconds. What is the length of the pendulum?

13. An object weighting 4 lbs is attached to the end of helical spring which stretches it by 6 inches. The spring oscillates and when it comes to rest, the object is pulled an additional 6 inches and let go. Show that it executes a simple harmonic motion. Find its

    (a) equation of motion

    (b) period

    (c) frequency

    (d) amplitude

Chapter 4

Nonhomogeneous Linear Equations

4.1 Introduction

In this chapter we continue our study of higher order linear equations which are not necessarily homogeneous, but whose coefficients are constants:

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx} + a_n y = Q(x). \tag{4.1}$$

Here $a_0, a_1, \ldots, a_n$ are constants and $Q(x)$ is a continuous function of x. Using the operator notation D, equation (4.1) can be written as

$$\left(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1}D + a_n\right) y = Q(x) \tag{4.2}$$

and we will employ this notation throughout this chapter (for the ease of writing as well as other advantages that will be apparent as we proceed). Notice that the left side of (4.2) is a polynomial in D of degree n, so we can write (4.2) simply as

$$p_n(D)y = Q(x), \tag{4.3}$$

omitting the subscript n when the degree of $p(D)$ is clear from the context. For example, the equation

$$2\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin x$$

can be written more simply as $p(D)y = \sin x$, where $p(D) \equiv 2D^3 + 3D^2 + D + 1$.

We will find the following theorem quite useful later.

Theorem 4.1. ***(Principle of Superposition)***
Let $p(D)$ be a polynomial in D, $f(x)$ be a solution of $p(D)y = U(x)$, and $g(x)$ a solution of $p(D)y = V(x)$. Then $f(x) + g(x)$ is a solution of $p(D)y = U(x) + V(x)$.

Proof: The Principle of Superposition says in an equation such as (4.3), where the right side $Q(x)$ is the sum of several terms, we can split it into several subproblems and combine their solutions to find a solution of the given equation. The proof simply depends on the property of derivatives. Since $D(f(x) + g(x)) = Df(x) + Dg(x)$, $D^2(f(x) + g(x)) = D^2f(x) + D^2g(x)$, and so on, it is easy to see that $p(D)(f(x) + g(x)) = p(D)f(x) + p(D)g(x) = U(x) + V(x)$. □

As an illustration of the above principle, consider the equation

$$(D^4 + D^2)y = 3x^2 + 4\sin x.$$

We will learn later in this chapter that a solution of $(D^4 + D^2)y = 3x^2$ is $f(x) = \frac{1}{4}x^4 - 3x^2$ (verify!) and a solution of $(D^4 + D^2)y = 4\sin x$ is $g(x) = 2x\cos x$ (verify!). Hence, a solution of the given equation is $f(x) + g(x) = \frac{1}{4}x^4 - 3x^2 + 2x\cos x$.

4.2 Solution of nonhomogeneous linear equations

We begin by noting that Theorem 3.1 of Chapter 3 on linear equations still applies to equation (4.3). In particular, given n arbitrary constants $k_1, k_2, \ldots, k_n$, there exists a unique solution $f(x)$ of (4.3).

Since $Q(x) = Q(x) + \mathbf{0}$, where $\mathbf{0}$ is the zero function $f(x) \equiv 0$, we can think of the solution (4.3) as arising from two equations

$$p_n(D)y = \mathbf{0} \tag{4.4}$$

and

$$p_n(D)y = Q(x) \tag{4.5}$$

and apply Theorem 4.1. Equation (4.4) is called the ***complementary part*** of the solution of (4.3). It is simply a homogeneous linear equation with constant coefficients and can be solved by the methods of Chapter 3. The general solution of (4.4), often denoted by y_c, is of the form

$$y_c = c_1y_1 + c_2y_2 + \ldots + c_ny_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants and $y_1, y_2, \ldots, y_n$ are linearly independent solutions of (4.4).

Of course we still don't know the *general* solution of (4.3) but assume that somehow we have managed to find a ***particular solution*** y_p of (4.5), which is *linearly independent* of $y_1, \ldots y_n$. The Principle of Superposition (Theorem 4.1) tells us that $y = y_c + y_p$ is a solution of equation (4.3). It is clear from these observations that a general solution of (4.3) has n arbitrary constants and can simply be obtained by solving the complementary part (4.4) and somehow finding a particular solution from (4.5) and adding the two. Thus, the general solution of (4.3) is of the form

$$y = y_c + y_p = c_1y_1 + c_2y_2 + \ldots + c_ny_n + y_p.$$

The rest of this chapter is devoted to finding a particular solution y_p of (4.3).

Remark 9. We do not include in y_p any term that already appears in y_c. Why? Because such a term, say $f(x)$, satisfies (4.4) and therefore cannot possibly satisfy (4.5). *The terms in y_p must therefore be linearly independent from the terms in y_c.*

4.3 Method of Undetermined Coefficients

In the method of Undetermined Coefficients (UC), one writes down y_p as a linear combination of functions

$$y_p = A_1f_1(x) + A_2f_2(x) + \cdots + A_mf_m(x),$$

for some integer m. The functions $f_1, f_2, \ldots, f_m$ are obtained by carefully examining all the functions that appear in y_c and $Q(x)$.

The coefficients $A_1, A_2, \ldots, A_m$ are determined (hence the name of the method) by substituting y_p for y in (4.5). This gives rise to m equations to determine the constants $A_1, A_2, \ldots, A_m$. As you might guess, the method is often laborious!

If you observe equation (4.3), it is evident that $Q(x)$ is a linear combination of y_p and its *linearly independent derivatives.* In fact, y_c, the solution of the homogeneous part of (4.3), is a linear combination of functions that have finitely many linearly independent derivatives. It follows that the UC method can be used *only if* $Q(x)$ *consists of functions that have finitely many linearly independent derivatives* as well. This restricts the use of the method to a small class of functions: exponentials (e^{mx}), basic circular functions ($\sin ax$, $\cos ax$) and polynomials (x^n). Indeed, e^{mx} has no linearly independent derivative, both $\sin ax, \cos ax$ have just one each, and x^n

has n linearly independent derivatives, namely $x^{n-1}, x^{n-2}, \ldots, x^2, x, 1$, omitting the constants (verify these assertions!). Luckily these are the functions that appear most in applications. On the other hand, $1/x$ has derivatives $1/x^2, 1/x^3, \ldots$ (omitting constants) which are all linearly independent. When $Q(x)$ involves functions such as $1/x, \tan x$, etc., that have infinitely many linearly independent derivatives, one has to take recourse to other methods.

Remark 10. In what follows, we say that the equation

$$p_n(x) = k_0x^n + k_1x^{n-1} + \cdots + k_n = 0$$

has a root $x = a$ of *multiplicity* r if

1. a is a multiple root of the equation, and
2. a occurs as a root r times.

In this case $(x - a)^r$ is a factor of the left side of the equation. For example, given the differential equation $y'' - 6y' + 9y = 0$. Its auxiliary equation $m^2 - 6m + 9 = 0$ has the root $m = 3$ with multiplicity $r = 2$ since $m^2 - 6m + 9 = (m - 3)^2$.

4.3.1 How the method of Undetermined Coefficients works

In the UC method, one builds a possible function y_p by considering every term $q(x)$ that appears in $Q(x)$ and examines it with the terms in y_c. We first determine which of the following three cases $q(x)$ belongs to and start adding terms to y_p accordingly.

Case 1: $q(x)$ is not a term in y_c
In this case y_p includes a linear combination of $q(x)$ and all of its linearly independent derivatives.

Example 4.1. *Find the form of the particular solution y_p for the equation*

$$y'' + 3y' + 2y = 2x^2$$

without solving it.
Solution: The auxiliary equation is $m^2 + 3m + 2 = 0$ and the roots are -1 and -2. Hence,

$$y_c = c_1e^{-x} + c_2e^{-2x}.$$

The only choice for $q(x)$ is $q(x) = x^2$ (omitting constants) and it is not a term in y_c. Thus, y_p is simply a linear combination of x^2 and all of its linearly independent derivatives, namely x and 1, and it can be written as

$$y_p = Ax^2 + Bx + C.$$

The constants A, B, C will have to be determined later.

Example 4.2. *Find the form of the particular solution y_p for the equation*

$$y'' - 2y' - 3y = 2e^x - 10\sin x$$

without solving it.

Solution: Here the auxiliary equation is $m^2 - 2m - 3 = 0,\ m = 3, -1$, so that

$$y_c = c_1 e^{3x} + c_2 e^{-x}.$$

Here, there are two choices for $q(x)$, namely $q(x) = e^x$ and $q(x) = \sin x$, and neither is a term of y_c. Thus, y_p consists of a linear combination of e^x and its linearly independent derivatives. Similarly for $\sin x$. But the derivative of e^x is itself so that it has no linearly independent derivative (why?). The only linearly independent derivative of $\sin x$ is $\cos x$ (why?). Hence,

$$y_p = Ae^x + B\sin x + C\cos x,$$

where the constants A, B, C are yet to be determined.

Case 2:

$$\begin{cases} \boldsymbol{q(x) = x^k f(x),\ k \geq 0} \\ \boldsymbol{f(x)} \textbf{ is a term in } \boldsymbol{y_c} \\ \boldsymbol{f(x)} \textbf{ does not arise from a multiple root of the auxiliary equation} \end{cases}$$

Here y_p includes a linear combination of $x^{k+1}f(x)$ and all of its linearly independent derivatives.

The reason for this should be clear. In the simple case when $k = 0$ (i.e. when $q(x)$ appears in y_c), it solves the complementary part (4.4). In Chapter 3, when dealing with repeated roots of the auxiliary equation, we handled this as follows: if $m = a$ repeats r times and the solution corresponding to $m = a$ is $f(x)$, then we added to y_c the terms $f(x), xf(x), x^2 f(x), \ldots, x^{r-1} f(x)$.

Let us consider some examples.

Example 4.3. *Find the form of the particular solution y_p for the equation*

$$(D^2 - 3D + 2)y = 2x^2 + e^x \tag{4.6}$$

without solving it.

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0$ with roots $m = 1, 2$. Hence,

$$y_c = c_1 e^x + c_2 e^{2x}.$$

Next we start building y_p.

Split the right hand side $Q(x) = 2x^2 + e^x$ of (4.6). Omitting the coefficients, let $q_1(x) = x^2$, $q_2(x) = e^x$. Consider the first term $q_1(x) = x^2$. This does not appear in y_c, so it belongs to Case 1. Hence, we include a linear combination of x^2 and its linearly independent derivatives, i.e. x and 1, to obtain

$$y_p = Ax^2 + Bx + C.$$

Next consider the second term $q_2(x) = e^x$. This is of the form $x^0 f(x)$, where $f(x) = e^x$ is a term in y_c and thus belongs to Case 2. We therefore include a linear combination of $x^{0+1}e^x = xe^x$ and all of its linearly independent derivatives, which is simply e^x. But the latter is already in y_c and need not be included (Remark 9). Thus, the final form of y_p is

$$y_p = \underbrace{Ax^2 + Bx + C}_{I} + \underbrace{Exe^x}_{II},$$

where the terms in group I come from $q_1(x) = x^2$ and those in group II from $q_2(x) = e^x$. This example shows that different terms in $Q(x)$ can belong to different cases! Note also that the constants A, B, C and E will have to be determined and we avoid using D to denote a constant as it is reserved for differential operator.

Example 4.4. *Find an appropriate form of y_p for the equation*

$$(D^2 - 5D + 6)\, y = xe^{2x}.$$

Solution: Here $y_c = c_1 e^{2x} + c_2 e^{3x}$. The only term $q(x)$ is xe^{2x}, which is of the form $x^1 f(x)$ with $k = 1$ and $f(x) = e^{2x}$ is a term in y_c. Hence, $q(x)$ belongs to Case 2 and y_p is a linear combination of $x^{(1+1)}e^{2x} = x^2 e^{2x}$ and its linearly independent derivatives, that is of, xe^{2x} and e^{2x}. But e^{2x} is already a term in y_c, thus need not be included (Remark 9). The particular solution is then given by

$$y_p = Ax^2e^{2x} + Bxe^{2x}.$$

Case 3:

$$\begin{cases} \boldsymbol{q(x) = x^k f(x),\ k \geq 0} \\ \boldsymbol{f(x)} \textbf{ is a term in } \boldsymbol{y_c} \\ \boldsymbol{f(x)} \textbf{ arises from a root of the auxiliary equation with multiplicity } r \end{cases}$$

Here y_p includes $x^{k+r}f(x)$ and all of its linearly independent derivatives.

Remark 11. For Case 3, two things must happen. The auxiliary equation must have a root k with multiplicity r, $f(x)$ is a solution corresponding to this root, and $q(x) = x^k f(x),\ k \geq 0$.

Example 4.5. *Find the form of particular solution y_p for the equation*

$$(D^4 + D^2)y = 3x^2 + 4\sin x.$$

Solution: The auxiliary equation is $m^4 + m^2 = 0$ with roots $m = 0, 0, \pm i$. The root $m = 0$ is of multiplicity 2 ($r = 2$) and the function corresponding to this solution is $f(x) = 1$. Here

$$y_c = (c_1 + c_2x) + c_3 \sin x + c_4 \cos x.$$

First consider $q_1(x) = x^2$. We can write this either as $x^2 f(x)$, where $k = 2$ and $f(x) = 1$ is in y_c. Hence, y_p will include $x^{k+r}f(x) = x^{2+2}\,1 = x^4$ and all of its linearly independent derivatives, namely $x^3, x^2, x, 1$. But x and 1 are already in y_c and therefore need not be included in y_p. So far we have

$$y_p = Ax^4 + Bx^3 + Cx^2.$$

Alternatively, we can also think of $q_1(x) = x^2$ as $x^1 f(x)$ where $k = 1$ and $f(x) = x$ is in y_c. From this point of view, y_p will consist of $x^{k+r}f(x) = x^{1+2}(x) = x^4$, just as before.

Finally consider $q_2(x) = \sin x$. This belongs to Case 2 since $\sin x = x^0 f(x)$, where $f(x) = \sin x$ is in y_c. Hence, we also need to include a linear combination of the derivatives of $x^{0+1}f(x)$, that is of $x\sin x, x\cos x, \sin x, \cos x$. However, the last two need not be included since they are included in y_c. Thus,

$$y_p = (Ax^4 + Bx^3 + Cx^2) + x(E\sin x + F\cos x).$$

Example 4.6. *Find the form of particular solution y_p for the equation*

$$(D-2)^2\, y = x^2 e^{2x}.$$

Solution: The auxiliary equation is $(m-2)^2 = 0$ and $m = 2, 2$. Thus, the root $m = 2$ has multiplicity 2 also. Hence,

$$y_c = c_1 e^{2x} + c_2 x e^{2x}.$$

Consider $q(x) = x^2 e^{2x}$, the only term in $Q(x)$. There are two possibilities. We can write this as $x^k f(x)$ with $k = 2, f(x) = e^{2x}$, where $f(x)$ of course is a term in y_c. Alternatively, we can also write $q(x) = x^k f(x), k = 1, f(x) = xe^{2x}$, $f(x)$ is a term in y_c. Since $q(x)$ belongs to Case 3, considering the first point of view we should include $x^{k+r} f(x) = x^{2+2}\, e^{2x} = x^4 e^{2x}$ and all of its linearly independent derivatives, namely $x^3 e^{2x}, x^2 e^{2x}, xe^{2x}, e^{2x}$. But the last two are in y_c and need not be included. You should verify you get the same result from the alternative point of view also. Thus,

$$y_p = (Ax^4 + Bx^3 + Cx^2)e^{2x}.$$

4.3.2 Determining coefficients

In the previous section we learned how to construct y_p with coefficients yet to be determined. In this section we shall see how one uses y_p to find the coefficients and thus write down the solution of (4.3). All one does is to actually substitute y_p for y in the left side of (4.3) and compare the coefficients of like terms on the left and right. This gives rise to a set of linear equations in an equal number of unknowns and they are solved by the usual method.

The process is best learned through some examples.

Example 4.7. *Solve the equation in Example 4.1: $y'' + 3y' + 2y = 2x^2$.*

Solution: From Example 4.1,

$$y_c = c_1 e^{-x} + c_2 e^{-2x} \quad \text{and} \quad y_p = Ax^2 + Bx + C$$

and hence,

$$y_p' = 2Ax + B, \quad y_p'' = 2A.$$

Substituting in the original equation we get

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= 2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) \\ &= 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) \\ &\equiv 2x^2. \end{aligned}$$

Comparing the coefficients of x^2, x and the constant terms, we have

$$2A = 2, \quad 6A + 2B = 0, \quad 2A + 3B + 2C = 0,$$

from which $A = 1, B = -3, C = 7/2$ and the complete solution is

$$\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{-x} + c_2 e^{-2x} + x^2 - 3x + 7/2. \end{aligned}$$

Example 4.8. *Solve the equation in Example 4.3:* $(D^2 - 3D + 2)y = 2x^2 + e^x$.
Solution: We found that

$$y_c = c_1 e^x + c_2 e^{2x} \text{ and } y_p = Ax^2 + Bx + C + Exe^x.$$

Hence,

$$\begin{aligned} Dy_p &= 2Ax + B + Exe^x + Ee^x \\ D^2 y_p &= 2A + Exe^x + 2Ee^x \end{aligned}$$

$$\begin{aligned} (D^2 - 3D + 2)y_p &= 2Ax^2 + (2B - 6A)x + (2A - 3B + 2C) - Ee^x \\ &\equiv 2x^2 + e^x \end{aligned}$$

and it follows that $A = 1, B = 3, C = \frac{7}{2}, E = -1$. Hence,

$$y_p = x^2 + 3x + \frac{7}{2} - xe^x$$

and the complete solution is $y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + \frac{7}{2} - xe^x$.

Example 4.9. *Solve the equation in Example 4.4:* $(D^2 - 5D + 6)\, y = xe^{2x}$.
Solution: From Example 4.4,

$$y_c = c_1 e^{2x} + c_2 e^{3x} \quad \text{and} \quad y_p = Ax^2 e^{2x} + Bxe^{2x}.$$

We thus get

$$\begin{aligned} Dy_p &= 2Ax^2 e^{2x} + (2A + 2B)xe^{2x} + Be^{2x} \\ D^2 y_p &= 4Ax^2 e^{2x} + (8A + 4B)xe^{2x} + (2A + 4B)e^{2x} \\ (D^2 - 5D + 6)\, y_p &= -2Axe^{2x} + (2A - B)e^{2x} \\ &\equiv xe^{2x} \end{aligned}$$

from which it follows that $A = -\frac{1}{2}$ and $B = -1$ and the complete solution is $y = c_1 e^{2x} + c_2 e^{3x} - \frac{1}{2}x^2 e^{2x} - xe^{2x}$.

Example 4.10. *Solve the equation in Example 4.5:* $(D^4 + D^2)\, y = 3x^2 + 4 \sin x$.
Solution: As we found earlier,

$$\begin{aligned} y_c &= (c_1 + c_2 x) + c_3 \sin x + c_4 \cos x \\ y_p &= (Ax^4 + Bx^3 + Cx^2) + x(E \sin x + F \cos x). \end{aligned}$$

Computing the derivatives of y_p,

$$\begin{aligned} Dy_p &= (4Ax^3 + 3Bx^2 + 2Cx) + (E \sin x + F \cos x) + x(E \cos x - F \sin x) \\ D^2 y_p &= (12Ax^2 + 6Bx + 2C) + 2(E \cos x - F \sin x) + x(-E \sin x - F \cos x) \\ D^3 y_p &= (24Ax + 6B) + 3(-E \sin x - F \cos x) + x(-E \cos x + F \sin x) \\ D^4 y_p &= 24A + 4(-E \cos x + F \sin x) + x(E \sin x + F \cos x) \end{aligned}$$

$$\begin{aligned} (D^4 + D^2)\, y_p &= (12Ax^2 + 6Bx + (2C + 24A)) + 4(-E \cos x + F \sin x) + x(E \sin x + F \cos x) \\ &\quad + 2(E \cos x - F \sin x) - x(E \sin x + F \cos x) \\ &= (12Ax^2 + 6Bx + (2C + 24A)) - 2(E \cos x - F \sin x) \\ &\equiv 3x^2 + 4 \sin x \end{aligned}$$

from which we get $A = \frac{1}{4}, B = 0, C = -3, E = 0, F = 2$ and

$$y_p = \frac{1}{4}x^4 - 3x^2 + 2x \cos x$$

and the complete solution is

$$y = (c_1 + c_2 x) + c_3 \sin x + c_4 \cos x + \frac{1}{4}x^4 - 3x^2 + 2x \cos x.$$

Example 4.11. *Solve the equation $y'' + a^2 y = \cos ax$.*

Solution: It is easy to see that $y_c = c_1 \cos ax + c_2 \sin ax$. To find y_p, here $q(x) = \cos ax = x^0 f(x)$ where $f(x) = \cos ax$ is a term in y_c. Hence, this is Case 2 and accordingly we assume

$$y_p = x(A \cos ax + B \sin ax).$$

Then it is easy to verify that

$$\begin{aligned} y_p'' &= x(-a^2 A \cos ax - a^2 B \sin ax) + 2(-aA \sin ax + aB \cos ax) \\ y_p'' + a^2 y &= 2(-aA \sin ax + aB \cos ax) \equiv \cos ax. \end{aligned}$$

Hence, $A = 0, B = 1/2a$ and

$$y_p = (x/2a) \sin ax.$$

Had the right hand side $Q(x)$ of the given equation been $\sin ax$, we would have $y_p = -(x/2a) \cos ax$. Thus,

$$y_p = \frac{x}{2a} \begin{Bmatrix} -\cos ax \\ +\sin ax \end{Bmatrix} \quad \text{according as} \quad Q(x) = \begin{Bmatrix} \sin ax \\ \cos ax \end{Bmatrix} \tag{4.7}$$

Final Comments: The UC method is not difficult since it involves carefully writing down the correct form of y_p, but is cumbersome, laborious and tiring. This is the bane of the method. In the next section we will learn how other methods would handle such problems more simply and elegantly. These methods automatically determine the correct form for y_p and then proceed to find the coefficients! Nevertheless when $Q(x)$ consists of polynomials of degree no higher than two, the UC method would be hard to beat.

4.4 Differential Operator Method

In this section we will still be concerned with determining y_p but we will be using the method of *Differential Operators*. This is both an elegant as well as far simpler method than the UC method. *This material is usually not found in most standard texts!*

The operator $D \equiv \frac{d}{dx}$ is versatile in the sense that it represents the process of taking the derivative and at the same time can be manipulated as if it were an algebraic symbol! For example, we can think of D as an ordinary algebraic symbol and write

$$(D+3)(D+2) \equiv (D^2+5D+6).$$

As a derivative operator, for any twice differentiable function $y = f(x)$,

$$\begin{aligned}
(D+3)(D+2)y &= (D+3)(Dy+2y) \\
&= (D+3)(y'+2y) \\
&= Dy' + 2Dy + 3y' + 6y \\
&= y'' + 5y' + 6y \\
&= (D^2y + 5Dy + 6y) \\
&= (D^2+5D+6)y.
\end{aligned}$$

It follows that D may be treated as just a symbol and at the same time acts as derivative operator (***Caution:*** *Only if the coefficients are constants!*). Note further that since the coefficients are all constants,

$$(D+3)(D+2) \equiv (D+2)(D+3),$$

that is, the factors commute. This is an important fact.

4.4.1 Meaning of the operator $\frac{1}{p\,(D)}$

Just as multiplication and division are "opposite" operations in the sense one nullifies the action of the other, so are D and $1/D$. Since D represents taking the derivative, *we can interpret* $1/D$ *as the opposite of taking derivative, that is, it represents integration.* Thus,

$$\frac{1}{D}x = \int x\,dx = x^2/2 + c.$$

Returning to equation (4.1) to fix ideas, let us assume that

$$p\,(D) = (D^2 + aD + b),$$

so that equation (4.1) can be written as

$$p\,(D)y = (D^2 + aD + b)y = Q(x).$$

If we treat $p\,(D)$ as a polynomial in the differential operator D, operating on both sides by $1/p\,(D)$, the *inverse operator of* $p\,(D)$ (note that we still don't know its meaning other than that it nullifies the action of $p\,(D)$ in the sense $p\,(D)\{1/p\,(D)\} = I$, where I is the identity operator), we get

$$y_p = \frac{1}{p\,(D)}\; Q(x). \tag{4.8}$$

In the rest of this chapter we will investigate the meaning of the right hand side of (4.8) for various functions $Q(x)$.

Note: *From now on, we will think of* $1/p\,(D)$ *as the operator which does the opposite of* $p\,(D)$*, that is, they are inverse operators in the sense*

$$\left\{p\,(D)\frac{1}{p(D)}\right\}y = \left\{\frac{1}{p(D)}p(D)\right\}y = y.$$

Remark 12. For any constant a, we have $D(ay) = aDy, D^2(ay) = aD^2y$, etc., and in general,

$$D^k(ay) = a(D^k y) \text{ and } p(D)ay = ap(D)y.$$

This property holds for inverse operators also, that is,

$$\frac{1}{p(D)}ay = a\frac{1}{p(D)}y.$$

To see this, operate on both sides by $p(D)$. On the left hand side we get ay at once and on the right,

$$p(D)\left\{a\frac{1}{p(D)}y\right\} = a\left\{p(D)\frac{1}{p(D)}\right\}y = ay$$

as well.

4.4.2 Exponential rule $Q(x) = e^{kx}$

It is easy to verify that $De^{kx} = ke^{kx}, D^2e^{kx} = k^2e^{kx}$, etc., so that

$$p\,(D)e^{kx} = (D^2 + aD + b)e^{kx} = (k^2 + ak + b)e^{kx} = p\,(k)e^{kx}. \tag{4.9}$$

Operating on both sides by $1/p(D)$ and remembering we can pull out all constants, we have from (4.9)

$$\frac{1}{p(D)}p(D)e^{kx} = \frac{1}{p(D)}p(k)e^{kx}$$
$$e^{kx} = p(k)\frac{1}{p(D)}e^{kx}$$

and dividing both sides by $p(k)$ which is simply a constant, we get the important rule:

$$\boxed{\frac{\mathbf{1}}{\boldsymbol{p(D)}}\boldsymbol{e^{kx}} = \frac{\mathbf{1}}{\boldsymbol{p(k)}}\boldsymbol{e^{kx}},\ \boldsymbol{p(k)} \neq \mathbf{0}.} \tag{4.10}$$

Example 4.12. *Solve the equation* $y'' + 5y' + 6y = 10e^{3x}$.
Solution: Here $k = 3$. The auxiliary equation is $m^2 + 5m + 6 = 0$ so that the complementary solution is $y_c = c_1e^{-2x} + c_2e^{-3x}$. Using the operator method and rule (4.10),

$$y_p = \frac{1}{(D^2 + 5D + 6)}10e^{3x} = \frac{10}{(3^2 + 5\cdot 3 + 6)}e^{3x} = \frac{e^{3x}}{3},$$

and the complete solution is $y = c_1e^{-2x} + c_2e^{-3x} + \frac{e^{3x}}{3}$.

Remark 13. The exponential rule fails if k is a root of the auxiliary equation $p(k) = 0$. This means $(D - k)$ is a factor of $p(D)$. This is taken care of in the next section.

4.4.3 Exponential shift $Q(x) = e^{kx}V(x)$

From the last section we see that the exponential rule (4.10) fails when we deal with $\frac{1}{p(D)}e^{kx}$, but $p(k) = 0$. This is taken care of by the *exponential shift.*

Consider now the equation

$$p(D)y = e^{kx}V(x),$$

where the right hand side has the additional factor $V(x)$ besides the exponential. We note that

$$D(e^{kx}V) = ke^{kx}V + e^{kx}DV = e^{kx}(D + k)V$$

$$\begin{aligned} D^2(e^{kx}V) &= D \cdot \{D(e^{kx}V)\} \\ &= D\{e^{kx}(D+k)V\} \\ &= ke^{kx}(D+k)V + e^{kx}(D+k)DV \\ &= e^{kx}(D+k)(D+k)V \\ &= e^{kx}(D+k)^2 V \end{aligned}$$

and in general

$$p\,(D)(e^{kx}V) = e^{kx}p\,(D+k)\,V,$$

which is usually referred to as the ***exponential shift***. We are however interested in the inverse operation $\frac{1}{p\,(D)}e^{kx}V(x)$. Let us write

$$V_1 = \left\{\frac{1}{p(D+k)}\right\}V$$

so that $p(D+k)V_1 = V$. Then using the exponential shift, we get

$$p(D)(e^{kx}V_1) = e^{kx}p(D+k)V_1 = e^{kx}V.$$

Operating on both sides by $1/p(D)$,

$$\boxed{\frac{1}{p\,(D)}e^{kx}V(x) = e^{kx}\frac{1}{p\,(D+k)}V(x).} \tag{4.11}$$

This means that we can move e^{kx} to the left of $\frac{1}{p\,(D)}$ as long as we replace D by $D+k$.

Caution: *One should be careful when using the exponential shift. It is important to remember that*

$$\frac{1}{p(D)}e^{kx}V \neq e^{kx}\,\frac{1}{p(k)}\,V.$$

That is, one cannot replace D by k. The correct usage is to replace D by $D+k$.

Remark 14. The following observations are very useful:

$$\frac{1}{p\,(D)}e^{kx} = e^{kx}\frac{1}{p\,(D+k)}\,1, \qquad \frac{1}{(aD+b)^n}\cdot 1 = \frac{1}{b^n} \quad \text{provided } b \neq 0.$$

The first is obvious. To see the second, note that

$$\frac{1}{(aD+b)}\cdot 1 = \frac{1}{(aD+b)}e^{0x} = \frac{e^{0x}}{a(0)+b} = \frac{1}{b}.$$

By repeated application of this fact, the second equation follows.
We consider the following example.

Example 4.13. *Find a particular solution y_p of $y'' - 5y' + 6y = e^{2x}$.*
Solution: We have

$$\begin{aligned}
y_p &= \frac{1}{D^2-5D+6}e^{2x}\\
&= \frac{1}{(D-2)(D-3)}e^{2x}\\
&= \frac{1}{(2-3)}\,\frac{1}{(D-2)}e^{2x} \quad (\textit{using exponential rule } (4.10))\\
&= -e^{2x}\frac{1}{((D+2)-2)}\cdot 1 \quad (\textit{using exponential shift } (4.11))\\
&= -e^{2x}\frac{1}{D}\cdot 1\\
&= -e^{2x}\int 1\,dx\\
&= -xe^{2x}.
\end{aligned}$$

To appreciate the power of the exponential shift consider the following example.

Example 4.14. *Solve the equation of Example 4.6: $(D-2)^2\,y = x^2e^{2x}$.*
Solution: As before $y_c = (c_1x + c_2)e^{2x}$. Using the UC method, we would have to solve three equations. But using the method developed so far in this chapter,

$$\begin{aligned}
y_p &= \frac{1}{(D-2)(D-2)}x^2e^{2x},\\
&= e^{2x}\frac{1}{D}\frac{1}{D}x^2 \quad (\textit{using exponential shift } (4.11))\\
&= e^{2x}\frac{1}{D}\int x^2dx,\\
&= e^{2x}\int \frac{x^3}{3}\,dx,\\
&= \frac{x^4e^{2x}}{12},
\end{aligned}$$

which is much faster.

Remark 15. If k is a root of $p(D)$ with multiplicity r, then $(D-k)^r$ is a factor of $p(D)$. Suppose $p(D) = \phi(D)(D-k)^r$. Then

$$\frac{1}{p(D)}e^{kx} = \frac{1}{\phi(D)}\frac{1}{(D-k)^r}e^{kx} = \frac{1}{\phi(D)}e^{kx}\frac{1}{D^r}\cdot 1 = \frac{e^{kx}}{\phi(k)}\frac{x^r}{r!}.$$

Example 4.15. *Solve* $y''' - 3y'' + 3y' - y = e^x$.

Solution: We can rewrite the equation as $(D^3 - 3D^2 + 3D - 1)y = e^x$. The auxiliary equation has three equal roots, viz., 1,1,1 (i.e. 1 has multiplicity 3). Hence

$$y_c = (c_1x^2 + c_2x + c_3)e^x.$$

Also

$$\begin{aligned} y_p &= \frac{1}{(D-1)^3}e^x \\ &= e^x\frac{1}{D^3}\cdot 1 \\ &= \frac{x^3e^x}{3!}. \end{aligned}$$

The complete solution is

$$y_c = (c_1x^2 + c_2x + c_3)e^x + \frac{x^3}{3!}e^x.$$

You should also try the UC method to solve this equation.

4.4.4 $Q(x)$ is a polynomial

This is one case where the operator method can get bogged down and in some cases it may be more easily done by the UC method if the number of equations to be solved is small.

Consider the equation $p(D)y = Q(x)$ where $Q(x)$ is a polynomial in x. To fix ideas, let us consider the equation

$$(D^2 - 4D + 3)y = x^2.$$

It is immediate that $y_c = c_1 e^x + c_2 e^{3x}$. Recall from calculus that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots, \quad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

To find y_p, we have

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 4D + 3} x^2 \\
&= \frac{1}{(D-1)(D-3)} x^2 \\
&= -\frac{1}{2}\left\{\frac{1}{D-1} - \frac{1}{D-3}\right\} x^2 \\
&= \frac{1}{2}\left\{\frac{1}{1-D} - \frac{1}{3-D}\right\} x^2 \\
&= \frac{1}{2}\Big\{(1 + D + D^2) - \frac{1}{3}(1 + D/3 + D^2/9)\Big\} x^2 \\
&= \frac{1}{2}\Big\{(2/3) + (8/9)D + (26/27)D^2\Big\} x^2 \\
&= \frac{1}{2}\Big\{(2/3)x^2 + (16/9)x + (52/27)\Big\} \\
&= (1/3)x^2 + (8/9)x + (26/27).
\end{aligned}$$

It is really not necessary to split $\frac{1}{D^2-4D+3}$ into partial fractions before expanding them in powers of D. Alternately we can use the power series method without really splitting $\frac{1}{D^2-4D+3}$ as shown below

$$\begin{aligned}
y_p &= \frac{1}{D^2 - 4D + 3}\, x^2 \\
&= \frac{1}{(D-1)(D-3)}\, x^2 \\
&= \left(\frac{1}{D-1}\right)\left(\frac{1}{D-3}\right) x^2 \\
&= \frac{1}{3}\left(\frac{1}{1-D}\right)\left(\frac{1}{1-\frac{D}{3}}\right) x^2 \quad \textit{(getting the denominators in the form } 1 \pm kD\textit{)} \\
&= \frac{1}{3}\left(\frac{1}{1-D}\right) \{1 + D/3 + D^2/9\} x^2 \\
&= \frac{1}{3}\left(\frac{1}{1-D}\right) \{x^2 + 2x/3 + 2/9\} \\
&= \tfrac{1}{3}\{1 + D + D^2\}\{x^2 + 2x/3 + 2/9\} \\
&= \tfrac{1}{3}\big\{(x^2 + 2x/3 + 2/9) + (2x + 2/3) + 2\big\} \\
&= \tfrac{1}{3}x^2 + \tfrac{8}{9}x + \tfrac{26}{27}
\end{aligned}$$

as before. In fact there are all kinds of variations possible. We don't have to factor at all! Write the denominator as $3\{1+\frac{D^2-4D}{3}\}$ and get

$$\begin{aligned} y_p &= \frac{1}{3\{1+\frac{D^2-4D}{3}\}}x^2 \\ &= \frac{1}{3}\left(1-\frac{D^2-4D}{3}+\frac{D^4-8D^3+16D^2}{9}\right)x^2 \\ &= \frac{1}{3}\left(1+\frac{4D}{3}+\frac{13D^2}{9}\right)x^2 \quad \textit{(dropping all powers of D higher than 2)} \\ &= \tfrac{1}{3}x^2+\tfrac{8}{9}x+\tfrac{26}{27}, \end{aligned}$$

which is the same as before.

Remark 16. Notice that there is a need to rewrite a term such as $(D-3)$ in the form $-3(1-\frac{D}{3})$ before applying the series expansion. Also, in the expansion of $\frac{1}{1-D}$ and $\frac{1}{-3(1-\frac{D}{3})}$ into power series, we only need take terms up to D^2 since $D^3x^2=0, D^4x^2=0$, etc.

Remark 17. If the right side is a polynomial, which method should one use? UC method or the operator method? It depends upon the form of $p(D)$. If $p(D)$ involves only factors such as $(D\pm1),(D\pm2)$, etc., where the coefficient of D is 1, the operator method would be still be useful. With factors of the form $aD\pm b$ it would be cumbersome and the UC method may be preferable. In any case if the right side involves cubic or higher order polynomials, the operator method should probably be the method of choice.

4.4.5 $Q(x)$ is a circular function: $Q(x)=\sin x,\cos x$

We now consider the equation

$$p(D)y=Q(x) \tag{4.12}$$

where $Q(x)=\sin ax$ or $\cos ax$. Let us assume $Q(x)=\sin ax$. We easily see that

$$\begin{aligned} D(\sin ax) &= a\cos ax \\ D^2(\sin ax) &= -a^2\sin ax \\ D^3(\sin ax) &= -a^3\cos ax \\ D^4(\sin ax) &= (D^2)^2(\sin ax)=(-a^2)^2\sin ax=a^4\sin ax \end{aligned}$$

and so on. It follows that

$$p\,(D^2)\sin ax = p\,(-a^2)\sin ax$$

and likewise

$$p\,(D^2)\cos ax = p\,(-a^2)\cos ax.$$

This gives us the important formula

$$\boxed{\frac{\mathbf{1}}{\boldsymbol{p\,(D^2)}}\begin{Bmatrix}\mathbf{sin}\,\boldsymbol{ax}\\ \mathbf{cos}\,\boldsymbol{ax}\end{Bmatrix} = \frac{\mathbf{1}}{\boldsymbol{p\,(-a^2)}}\begin{Bmatrix}\mathbf{sin}\,\boldsymbol{ax}\\ \mathbf{cos}\,\boldsymbol{ax}\end{Bmatrix}} \tag{4.13}$$

Hence in finding y_p for equation* (4.3)*, all we need to do is to replace D^2 by $(-a^2)$. In particular, we replace $D^{2n} = (D^2)^n$ by $(-a^2)^n$

Remark 18. Equation (4.13) can fail when $p(-a^2) = 0$, which happens when $D^2 + a^2$ is a factor of $p(D^2)$ and in which case $\sin ax$ and $\cos ax$ are part of the complementary solution. The easiest way to handle this is through the UC method (equation (4.7)).

In the notation of differential operators we can rewrite equation (4.7) as

$$\frac{1}{D^2 + a^2}\begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix} = \frac{x}{2}\int \begin{Bmatrix}\cos ax\\ \sin ax\end{Bmatrix}\,dx. \tag{4.14}$$

(For an elegant way of treating circular functions using operator methods, see Appendix B: Operator Methods with Complex Coefficients)

4.4.6 Substitution for D^2

In solving the equation $p\,(D)y = Q(x)$ where $Q(x)$ is a sine, cosine or both, when do we substitute $-a^2$ for D^2? Actually this can be done any time. Suppose that

$$p\,(D) = g(D^2) + h(D),$$

where $h(D)$ consists of all odd powers of D. Then from

$$\begin{aligned} p\,(D)\sin ax &= g(D^2)\sin ax + h(D)\sin ax \\ &= (h(D) + g(-a^2))\sin ax, \end{aligned}$$

it follows that

$$\frac{1}{p\,(D)}\sin ax = \frac{1}{h(D)+g(-a^2)}\sin ax,$$

showing that we can substitute $-a^2$ for D^2 at any time.

After the substitution for the even powers of D, this leaves only *odd* powers of D in $p\,(D)$, but this can be further simplified. We can write

$$p\,(D) = D\,g(D^2) + c,$$

where g contains only even powers of D. Then

$$\begin{aligned} p\,(D)\sin ax &= (D\,g(D^2)+c)\sin ax \\ &= (D\,g(-a^2)+c)\sin ax \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{p\,(D)}\sin ax &= \frac{1}{D\,g(-a^2)+c}\sin ax \\ &= \frac{1}{g(-a^2)D+c}\sin ax \end{aligned}$$

and this is of the form

$$\frac{1}{pD+q}\sin ax,$$

where p and q are constants. But this is taken care of by the trick

$$\begin{aligned} \frac{1}{pD+q}\sin ax &= \frac{1}{pD+q}\cdot\frac{pD-q}{pD-q}\sin ax \\ &= \frac{pD-q}{p^2D^2-q^2}\sin ax \\ &= \frac{p\,a\cos ax - q\sin ax}{p^2(-a^2)-q^2}. \end{aligned}$$

Example 4.16. *Find a particular solution for the equation* $(D^3+D^2+D)y = \sin 2x$.
Solution:

$$\begin{aligned} y_p &= \frac{1}{D^3+D^2+D}\sin 2x \\ &= \frac{1}{D^2\cdot D+D^2+D}\sin 2x \\ &= \frac{1}{(-4)D-4+D}\sin 2x \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{-3D-4}\sin 2x \\
&= \frac{-3D+4}{9D^2-16}\sin 2x \\
&= \frac{-3D+4}{-36-16}\sin 2x \\
&= \frac{6\cos 2x - 4\sin 2x}{52} \\
&= \frac{3\cos 2x - 2\sin 2x}{26}.
\end{aligned}
$$

4.5 Exercises

Solve problems 1 - 7 using UC method. Solve *all* problems using the operator method

1. $y'' + 3y' - 10y = 6e^{4x}$
2. $y'' + 10y' + 25y = 14e^{-5x}$
3. $y'' - y' - 6y = 20e^{-2x}$
4. $y'' + y = 2\cos x$
5. $y'' - 2y' + y = 6e^{x}$
6. $y'' + y' = 10x^4 + 2$
7. Use the Principle of Superposition to solve
   $y'' + 4y = 4\cos 2x + 6\cos x + 8x^2 - 4x$. (Hint: Use Theorem 4.1).
8. $y'' - 4y = e^{2x}$.
9. $y'' + 4y' + 4y = 10x^3e^{-2x}$
10. $(D^2 - 1)y = e^{-x}$.
11. $y'' - y' + y = x^3 - 3x^2 + 1$
12. $y'' - 4y' + 3y = x^3e^{2x}$.
13. $y''' - 8y = 16x^2$.
14. $(D-2)^3y = e^{2x}$
15. $(D-2)^2 = e^{2x}\sin x$.

Chapter 5

Linear Equations with Variable Coefficients

5.1 Introduction

In Chapters 3 and 4 we considered linear equations where the coefficients of y and its derivatives were constants. We now consider three methods dealing with equations with variable coefficients. The first two deal with equations of second degree, i.e.,

$$\big(D^2 + p(x)D + q(x)\big)y = Q(x), \tag{5.1}$$

where the coefficients $p(x), q(x)$ need not necessarily be constants but are continuous functions of x, usually for all real x.

The methods we have studied so far are inapplicable in this situation. Of the first two methods considered here, the first one deals with the homogeneous case and the second one finds a particular solution in the nonhomogeneous case (and sometimes both methods may be needed). The second method is useful in particular when the coefficients are constants but the right hand side of (5.1) has functions such as $\ln x, \sec x$, etc. Recall that in the latter case, the methods of Chapter 4 are not applicable. Thirdly, we consider a class of equations where the coefficients have a special form, often called *Cauchy–Euler equations*:

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x),$$

where $a_0, a_1, \cdots, a_n$ are all constants. Note that each term is of the form $x^k \dfrac{d^k y}{dx^k}$.

Here the order of equation can be higher than two. Although the coefficients do not *appear* to be constants, they can be made so by a simple change of the independent variable. The resulting equation is usually solved by an interesting variation of the operator method and sometimes by the first two methods to be discussed here.

Although the methods of this chapter may be applied to the types of equations we have considered in Chapters 3 and 4 (constant coefficients and the right side of (5.1) a combination of exponentials, polynomials, sines and cosines), the methods considered in those chapters may be simpler. A word of caution: The formulas developed here are not particularly simple and need to be committed to memory!

Note that Theorem 3.1 is applicable to (5.1). In particular, any two arbitrary constants uniquely define a solution of (5.1). When (5.1) is homogeneous, that is $Q(x) = 0$, all we need is two linearly independent solutions of (5.1), say y_1, y_2 and the general solution of (5.1) can be written as $y = c_1y_1 + y_2$. Of the two methods considered below, the *reduction of order* applies to homogeneous equations but with variable coefficients. In this case, given one solution y_1 of (5.1), the method produces another solution y_2 linearly independent from y_1.

The second method, *variation of parameters*, applies to those types of (5.1), where the solution of the homogeneous part of (5.1) is known (e.g., when the coefficients are constants), and $Q(x)$ contains functions with possibly *infinitely many* linearly independent derivatives. The UC method and the method of operators are often useless in such cases. In this case, one first computes the complementary part of the solution of (5.1) somehow (we denoted this in Chapter 4 by y_c) and uses this to find y_p for (5.1).

5.2 Homogeneous equations – Method of Reduction of Order

This method derives its name from the fact that a second order equation is replaced by a first order equation for its solution. But one needs to know at least one solution of (5.1) somehow, which can often be guessed! Essentially, given one solution of equation (5.1), the reduction of order method manufactures another solution linearly independent from the first. When it works, the method is quite powerful.

Suppose that we are given the following homogeneous second order linear differential equation

$$y'' + p\,(x)y' + q\,(x)y = 0, \tag{5.2}$$

where $p(x)$ and $q(x)$ are functions of x. Suppose also that we are given a solution $y = y_1$ of (5.2), we assume that

$$y_2 = v(x)y_1$$

is another solution of (5.2). Our aim is to find $v(x)$. Since, by assumption, $y = y_1$ is a solution of (5.2), we have

$$y_1'' + py_1' + qy_1 = 0. \tag{5.3}$$

Also since $y_2 = vy_1$ we have

$$y_2' = v'y_1 + vy_1' \tag{5.4}$$

$$\begin{aligned} y_2'' &= v''y_1 + v'y_1' + v'y_1' + vy_1'' \\ &= v''y_1 + 2v'y_1' + vy_1''. \end{aligned} \tag{5.5}$$

Substituting (5.4), (5.5) in (5.2) we get

$$v''y_1 + v'(2y_1' + py_1) + v(y_1'' + py_1' + qy_1) = 0, \tag{5.6}$$

and because of (5.3), the last term in (5.6) is zero. Hence, equation (5.6) now becomes

$$v''y_1 + v'(2y_1' + py_1) = 0,$$

which we rewrite as

$$\frac{v''}{v'} = -\frac{2y_1' + py_1}{y_1} = \frac{-2y_1'}{y_1} - p. \tag{5.7}$$

Integrating with respect to x throughout we get

$$\ln v' = -\ln y_1^2 - \int p\,dx$$

which can be rewritten as

$$v'y_1^2 = \exp\left(-\textstyle\int p\,dx\right) \tag{5.8}$$

and from which it is immediate that

$$v = \int \frac{\exp\left(-\int p\,dx\right)}{y_1^2} dx. \tag{5.9}$$

The new solution then is

$$y_2 = y_1 \cdot v,$$

where v is given by (5.9). Notice the appearance of $\exp\left(-\int p\,dx\right)$ as in the case of linear equations of first order. The name *Reduction of Order* comes from equation (5.8) which shows that the solution of the second order equation (5.2) has been reduced to the solution of the first order equation (5.8).

Linear independence of y_1 and y_2

How do we know that the given solution y_1 is linearly independent from the new computed solution y_2? We compute their Wronskian (Theorem 3.2):

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & y_1 v \\ y_1' & y_1' v + y_1 v' \end{vmatrix} = y_1^2 v' = \exp\Big(-\int p\,dx\Big) \neq 0$$

from equation (5.8) and the fact that an exponential function is never zero.

Example 5.1. *Given that x is a solution of*

$$(x^2+1)y'' - 2xy' + 2y = 0$$

find a linearly independent solution by reducing the order.

Solution: Note that the coefficient of y'' in equation (5.2) is 1. Hence, we first divide throughout by (x^2+1) to get

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = 0.$$

Here $y_1 = x,\ p(x) = -2x/(x^2+1)$, so

$$\exp\,(-\int p\,dx) = \exp(\ln(x^2+1)) = x^2+1.$$

Hence,

$$v = \int \frac{x^2+1}{x^2}\,dx = \int 1 + \frac{1}{x^2}\,dx = x - \frac{1}{x}$$

and the required solution is $y = v \cdot y_1 = x(x - 1/x) = x^2 - 1$. Thus the complete solution to the given equation is $y = c_1 x + c_2(x^2-1)$.

Example 5.2. *Solve the equation*

$$y'' - 4y' + 4y = 0.$$

Solution: We have already seen this and similar equations earlier. The auxiliary equation has two equal roots 2, 2 and therefore one solution is e^{2x}. Here $p(x) = -4$ and to find the other solution, use (5.9) to get

$$v = \int \frac{\left(e^{-\int -4dx}\right)}{(e^{2x})^2}\, dx = \int \frac{e^{4x}}{e^{4x}}\, dx = x$$

so that the second solution is $y_2 = xe^{2x}$ as we have seen before.

Remark 19. The method derives its name from the fact that the solution of a second order equation such as (5.2) is reduced to the solution of a first order (linear) differential equation (5.6) (with the substitution $w = v', w' = v''$ and all functions involved are functions of x).

Remark 20. The method requires *one solution be known*! Sometimes $y = x$ is an obvious choice. But the cases where one solution is known a priori are not common. In cases where no solution is known the method is useless.

Remark 21. Notice that equation (5.9) involves only the function $p(x)$ (the coefficient of y') only!

5.3 Nonhomogeneous equations – Method of Variation of Parameters

This method is very much similar to the last method except that it especially applies to nonhomogeneous equations. Given a second order equation

$$y'' + p(x)y' + q(x)y = Q(x), \tag{5.10}$$

we assume we know two linearly independent solutions of the homogeneous part, that is y_c. The method is mostly used when $p(x)$ and $q(x)$ are constants which we have seen before. Once this is known, one replaces the arbitrary constants in y_c by *functions* to manufacture y_p! The method of variation of parameters, due to Lagrange, is quite powerful especially when $Q(x)$ is a complicated function. We assume that we can solve the homogeneous part of equation (5.10)

$$y'' + p(x)y' + q(x)y = 0. \tag{5.11}$$

Let y_1 and y_2 be two linearly independent solutions of (5.11) so that the general solution of (5.11) is

$$y_c = c_1 y_1 + c_2 y_2. \tag{5.12}$$

Since y_1, y_2 are linearly independent, their Wronskian is nonzero, i.e.,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq 0. \tag{5.13}$$

We replace the constants c_1 and c_2 in (5.12) by the functions $v_1(x)$ and $v_2(x)$, and wish to determine v_1, v_2 so that a particular solution y_p exists,

$$y_p = v_1 y_1 + v_2 y_2$$

which is a solution of (5.10). Since we have to determine two functions, we have the freedom to impose two conditions. It is easy to verify that

$$y_p' = \{v_1 y_1' + v_2 y_2'\} + \{v_1' y_1 + v_2' y_2\}. \tag{5.14}$$

As our first condition we impose that the quantity inside the second braces be zero, i.e.,

$$v_1' y_1 + v_2' y_2 = 0. \tag{5.15}$$

With this simplification, equation (5.14) becomes

$$y_p' = v_1 y_1' + v_2 y_2'. \tag{5.16}$$

We differentiate this to get

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''. \tag{5.17}$$

Since y_p is assumed to be a solution of (5.10), we substitute for y_p'', y_p', y_p from equations (5.16) and (5.17) in (5.10):

$$\begin{aligned} y_p'' + p y_p' + q y_p &= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' \\ &\quad + p\, v_1 y_1' + p\, v_2 y_2' + q v_1 y_1 + q v_2 y_2 \\ &= v_1(y_1'' + p\, y_1' + q y_1) + v_2(y_2'' + p\, y_2' + q y_2) + v_1' y_1' + v_2' y_2' \\ &= Q(x). \end{aligned} \tag{5.18}$$

In equation (5.18), the expressions inside parentheses are zero since $y_1,\ y_2$ are assumed to be solutions of (5.11). The last equality follows since y_p is a solution of (5.10). We have from equations (5.15) and (5.18) that

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0 \\ v_1'y_1' + v_2'y_2' &= Q. \end{aligned} \tag{5.19}$$

Solving equations (5.19) it is easy to see that we get

$$v_1' = \frac{-y_2Q}{W(y_1, y_2)}, \quad v_2' = \frac{y_1Q}{W(y_1, y_2)}$$

and it follows that

$$v_1 = \int \frac{-y_2Q}{W(y_1, y_2)}dx, \quad v_2 = \int \frac{y_1Q}{W(y_1, y_2)}dx, \tag{5.20}$$

and of course

$$y_p = v_1y_1 + v_2y_2.$$

Example 5.3. *Solve the equation*

$$y'' + y = \tan x.$$

Solution: This cannot be done by any of the methods we have studied in Chapter 4. The complementary solution to this equation is $y_c = c_1 \cos x + c_2 \sin x$ giving $y_1 = \cos x$ and $y_2 = \sin x$. Also $W(y_1, y_2) = \cos^2 x + \sin^2 x = 1$. By the formulas in (5.20)

$$v_1 = \int \{-\sin x \tan x\}dx, \quad v_2 = \int \{\cos x \tan x\}dx.$$

It is easy to see that $v_2 = \int \sin x dx = -\cos x$ and

$$\begin{aligned} v_1 &= \int (-\sin^2 x/\cos x)dx \\ &= \int (\cos^2 x - 1)/\cos x dx \\ &= \int (\cos x - \sec x)dx = \sin x - \ln|\sec x + \tan x|. \end{aligned}$$

Hence,

$$\begin{aligned} y_p &= \cos x(\sin x - \ln|\sec x + \tan x|) - \sin x \cos x \\ &= -\cos x \ln|\sec x + \tan x|, \end{aligned}$$

and the complete solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|.$$

Remark 22. As is obvious from the last problem, although the formulas (5.20) may seem simple, the integrals may not be particularly easy!

5.4 Cauchy–Euler equations

A typical such equation is of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x), \quad (5.21)$$

where $a_0, a_1, \ldots, a_n$ are all constants. Note that each term is of the form $x^k \dfrac{d^k y}{dx^k}$.

Equation (5.21) may not seem like a linear equation with constant coefficients, but a simple change of independent variable can make it so. We define a new independent variable t by

$$x = e^t,\ x > 0,$$

so that

$$t = \ln x \text{ and } \frac{dx}{dt} = x.$$

Thus, y is a functions of x which is now a function of t. We now compute the derivatives of y with respect to the new independent variable t. By chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = x\frac{dy}{dx}. \quad (5.22)$$

Since we use D to denote derivatives with respect to x, we will use the symbol θ to denote derivatives with respect to t. Observe that θ has all the properties of D and can be used freely when we use operator methods. We can therefore write

$$\frac{dy}{dt} = \theta y, \quad \frac{d^2 y}{dt^2} = \theta^2 y, \ldots, \text{ etc.}$$

Thus, (5.22) can be written as

$$\theta y = xDy,$$

and more generally,

$$\theta f(x) = x \cdot Df(x). \tag{5.23}$$

From equation (5.23) we now have

$$\begin{aligned}\theta^2 y &= \theta(\theta y)\\ &= \theta(xDy)\\ &= \frac{d}{dx}(xDy)\cdot\frac{dx}{dt}\\ &= (xD^2y + Dy)\cdot x\\ &= x^2D^2y + xDy\end{aligned}$$

from which

$$x^2D^2y = \theta^2y - xDy = \theta^2y - \theta y = \theta(\theta-1)y. \tag{5.24}$$

Likewise,

$$\begin{aligned}\theta^3 y &= \theta(\theta^2 y)\\ &= \frac{d}{dx}(x^2D^2y + xDy)\cdot\frac{dx}{dt}\\ &= x\cdot(2xD^2y + x^2D^3y + Dy + xD^2y)\\ &= x^3D^3y + 3x^2D^2y + xDy\end{aligned}$$

from which we get

$$\begin{aligned}x^3D^3y &= \theta^3y - 3x^2D^2y - xDy\\ &= \theta^3y - 3(\theta^2y - \theta y) - \theta y\\ &= \theta^3y - 3\theta^2y + 2\theta y\\ &= \theta(\theta-1)(\theta-2)y, \end{aligned}\tag{5.25}$$

and likewise one can easily show that

$$x^nD^ny = \theta(\theta-1)(\theta-2)\dots(\theta-(n-1))y. \tag{5.26}$$

To solve equation (5.21) then, all one needs to do is to replace $x\dfrac{dy}{dx}$, $x^2\dfrac{d^2y}{dx^2}$, etc., as per equation (5.26). The resulting equation is in terms of the operator θ with independent variable t and coefficients that are constants. We solve this by methods we have studied so far (including operator methods!) and in the final result replace t by $\ln x$.

Example 5.4. *Solve*

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3, \quad x > 0.$$

Solution: With our usual notation the given equation is

$$x^2D^2y - 2xDy + 2y = x^3.$$

We substitute $x = e^t$ and replace x^2D^2y, xDy, etc., as per equation (5.26) to get

$$(\theta)(\theta - 1)y - 2\theta y + 2y = e^{3t}.$$

Simplifying,

$$\theta^2 y - 3\theta y + 2y = e^{3t}.$$

The auxiliary equation is $m^2 - 3m + 2 = 0$ whose roots are $m = 1, 2$. Hence, the complementary solution is

$$y_c = c_1e^t + c_2e^{2t}.$$

To find the particular solution y_p,

$$y_p = \frac{1}{\theta^2 - 3\theta + 2}e^{3t} = \frac{e^{3t}}{3^2 - 3(3) + 2} = \frac{e^{3t}}{2},$$

so the complete solution is

$$y = c_1e^t + c_2e^{2t} + \frac{e^{3t}}{2}.$$

We substitute for $t = \ln x$ to finally get

$$y = c_1x + c_2x^2 + \frac{x^3}{2}.$$

Example 5.5. *Solve*

$$x^2y'' + xy' + y = 4\sin(\ln x).$$

Solution: Writing the equation in terms of operator D we have

$$(x^2D^2 + xD + 1)y = 4\sin(\ln x).$$

Using equation (5.26) we rewrite the last equation in terms of t using the operator θ to get

$$(\theta(\theta - 1) + \theta + 1)y = 4\sin t,$$

or

$$(\theta^2 + 1)y = 4\sin t. \tag{5.27}$$

The auxiliary equation here is $m^2 + 1 = 0$ and the complementary solution is

$$y_c = c_1\cos t + c_2\sin t.$$

Note that $\sin t$ is part of the complementary solution. From Remark 18, the particular solution is given by

$$y_p = 4\left\{\frac{t}{2}\right\}\int \sin t\,dt,$$

that is,

$$y_p = -2t\cos t.$$

The complete solution is thus

$$y = c_1\cos t + c_2\sin t - 2t\cos t,$$

which after substituting back $t = \ln x$ gives us

$$y = c_1\cos(\ln x) + c_2\sin(\ln x) - 2\ln x\cos(\ln x), \quad x > 0.$$

Remark 23. It should be noted that some equations that appear to be of the Cauchy-Euler type can be solved by previously learned methods. For example, in Example 5.4, it is easy to verify that $y = x$ is an obvious solution of the homogeneous equation $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ and we can use reduction of order to find another linearly independent solution and hence y_c. One can then use variation of parameters to find a particular solution. However, the methods discussed here are usually much simpler to use!

5.5 Exercises

In problems 1 –4, you are given one solution of the equation. You are to find the complete solution.

1. $x^2y'' - 4xy' + 4y = 0$ and $y_1 = x$.

2. $y_1 = x, (x^2 - 1)y'' - 2xy' + 2y = 0$.

3. $y_1 = e^{2x}, (2x + 1)y'' - 4(x + 1)y' + 4y = 0$.

4. $y = x, (x^2 - 2x + 2)y'' - x^2y' + x^2y = 0$.

5. Prove that if $1 + p + q = 0$ than $y = e^x$ is a solution of $y'' + p(x)y' + q(x)y = 0$. Use this fact to solve $(x - 1)y'' - xy' + y = 0$. $\quad (y = c_1x + c_2e^x)$

Solve the following equations:

6. $(x^2 + 1)y'' - 2xy' + 2y = 6(x^2 + 1)^2$.

7. $y'' + y = \sec x$.

8. $x^2y'' - 6xy' + 10y = 3x^4 + 6x^3$, given that x^2 and x^5 are linearly independent solutions.

9. $y'' - 2y' = 8xe^{2x}$, and by two other methods.

10. $y'' - 2y' + y = 2x$.

11. $y'' + 2y' + y = e^{-x} \ln x$.

12. $y'' - 3y' + 2y = (1 + e^{-x})^{-1}$.

13. $y'' + y = \sec x$.

14. $y'' + y = \sec x \tan x$.

15. $y''(x^2 - 1) - 2x\,y' + 2\,y = (x^2 - 1)^2$.

Solve the following problems using the Cauchy–Euler method:

16. $2x^2y'' - 5xy' + 3y = 0$.

17. $x^2y'' - 3xy' + 6y = 0$.

18. $9\,x^2y'' + 3xy' + y = 0$.

19. $x^3y''' + 2x^2y'' - 10xy' - 8y = 0.$

20. $x^2y'' + xy' + y = 4\cos\ln x.$

21. $x^2y'' - 3xy' + 5y = 5x^2.$

Chapter 6

Power Series Solutions

6.1 Introduction

The linear differential equations we have studied so far all had *closed form solutions*, that is, their solutions could be expressed in terms of *elementary functions*, viz. exponential, trigonometric (including inverse trigonometric), polynomial, and logarithmic functions. As we know from calculus courses, most such elementary functions have expansions in terms of power series. Some famous functions with their corresponding power series are:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \end{aligned}$$

But there are a whole class of functions, called *special functions*, which are not elementary functions and which occur frequently in mathematical physics. They usually satisfy second order homogeneous linear differential equations. These equations can sometimes be solved by discovering a power series that satisfies the differential equation but the solution series may not be summable to an elementary function. In this chapter we study the methods of solution to such equations.

Example 6.1. *Solve the equation*

$$y'' + y = 0 \tag{6.1}$$

by using the power series method.

Solution: This equation, as we know, has the solution $y = c_1 \cos x + c_2 \sin x$. Let us see how to solve this using power series method. We assume that the solution has a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n. \tag{6.2}$$

To solve the equation means to find the coefficients a_n. And in most cases what we really find is a *recurrence relation* between the a_n's, that is, an equation that defines a_n in terms of finitely many previous coefficients a_{n-1}, a_{n-2}, etc.

It is easy to see that

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n, \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots = \sum_{n=1}^{\infty} n a_n x^{n-1}, \\
y'' &= 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \ldots = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n x^{n-2}.
\end{aligned}$$

We now substitute this to equation (6.1) to get

$$\sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0. \tag{6.3}$$

The idea is to combine the two summation symbols but that seems difficult since the first series starts from $n = 2$ while the second starts from $n = 0$, and the powers of x are different. *Our aim is to rewrite the summations so that the exponent of x is the same in both.* Since the power of x is n in the second series, we need to modify the first series.

Consider the first series in equation (6.3). We define a new summation index k and set $k = n - 2$. Then $n = k + 2$. Also $n = 2 \Rightarrow k = 0$, so that k now ranges from 0 to ∞. We also change the subscripts of the coefficients (most important step). We then get

$$\begin{aligned}
\sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n x^{n-2} &= \sum_{k=0}^{\infty} (k+2) \cdot (k+1) \cdot a_{k+2}\, x^k \\
&= 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + 4 \cdot 3 \cdot a_4 x^2 + \ldots,
\end{aligned}$$

and the equality shows that the series still is the same although its summation representation has changed.

We change k back to n since the second series has summation index n and substitute this in (6.3):

$$\sum_{n=0}^{\infty}(n+2)\cdot(n+1)\cdot a_{n+2}\,x^n + \sum_{n=0}^{\infty} a_n x^n$$
$$= \sum_{n=0}^{\infty}\Big\{(n+2)\cdot(n+1)\cdot a_{n+2} + a_n\Big\}x^n = 0. \qquad (6.4)$$

Remark 24. When we changed the summation index in the first summation so that the power of x is the same in both summations, the first series started from $k = 0$ as it is in the second. This is purely a coincidence and we will see examples where this is generally not the case.

Since the right hand side of (6.4) is zero, we equate the coefficient of x^n to zero in (6.4) and obtain

$$(n+2)\cdot(n+1)\cdot a_{n+2} + a_n = 0,$$

from which it is immediate that

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \quad n \geq 0. \qquad (6.5)$$

Equation (6.5) is a ***recurrence relation*** between the coefficients and to "solve" an equation such as (6.5) is to get such a relation. A *recurrence relation* among the values of a sequence (like $\{a_n\}$ here) does not explicitly provide the values of the terms of the sequence. In equation (6.5), a_{n+2} is given in terms of a_n and to compute the latter, one has to know a_{n-2} and so on.

Remark 25. Generally, an explicit formula for a_n in terms of n, is cumbersome and often difficult. For this reason, one often computes the first few terms using the recurrence relation. This is all that is required in most applications.

From (6.5), given a_0 we can find $a_2, a_4, a_6, \ldots$ and given a_1 we can compute $a_3, a_5, a_7, \ldots$ etc., but equation (6.5) does not define a_0 or a_1! These are arbitrary constants (since we have a second order equation). Thus, we quickly get

$$a_2 = (-1)\frac{a_0}{2\cdot 1}$$
$$a_4 = (-1)\frac{a_2}{4\cdot 3} = (-1)^2\frac{a_0}{4\cdot 3\cdot 2\cdot 1}, \quad \text{etc.}$$

and likewise

$$a_3 = (-1)\frac{a_1}{3 \cdot 2}$$
$$a_5 = (-1)\frac{a_3}{5 \cdot 4} = (-1)^2\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \quad \text{etc.}$$

In this example it is easy to see that

$$a_{2n} = (-1)^n\frac{a_0}{(2n)!}, \qquad a_{2n+1} = (-1)^n\frac{a_1}{(2n+1)!}.$$

Substituting this in (6.2) we finally get

$$y = a_0\Big\{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\Big\} + a_1\Big\{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\Big\}$$

which we recognize as

$$y = a_0 \cos x + a_1 \sin x.$$

This is the essence of the power series method. It essentially consists of assuming a power series expansion for y, computing corresponding series for y', y'', etc., and substituting all these in the given equation. The process of manipulating and shifting the indices is, one must admit, rather tedious work and mistakes are likely. Later in this chapter we consider a more elegant method that dispenses with this drudgery work!

Note: *For the rest of this and the next chapter, we will be concerned with second order homogeneous equations only.*

6.2 Some theory - Analytic functions and ordinary points

Given a function $f(x)$ which is differentiable n times at a point $x = a$, its Taylor polynomial is

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R(x),$$

where $R(x)$ is called the "error term". If $f(x)$ has derivatives of all orders at $x = a$, and if the error term goes to zero as $n \to \infty$, we get the *Taylor series* of $f(x)$ around $x = a$:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!}.$$

A function $f(x)$ that has a Taylor series at $x = a$ is said to be *analytic at* $x = a$. Associated with the Taylor series for f at $x = a$ is a real number $r > 0$ with the following property: The Taylor series for f converges absolutely within the interval $(a - r, a + r)$ and diverges outside. We call r the *radius of convergence*.

In the special case $a = 0$, the Taylor series is often called the *Maclaurin series* which is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0).$$

For many important functions (e.g., $e^x, \sin x, \cos x$), their Maclaurin series converges for all real x. We will mostly be concerned with Maclaurin series only in this chapter.

Given a second order homogeneous linear differential equation,

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \tag{6.6}$$

a point $x = a$ is called an *ordinary point* if the two functions

$$P(x) = \frac{a_1(x)}{a_0(x)} \quad \text{and} \quad Q(x) = \frac{a_2(x)}{a_0(x)}$$

are *both analytic at* $x = a$. If $P(x)$, $Q(x)$ are not both analytic at $x = a$, then $x = a$ is called a *singular point*. However, we will only be concerned with ordinary points in this chapter.

Theorem 6.1. *The solution space of* (6.6) *is a vector space of dimension two. Assume that* $x = a$ *is an ordinary point of* (6.6)*. Then there are two nontrivial linearly independent solutions of* (6.6) *of the form*

$$\sum_{n=0}^{\infty} a_n (x - a)^n$$

which converge in some interval $|x - a| < r,\ r > 0$. *Any solution of* (6.6) *is of the form*

$$f(x) = c_1 f_1(x) + c_2 f_2(x),$$

where c_1, c_2 *are arbitrary constants.*

We note that the statement about the dimension of the solution space is from Theorem 3.1. We consider another example.

Example 6.2. *Solve the equation*

$$y'' + xy' + y = 0 \tag{6.7}$$

using the power series method.

Solution: Here $P(x) = x, Q(x) = 1$ and both are analytic everywhere, in particular at the origin. Hence, equation (6.7) has two linearly independent Maclaurin series for its solution. We assume a power series expansion for y as in equation (6.2), and get

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}. \tag{6.8}$$

Substituting this in (6.7) one gets

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0, \tag{6.9}$$

where the power of x in the second series is now n because of the coefficient x for y' in (6.7). Our aim is to combine all this into one sum. As a first step, we want to make sure the power of x is n in all of them. This is the case for the second and third sums, but not in the first.

By making the substitution $k = n - 2$, we get $n = k + 2$. Since n goes from 2 to ∞ it follows that k goes from 0 to ∞. The first sum now becomes

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

For uniformity we change k back to n and substitute back in (6.9) to get

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Although the power of x is n in all sums now, the second starts from $n = 1$ while the others start from $n = 0$. We take care of this by removing the first term in the first and third sums from the summation symbols. This effectively makes all of them start from $n = 1$. We thus get

$$2\,a_2 + a_0 + \sum_{n=1}^{\infty} \Big\{ (n+2)(n+1) a_{n+2} + n a_n + a_n \Big\} x^n = 0, \tag{6.10}$$

where the first two terms are terms corresponding to $n = 0$ in the first and third summations.

Since the right hand side of (6.10) is zero, we equate the constant terms and coefficients of x^n, $n \geq 1$ in (6.10) to zero to finally get

$$a_2 = -\frac{1}{2}\,a_0 \tag{6.11}$$
$$a_{n+2} = -\frac{a_n}{n+2}, \quad n \geq 1. \tag{6.12}$$

Purely by chance, for $n = 0$, the expression (6.12) for a_{n+2} already contains the equation (6.11), so that we might as well write

$$a_{n+2} = -\frac{a_n}{n+2}, \quad n \geq 0. \tag{6.13}$$

Equation (6.13) is the recurrence relation for the coefficients and the main result we are after! From it, by setting $n = 1, 3, 5, \ldots$, we immediately get

$$\begin{aligned}
n = 1, \quad & a_3 = -\frac{a_1}{3} = (-1)\frac{a_1}{1\cdot 3}\\
n = 3, \quad & a_5 = -\frac{a_3}{5} = (-1)^2\frac{a_1}{5\cdot 3\cdot 1}\\
n = 5, \quad & a_7 = -\frac{a_5}{7} = (-1)^3\frac{a_1}{7\cdot 5\cdot 3\cdot 1}\\
& \quad\vdots \qquad\qquad \vdots\\
& a_{2n+1} = (-1)^n\frac{a_1}{1\cdot 3\cdot 5\cdots(2n+1)}, \quad n \geq 1.
\end{aligned} \tag{6.14}$$

Similarly, by taking $n = 2, 4, 6, \ldots$, etc., we get

$$\begin{aligned}
a_2 &= (-1)\frac{a_0}{2}\\
a_4 &= (-1)\frac{a_2}{4} = (-1)^2\frac{a_0}{2\cdot 4}\\
a_6 &= (-1)\frac{a_4}{6} = (-1)^3\frac{a_0}{2\cdot 4\cdot 6}\\
&\ \ \vdots \qquad\qquad \vdots\\
a_{2n} &= (-1)^n\frac{a_0}{2\cdot 4\cdot 6\cdot 8\cdots 2n}, \quad n \geq 1.
\end{aligned} \tag{6.15}$$

Since a_0 and a_1 are not defined, they are arbitrary and independent. By substituting (6.14), (6.15) to (6.2), the final solution is

$$y = a_0 + a_1\,x + a_1 \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot \ldots (2n+1)} + a_0 \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{2 \cdot 4 \cdot 6 \cdot \ldots (2n)},$$

which we can rewrite as

$$y = a_0 \left\{ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2 \cdot 4 \cdot 6 \cdot \cdots (2n)} \right\} + a_1 \left\{ \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdot \cdots (2n+1)} \right\}.$$

Remark 26. Notice that the two series inside the braces are the two guaranteed by Theorem 6.1. Most problems are pretty similar to equation (6.7). And the series for y, y', y'' are as in (6.8). Often, y, y', y'' have coefficients which are simple polynomials in x. With such coefficients, the power of x often changes and we adjust this so that they all have the same power of x by manipulating the start value of the summation index. Eventually, we remove the first few terms in one or all so that all the summations start from the same value and have the same power of x.

Example 6.3. *Given the differential equation*

$$(x^2+1)y'' + y' = 0,$$

and that y admits a Maclaurin series of the form $y = \sum_{n=0}^{\infty} a_n x^n$, obtain a recurrence relation for the coefficients a_n.

Solution: Here $P(x) = 1/(x^2+1), Q(x) = 0$. The origin $x = 0$ is an ordinary point of the given equation. As in the previous examples,

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Rewriting the given equation as $x^2 y'' + y'' + y' = 0$ and substituting for y', y'' we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0.$$

We adjust the power of x in second and third summations to be n,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 0.$$

We make the second and third summations start from $n = 2$ by removing terms corresponding to $n = 0, 1$:

$$\underbrace{(2a_2 + a_1)}_{n=0} + \underbrace{(6a_3 + 2a_2)x}_{n=1} + \sum_{n=2}^{\infty} \Big\{ n(n-1)a_n + (n+2)(n+1)a_{n+2} + (n+1)a_{n+1} \Big\} x^n = 0.$$

Equating the various powers of x to zero we get

$$\begin{aligned} 2a_2 + a_1 &= 0 \\ 3a_3 + a_2 &= 0 \\ n(n-1)a_n + (n+2)(n+1)a_{n+2} + (n+1)a_{n+1} &= 0 \end{aligned}$$

Hence, the recurrence relation sought is

$$\begin{aligned} a_2 &= -\frac{1}{2}\, a_1 \\ a_3 &= -\frac{1}{3}\, a_2 = \frac{1}{6}\, a_1 \\ a_{n+2} &= -\frac{1}{(n+2)}\, a_{n+1} - \frac{n(n-1)}{(n+2)(n+1)}\, a_n, \quad n \geq 2 \\ & a_0 \text{ and } a_1 \text{ arbitrary .} \end{aligned} \tag{6.16}$$

In (6.16), $n \geq 2$ since this is the lower limit of the summation. You would notice that the relation $3a_3 + a_2 = 0$ is already contained in (6.16). Hence, the limit for n in (6.16) can be changed to $n \geq 1$. But this is coincidental!

Remark 27. The power series method to find the crucial recurrence relation between the various a_n's that is presented here is admittedly cumbersome and laborious. In the next section we look at this from a different point of view and consider a method that avoids manipulations of summation indices, etc. It is very elegant and fast, just like the operator method. In the process we will learn a theorem that has very many uses.

6.3 Relationship between $\{a_n\}$ and $\{f^{(n)}(0)\}$[1]

Given the second order differential equation (6.6), and assuming origin is an ordinary point, it is clear from the previous sections and examples, that the aim of the

[1] Much of the material in this section appears in P. K. Subramanian, *Successive Differentiation and Leibniz's Theorem*, The College Mathematical Journal, Vol 35, No. 4, pp 274–282.

power series method is to find a series of the form $y = \sum_{n=0}^{\infty} a_n x^n$ that satisfies the differential equation. We noted that one really finds a recurrence relation satisfied by the coefficients $\{a_n\}$. The series $\sum_{n=0}^{\infty} a_n x^n$ contains only powers of x and hence is the Maclaurin series for $y = f(x)$. In particular, it is assumed that the origin is an ordinary point of (6.6).

We saw in the last section that the Maclaurin series for $y = f(x)$ is of the form $\displaystyle\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$. Hence,

$$y = f(x) = \sum_{n=0}^{\infty} a_n x^n \equiv \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!},$$

and comparing the coefficients of x^n in both series, we see that

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}, \quad f^{(n)}(0) = n!\, a_n} \tag{6.17}$$

Remark 28. A word about notations. The n^{th} derivative of f at the origin, $f^{(n)}(0)$, has many notations in the literature. Some denote this by $y_n(0)$, some others by $y^n(0)$, and many others by $(y_n)_0$. We will dispense with these cumbersome notations, allow ourselves to be sloppy and simply write y_n for $f^{(n)}(x)$, the n^{th} derivative of $y = f(x)$ at any point x, *as well as at the origin*, $f^{(n)}(0)$. This should cause no confusion; the meaning would be quite clear from the context. If an equation containing y_n also involves x, clearly $y_n(x)$ is meant. Otherwise $y_n(0)$.

Using our notation it follows from (6.17) that,

$$\boxed{a_n\, n! = y_n} \tag{6.18}$$

Remark 29. From equation (6.18) it is clear that *to find a recurrence relation between the coefficients* $\{\boldsymbol{a_n}\}$*, it suffices to find a recurrence relation amongst the derivatives at the origin, that is between* $\{\boldsymbol{y_n}\}$*.*

Example 6.4. *Consider the equation*

$$y'' + y = 0 \tag{6.19}$$

from Example 6.1. Obtain a recurrence relation for the coefficients $\{a_n\}$ *of the solution to* (6.19)*.*

Solution: Using our new notation, we rewrite this as $y_2 + y = 0$. By differentiating this repeatedly, one gets $y_3 + y_1 = 0,\ y_4 + y_2 = 0, \dots$ and in general

$$y_{n+2} + y_n = 0. \tag{6.20}$$

Equation (6.20) is the recurrence relation between the y_n's. Using (6.18) we immediately get

$$a_{n+2}(n+2)! + a_n n! = 0.$$

Since $(n+2)! = (n+2)(n+1)n!$, simplification of the last equation yields

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, \tag{6.21}$$

which is the same as (6.5), and we did not have to substitute power series or manipulate coefficients! What are the limits on n? The limit for n is $n \geq 0$ since equation (6.21) for a_{n+2} is valid for all $n \geq 0$. However, (6.21) does not define a_0 or a_1. They are arbitrary.

But not all equations are as simple as (6.1). Consider the equation of Example 6.3,

$$(x^2+1)y'' + y' = 0.$$

First of all observe that the given equation itself is a relation between y'' and y'. To get similar relations among higher order derivatives we differentiate the given equation:

$$\begin{aligned}(x^2+1)y_2 + y_1 &= 0\\ (x^2+1)y_3 + (2x+1)y_2 &= 0\\ (x^2+1)y_4 + (4x+1)y_3 + 2y_2 &= 0.\end{aligned}$$

Letting $x = 0$ (remember we need derivatives at the origin), these equations become

$$y_2 + y_1 = 0, \quad y_3 + y_2 = 0, \quad y_4 + y_3 + 2y_2 = 0,$$

etc., but there is no way we can discern any pattern here. The process of differentiating the equations becomes cumbersome as we continue *because of the product term* $(x^2+1)y''$ *in the original equation. What we need is a method for differentiating the product of two functions n-times without having to proceed step by step!* Enter Gottfried Wilhelm Leibniz, the co-inventor of the calculus! With his theorem, we can differentiate the given differential equation n–times. Leibniz's theorem has many applications besides those to differential equations.

6.4 Successive differentiation and Leibniz's Theorem

An unusually powerful theorem, due to Leibniz, shows how to compute derivatives of *any* order for products of functions! In other words, it is not necessary to compute derivatives of products of functions, one step at a time. For convenience, we use the differential operator D to denote derivatives. In particular D^n denotes the n^{th} derivative.

Theorem 6.2. ***(Leibniz) Higher order derivatives of the product of two functions*[2]*:*** *If f and g are functions of x that are differentiable n times, then*

$$\begin{aligned} D^n(fg) &= \binom{n}{0}\{D^n(f)\}\{g\} + \binom{n}{1}\{D^{n-1}(f)\}\{D(g)\} + \binom{n}{2}\{D^{n-2}(f)\}\{D^2(g)\} \\ &\quad + \ldots + \binom{n}{n}\{f\}\{D^n(g)\} \\ &= \sum_{r=0}^{n} \binom{n}{r} (D^{n-r}(f))\,(D^r(g)). \end{aligned}$$

Notice that the theorem resembles the Binomial Theorem for the n^{th} power of the sum of two terms, where the binomial coefficient $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$. The following observations are important in applying this theorem:

1. In every term, the sum of the derivatives of f and g is always n.

2. We start with the n^{th} derivative of f and reduce it by one in each successive term. In the last term there is no derivative of f.

3. The formula is symmetrical with respect to f and g, that is, they can be interchanged if we remember that $\binom{n}{r} = \binom{n}{n-r}$.

To appreciate the power of the theorem, let us consider $h(x) = e^x/x$. Although the computation of $h'(x) = e^x(x-1)/x^2$ is straightforward, finding $h''(x)$ and $h'''(x)$ gets laborious. On the other hand we can use Leibniz's theorem. We choose $f(x) =$

[2]This theorem is easily proved using mathematical induction.

e^x, $g(x) = 1/x$. This gives

$$h''(x) = \binom{2}{0}(e^x)(1/x) + \binom{2}{1}(e^x)(-1/x^2) + \binom{2}{2}(e^x)(2/x^3)$$
$$= \frac{e^x}{x} - \frac{2e^x}{x^2} + \frac{2e^x}{x^3},$$

where we have used e^x as the first function and $1/x$ as the second function. (The choice of the first function is often dictated by the ease with which its higher derivatives can be computed, and in this case $D^n e^x = e^x$.) Simplified, this yields $h''(x) = e^x(x^2 - 2x + 2)/x^3$.
Likewise,

$$h'''(x) = \binom{3}{0}(e^x)(1/x) + \binom{3}{1}(e^x)(-1/x^2) + \binom{3}{2}(e^x)(2/x^3) + \binom{3}{3}(e^x)(-6/x^4)$$
$$= \frac{e^x}{x} - \frac{3e^x}{x^2} + \frac{6e^x}{x^3} - \frac{6e^x}{x^4}$$

which simplifies to $e^x(x^3 - 3x^2 + 6x - 6)/x^4$.

6.5 Leibniz's Theorem and differential equations

How is Leibniz's Theorem useful in differential equations, in particular finding n^{th} derivative? To understand this, let us see how to find the n^{th} *derivative of the product of two functions.*

Example 6.5. *If*

$$f(x) = x^4 + 1, \quad g(x) = e^x,$$

find the n^{th} derivative of fg.
Solution: By Leibniz's Theorem and noting that

$$g^{(n)} = e^x, \quad f' = 4x^3$$
$$f'' = 12x^2$$
$$f''' = 24x$$
$$f^{(4)} = 24$$
$$f^{(5)} = f^{(6)} = \ldots = 0,$$

we have

$$\begin{aligned}D^n(fg) &= \binom{n}{0}(e^x)(x^4+1) + \binom{n}{1}(e^x)(4x^3) + \binom{n}{2}(e^x)(12x^2)\\ &\quad + \binom{n}{3}(e^x)(24x) + \binom{n}{4}(e^x)(24)\\ &= e^x\{(x^4+1) + 4nx^3 + 6n(n-1)x^2\\ &\quad + 4n(n-1)(n-2)x + n(n-1)(n-2)(n-3)\}.\end{aligned}$$

We consider the equation of Example 6.3 again.

Example 6.6. *Find a power series solution around the origin for the equation*

$$(x^2+1)y'' + y' = 0 \tag{6.22}$$

using the Leibniz's theorem.

Solution: We rewrite the equation as $(x^2+1)y_2 + y_1 = 0$ and differentiate it n-times term by term.

$$\begin{aligned}D^n\{(x^2+1)y_2\} &= \binom{n}{0}(x^2+1)y_{n+2} + \binom{n}{1}(2x)y_{n+1} + \binom{n}{2}(2)y_n\\ D^n\{y_1\} &= y_{n+1}.\end{aligned}$$

We now have the n-th derivative of the given equation:

$$\binom{n}{0}(x^2+1)y_{n+2} + \binom{n}{1}(2x)y_{n+1} + \binom{n}{2}(2)y_n + y_{n+1} = 0$$

and since we need derivatives at the origin, put $x = 0$ in the last equation to get

$$y_{n+2} + y_{n+1} + n(n-1)y_n = 0, \quad n \geq 1.$$

This is the recurrence relation satisfied by y_n's. To get the corresponding relation for a_n, we use (6.18):

$$a_{n+2}(n+2)! + a_{n+1}(n+1)! + n(n-1)a_n\, n! = 0 \tag{6.23}$$

from which

$$a_{n+2} = -\frac{1}{(n+2)}\,a_{n+1} - \frac{n(n-1)}{(n+2)(n+1)}\,a_n, \quad n \geq 1.$$

and this is the same as (6.16). Again, no infinite series to substitute or coefficients to manipulate! The limits for n are determined by those values for which a_{n+2} is well defined in the last equation; in this case, $n \geq 1$. But then the formula for a_{n+2} above does not define a_2. Remember the given equation itself defines relations between derivatives. We let $x = 0$ in the equation (6.22)! This gives $y_2 = -y_1$ and by (6.18) we get $a_2 = -\frac{1}{2}a_1$. The coefficients a_0 and a_1 are arbitrary.

Example 6.7. *Find the power series solution around the origin for the initial value problem*

$$y'' + xy' + (x^2 + 2)y = 0, \tag{6.24}$$

given that $y(0) = y'(0) = 1$. *Find the first six coefficients of the series.*
Solution: Rewrite the equation as $y_2 + xy_1 + (x^2 + 2)y = 0$ and differentiate it term by term n–times by Leibniz's theorem to get

$$\begin{aligned}
\big\{y_{n+2}\big\} &+ \left\{\binom{n}{0} y_{n+1} x + \binom{n}{1} y_n (1)\right\} \\
&+ \left\{\binom{n}{0}(x^2+2)y_n + \binom{n}{1}(2x)y_{n-1} + \binom{n}{2} 2y_{n-2}\right\} = 0 \\
\big\{y_{n+2}\big\} &+ \big\{xy_{n+1} + ny_n\big\} + \big\{(x^2+2)y_n + (n)2xy_{n-1} + n(n-1)y_{n-2}\big\} = 0.
\end{aligned}$$

At $x = 0$ this reduces to the recurrence relation

$$y_{n+2} + (n+2)y_n + n(n-1)y_{n-2} = 0.$$

From (6.18) we can translate the last equation to

$$a_{n+2}(n+2)! = -(n+2)n!a_n - n(n-1)(n-2)!a_{n-2},$$

which simplifies to

$$a_{n+2} = -\frac{1}{n+1}\, a_n - \frac{1}{(n+2)(n+1)} a_{n-2}, \quad n \geq 2. \tag{6.25}$$

The last equation contains a_{n-2} which is meaningful only for $n \geq 2$; hence the limits on n.

We cannot use (6.25) to find a_2 or a_3, but we can use equation (6.24)! First of all, note that the initial conditions $y(0) = y'(0) = 1$ simply mean that $a_0 = a_1 = 1$. Putting $x = 0$ in (6.24),

$$y_2 = -2\,y \iff 2!\,a_2 = -2\,a_0 \iff a_2 = -a_0 = -1.$$

To find a_3, we differentiate (6.24) once to obtain

$$y_3 + xy_2 + y_1 + (x^2+2)y_1 + 2xy = 0$$

and we get immediately (by letting $x = 0$ and use (6.18)) that

$$y_3 = -3\,y_1 \iff 3!\,a_3 = -3\,a_1 \iff a_3 = -\frac{1}{2}a_1 = -\frac{1}{2}.$$

Plugging $n = 2, n = 3$ in (6.25) it is easy to compute $a_4 = \frac{1}{4}$, $a_5 = \frac{3}{40}$, giving the value of the first six coefficients of the series for y. Thus, the solution is given by

$$y = 1 + x - x^2 - \tfrac{1}{2}x^3 + \tfrac{1}{4}x^4 + \tfrac{3}{40}x^5 + \ldots.$$

That is all there is to solving second order linear equations by power series. Leibniz has taken the drudgery out of the computations and even made it fun. Now try doing this problem the long way!

6.6 Exercises

1. Rewrite $\displaystyle\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3}$ so that the power of x is n instead of $n-3$.

In the following problems prove that the origin is an ordinary point and find a power series solution for y by finding a recurrence relations among the coefficients.

2. $y'' + xy' + (2x^2 - 1)y = 0.$
3. $y'' + xy' + (3x - 2)y = 0.$
4. $y' + 2xy = 0.$
5. $2y'' + xy' + y = 0.$
6. Differentiate the function $f(x) = (x+1)^3 \ln(1+x)$ five times using Leibniz's theorem. $(f^5(x) = -6!/(1+x)^2)$
7. Differentiate the equation $(1+x^2)y'' - xy' + xy = 0$ n-times by Leibniz's theorem.

Use Leibniz theorem to find a power series solution for the following equations by finding a recurrence relation between the coefficients

8. $(x^2+2)y''+2xy'+3y=0$ given that $y(0)=1, y'(0)=1$.

9. $y''+xy'+(2x^2-1)y=0$.

10. $y''+xy'+(3x-2)y=0$.

11. $y''-y'+2xy=0$.

12. $y'+2xy=0$.

13. $2y''+xy'+y=0$.

14. $y''-xy=0$. *This is called Airy's equation and is important in mathematical physics.*

Chapter 7

Systems of Linear Equations

7.1 Introduction

Thus far we have considered solutions of one differential equation at a time. In this chapter we consider a system of two differential equations, with constant coefficients, in which we have an independent variable t and *two* dependent variables $x(t)$ and $y(t)$. A typical example is:

$$\begin{aligned} 2x' - 2y' - 3x &= e^t \\ 2x' + 2y' + 3x + 8y &= e^t, \end{aligned} \tag{7.1}$$

where, for convenience, primes denote derivatives with respect to t. Similarly one could consider a system of three equations in three unknown functions. Such systems occur frequently in mathematical models of biological systems.

7.2 A mathematical model

Two storage tanks A and B, with equal capacity of C liters, are full with brine and are connected to each other. See Figure 7.1. Fluid can be pumped to each other as well as out of the tanks. Initially tank A contains 5 kg of salt (in C liters), whereas tank B contains 2 kg of salt. At time $t = 0$, pure water flows into A (from outside) at 6 li/m (liters per minute), brine flows from A to B at 8 li/m, brine is pumped *back from B into A* at 2 li/m, and finally brine is pumped out of B at 6 li/m. The problem is to determine the amount of salt in each tank at any given time t.

At any given time t, tank A gets 6 li/m water from outside, 2 li/m brine from B and pumps out 8 li/m to tank B. Hence, it still has C liters of solution at all time.

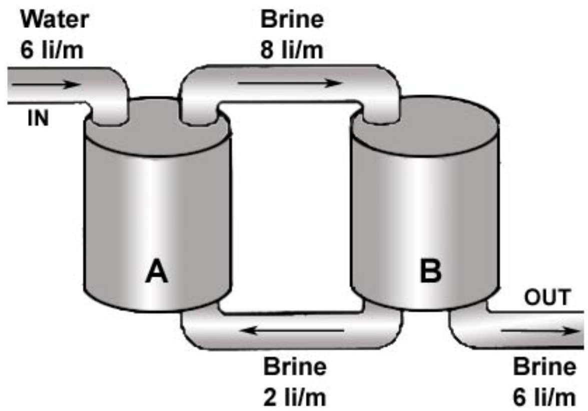


Figure 7.1: Two-tanks model

On the other hand, tank B gets 8 li/m brine from A and pumps out 2 li/m to tank A and 6 li/m outside. Thus, tank B also maintains C liters of fluid at all time in it. Let x and y be the amount of salt at time t in tanks A and B, respectively. The concentration of salt in A is x/C kg/li at time t (since A has C liters of fluid at any time t) and the concentration in B is y/C kg/li.

Consider tank A. The only brine coming is from B, *at the rate* of 2 li/m with concentration y/C kg/li, so that salt enters A *at the rate* of $2y/C$ kg/m. Similarly, the amount of salt leaving A to be pumped to B is $8x/C$ kg/m. Thus, the *rate* at which the amount of salt changes in A ($dx/dt = x'$) is its *input rate – output rate* and is therefore given by the equation,

$$x' = \frac{2y}{C} - \frac{8x}{C}$$

and similarly for tank B,

$$y' = \frac{8x}{C} - \frac{8y}{C}.$$

This is a system of two linear equations with constant coefficients and the solution of this system provides $x(t)$ and $y(t)$.

7.3 How do we solve such a system?

Different authors use different notations, the method of undetermined coefficients, and high powered linear algebra (matrices, eigenvalues, etc.,), but we will consistently use our friend the operator D and where possible use the operator method to find particular solutions. Let us consider the simultaneous system (7.1) again.

Example 7.1. *Solve the system of equations* (7.1).
Solution: Using the operator D we can rewrite this system as

$$\begin{aligned}(2D-3)x-2Dy&=e^t\\(2D+3)x+(2D+8)y&=e^t,\end{aligned}\tag{7.2}$$

which is of the form

$$\begin{aligned}p_1(D)x+p_2(D)y&=g_1\\p_3(D)x+p_4(D)y&=g_2,\end{aligned}\tag{7.3}$$

where $p_i(D)$ are first degree polynomials in D with constant coefficients, and g_i are functions of t. We can eliminate y (or x, whichever is convenient) by operating on the first equation in (7.3) by p_4, the second by p_2 (remember these operators are polynomials in D and commute) and subtract to get

$$\big(p_1(D)p_4(D)-p_2(D)p_3(D)\big)x=p_4(D)g_1-p_2(D)g_2.\tag{7.4}$$

Using matrix determinant, this is easier to remember as:

$$\begin{vmatrix}p_1(D) & p_2(D)\\ p_3(D) & p_4(D)\end{vmatrix}x=-\begin{vmatrix}p_2(D) & g_1\\ p_4(D) & g_2\end{vmatrix}.\tag{7.5}$$

Had we eliminated x instead of y we would have obtained

$$\begin{vmatrix}p_1(D) & p_2(D)\\ p_3(D) & p_4(D)\end{vmatrix}y=\begin{vmatrix}p_1(D) & g_1\\ p_3(D) & g_2\end{vmatrix}.\tag{7.6}$$

In equations (7.5) and (7.6) the determinant on the left hand side is the same. On the right, the first column of the determinant is simply the coefficients of y (the variable that is being eliminated) for equation (7.5), and that of x in equation (7.6). These would be very useful later. *Note that there is no negative sign on the right hand side in equation* (7.6) *when we eliminate* x.

Theorem 7.1. *The number of arbitrary constants in the solution of the system of equations* (7.3) *equals the order of*

$$\begin{vmatrix}p_1(D) & p_2(D)\\ p_3(D) & p_4(D)\end{vmatrix}=p_1(D)p_4(D)-p_2(D)p_3(D)$$

provided this is not zero. If this is zero but the right hand side of (7.4) *is non-zero, there is no solution. If the right side of* (7.4) *is also zero, there are infinitely many solutions with* $x(t)$ *arbitrary and* $y(t)$ *is determined by one of the system equations.*

Since p_i are all first degree polynomials, the order of the determinant is usually two, and there are two constants in the solution of equation (7.3). But this is not always the case! The order can be one sometimes. We will see examples later.

Let us solve equation (7.2) by eliminating y. Substituting in equation (7.5) we get

$$\begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} x = -\begin{vmatrix} -2D & e^t \\ 2D+8 & e^t \end{vmatrix}, \tag{7.7}$$

which gives us

$$\{(2D-3)(2D+8)-(-2D)(2D+3)\}\, x = -\{(-2D)e^t-(2D+8)e^t\}$$

$$\begin{aligned} (8D^2+16D-24)\, x &= 4De^t+8e^t \\ (D^2+2D-3)x &= \tfrac{3}{2}e^t. \end{aligned} \tag{7.8}$$

Equation (7.8) is linear of second degree and there should be two constants in its solution. Using our usual methods we obtain

$$x = \underbrace{c_1e^{-3t}+c_2e^t}_{x_c} + \underbrace{\tfrac{3}{8}te^t}_{x_p}, \tag{7.9}$$

where x_c and x_p are the complementary and particular solutions, respectively, and c_1, c_2 are the only arbitrary constants.

We could find y the same way to first obtain the determinant equation

$$\begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} y = \begin{vmatrix} 2D-3 & e^t \\ 2D+3 & e^t \end{vmatrix}, \tag{7.10}$$

and after simplification get the equation

$$(D^2+2D-3)y = -\tfrac{3}{4}e^t, \tag{7.11}$$

etc. But we would be duplicating our efforts and furthermore, since there are only two constants for this system (Theorem 7.1) and they already appear in the solution of x, *there can be no other arbitrary constants in the solution of* y. We should exploit the fact we have a system of equations and determine y from the system just as we do in basic algebra. Indeed, if we add the two equations in (7.1),

$$4x' + 8y = 2e^t,$$

that is,

$$\begin{aligned} y &= \tfrac{1}{4}e^t - \tfrac{1}{2}x' \\ &= \tfrac{1}{4}e^t - \tfrac{1}{2}\big\{ -3c_1e^{-3t} + c_2e^t + \tfrac{3}{8}te^t + \tfrac{3}{8}e^t \big\} \\ &= \tfrac{3}{2}c_1e^{-3t} - \tfrac{1}{2}c_2e^t - \tfrac{3}{16}te^t + \tfrac{1}{16}e^t. \end{aligned}$$

7.4 Pathologies - When things don't go according to plan

One needs to be careful in using the determinant notation (equations (7.5) or (7.6)). It is possible that entries in one of the columns may be *identical*! This has several implications.

1. *An attempt to eliminate y often eliminates y' and the resulting equation is of first order, not of order 2. Similarly for x.*
2. *x and y are independent.*
3. *x (as well as y) is the solution of a linear first order equation with a single constant.*

So what do we do? Consider the following example.

Example 7.2. *Solve*

$$\begin{aligned} 2x' + y' - x - y &= -2t, \\ x' + y' + x - y &= t^2. \end{aligned} \tag{7.12}$$

Solution: We rewrite the given equations using the operator D:

$$\begin{aligned} (2D-1)x + (D-1)y &= -2t \\ (D+1)x + (D-1)y &= t^2. \end{aligned}$$

In determinant form this is

$$\begin{vmatrix} 2D-1 & D-1 \\ D+1 & D-1 \end{vmatrix} x = - \begin{vmatrix} D-1 & -2t \\ D-1 & t^2 \end{vmatrix}, \tag{7.13}$$

which we can write as

$$\big\{(2D-1)(D-1)-(D+1)(D-1)\big\}\,x = -\big\{(D-1)t^2-(D-1)(-2t)\big\}, \quad (7.14)$$

or

$$(D-1)\big\{(2D-1)-(D+1)\big\}\,x = (D-1)\{-t^2-2t\}. \quad (7.15)$$

First of all note that the factor $(D-1)$ on both sides of (7.15). This occurs because the entries in the second column on the left (and the first column on the right) in equation (7.13) both containing coefficients of y, *are identical*. This is usually a tell-tale sign that warns us not to use determinants to eliminate y (or x).

If we now proceed blindly to find the complementary solution x_c we would be dealing with the equation

$$(D-1)\big\{(2D-1)-(D+1)\big\}x = 0.$$

However, $(D-1)$ is an extraneous factor and this introduces an additional extraneous solution c_1e^t corresponding to $(D-1)x = 0$ in x_c. Instead of using (7.15) then, we simply use elementary methods to eliminate y (and y')! In (7.12), we subtract the second equation from the first to get

$$x' - 2x = -2t - t^2,$$

that is,

$$(D-2)x = -2t - t^2,$$

which is a first order equation in x as we suspected earlier (one could also get to this equation by canceling the extraneous factor $(D-1)$ in (7.15) provided one looks for the common factor on both sides). Hence,

$$\begin{aligned}
x &= c_1e^{2t} + \frac{1}{D-2}(-2t-t^2) \\
&= c_1e^{2t} + \frac{1}{2}\frac{1}{1-\frac{D}{2}}(2t+t^2) \\
&= c_1e^{2t} + \tfrac{1}{2}\big(1+\tfrac{D}{2}+\tfrac{D^2}{4}\big)(2t+t^2) \\
&= \underbrace{c_1e^{2t}}_{x_c} + \underbrace{\tfrac{1}{2}\big(t^2+3t+\tfrac{3}{2}\big)}_{x_p}.
\end{aligned}$$

We have used the operator method to find the particular solution, but the method of undetermined coefficients would have been quicker.

Remark 30. Note that x has only one arbitrary constant.

Remark 31. The factor $(D-1)$ in equation (7.15) affects only x_c, but not x_p. This is because from (7.15),

$$\begin{aligned} x_p &= \frac{1}{(D-1)\{(2D-1)-(D+1)\}}\{(D-1)(-t^2-2t)\} \\ &= \frac{1}{(2D-1)-(D+1)}\{-t^2-2t\} \end{aligned}$$

and the additional factor $(D-1)$ cancels out!

We continue to find y by eliminating x' in (7.12) to get

$$y' + 3x - y = 2t^2 + 2t.$$

Hence,

$$\begin{aligned} (D-1)y &= 2t^2 + 2t' - 3x \\ &= -3c_1e^{2t} + \tfrac{1}{2}t^2 - \tfrac{5}{2}t - \tfrac{9}{4} \end{aligned}$$

after simplification. Hence

$$\begin{aligned} y &= c_2e^t + \frac{1}{D-1}\big\{-3c_1e^{2t} + \tfrac{1}{4}(2t^2-10t-9)\big\} \\ &= c_2e^t - 3c_1e^{2t} - (1+D+D^2)\big\{\tfrac{1}{4}(2t^2-10t-9)\big\} \\ &= c_2e^t - 3c_1e^{2t} - \tfrac{1}{4}(2t^2-6t-15). \end{aligned}$$

Remark 32. Although there are two constants in the solution of y, only c_2 is arbitrary.

In dealing with system of equations then, it is wise to check if the variables x and y are independent before proceeding with evaluating the determinant in (7.5) when we attempt to solve for x. But this would be self evident in when one examines this determinant. The entries in the second column on the left hand side (and the first column on the right) are identical. In that case one could do what we did in the last example. Alternately, one could eliminate x and solve for y (equation (7.6)). Indeed, in the last problem, there are no extraneous factors:

$$\begin{vmatrix} 2D-1 & D-1 \\ D+1 & D-1 \end{vmatrix} y = -\begin{vmatrix} 2D-1 & -2t \\ D+1 & t^2 \end{vmatrix}. \tag{7.16}$$

Although $(D-1)$ is a common factor on the left hand side, just like in (7.15), it does not occur on the right, and we do get a second order differential equation for y. We could proceed as we did in the Example 7.1 but obviously this involves much more labor!

Example 7.3. *Solve the system of equations*

$$\begin{aligned} 2x' - x + y' - y &= t, \\ 2x' + 2x + y' + 2y &= t^2. \end{aligned}$$

Solution: Using the operator D and writing the equations in determinant form,

$$\begin{vmatrix} 2D-1 & D-1 \\ 2(D+1) & D+2 \end{vmatrix} x = -\begin{vmatrix} D-1 & t \\ D+2 & t^2 \end{vmatrix},$$

that is,

$$3Dx = t^2 + 1. \tag{7.17}$$

It follows that the left hand side of (7.17) is of degree 1, hence there is only one arbitrary constant. Solving the last equation we get

$$x = \tfrac{1}{9}t^3 + \tfrac{1}{3}t + c_1.$$

If we subtract the top equation from the bottom in the given system we get

$$3x + 3y = t^2 - t,$$

from which

$$3y = t^2 - t - \tfrac{1}{3}t^3 - t - 3c_1,$$

that is,

$$y = -\tfrac{1}{9}t^3 + \tfrac{1}{3}t^2 - \tfrac{2}{3}t - c_1.$$

Thus,

$$x = \tfrac{1}{9}t^3 + \tfrac{1}{3}t + c_1, \quad y = -\tfrac{1}{9}t^3 + \tfrac{1}{3}t^2 - \tfrac{2}{3}t - c_1.$$

7.5 Exercises

Solve the following systems of equations:

1. $x' + y' - 2x - 4y = e^{4t}$
   $x' + y' - y = e^{4t}$

2. $5x' + y' - 3x + y = 0$
   $4x' + y' - 3x = -t.$

3. $x' + y' - x - 4y = e^{t}$
   $x' + y' + x = e^{3t}$

4. $x' + y' + y = \sin t$
   $x' + y' - x - y = 0$

5. $5x' + y' - 5x - y = 0$
   $4x' + y' - 3x = t$

6. $x' + y' - x - 6y = e^{2t}$
   $x' + 2y' - 2x - 6y = t$

7. $3x' + 2y' - x + y = t - 3$
   $x' + y' - x = t + 1$

8. $2x' + y' + x + 5y = 4t$
   $x' + y' + 2x + 2y = 2t$

9. $2x' + y' - x - y = 2e^{t}$
   $x' + y' + x - y = e^{t}$

10. $2x' + y' - x - y = 0$
    $x' + y' + 2x - y = t$

11. $x' = x + 3y$
    $y' = 3x + y$

12. $x'' + y' - x + y = \sin t$
    $y'' + x' - x + y = \cos t$

Chapter 8

The Laplace Transform

8.1 Introduction

In this chapter we will discuss another technique, called the *Laplace transform*, that is especially useful in solving initial value problems. The Laplace transform, as the name suggests, takes a function f and turns it into another function F. In engineering-related areas, such as signal processing and control system, one can see $f(t)$ as a function in "time" domain, and $F(s)$ as the equivalent function in "frequency" domain. For our purposes in relation to differential equation, the Laplace transform converts a linear differential equation into an algebraic equation, which in general is easier to solve. Applying the inverse mapping (*inverse transform*) to the solution of the algebraic equation then gives us the solution to the differential equation.

Definition 8.1. *Let f be a real-valued function in real variable t defined for $t > 0$. The* ***Laplace transform*** *F of f is the function*

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t) dt, \tag{8.1}$$

provided that the limit exists.

Notice that (8.1) is an improper integral and it is computed as the limit

$$\int_0^\infty e^{-st} f(t) dt = \lim_{R\to\infty} \int_0^R e^{-st} f(t) dt.$$

Let us consider the following examples.

Example 8.1. *Find the Laplace transform of $f(t) = t, \ t \geq 0$.*

Solution: Using Definition 8.1 and integration by parts,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty te^{-st}dt \\
&= \lim_{R\to\infty}\int_0^R te^{-st}dt \\
&= \lim_{R\to\infty} -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^R \\
&= \lim_{R\to\infty}\Big(-\frac{R}{s}e^{-sR} - \frac{1}{s^2}e^{-sR}\Big) - \Big(0 - \frac{1}{s^2}\Big) \\
&= \frac{1}{s^2}, \quad \text{for } s > 0.
\end{aligned}$$

Note that when $s \le 0$, the integral diverges.

Example 8.2. *Let $f(t) = e^{at}$, where a is a constant and $t > 0$. Find $\mathcal{L}\{f(t)\}$.*
Solution: Again by Definition 8.1, we have

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}e^{at}dt \\
&= \lim_{R\to\infty}\int_0^R e^{(a-s)t}dt \\
&= \lim_{R\to\infty}\frac{e^{(a-s)t}}{a-s}\Big|_0^R \\
&= \lim_{R\to\infty}\frac{e^{(a-s)R}}{a-s} - \frac{1}{a-s} \\
&= \frac{1}{s-a}, \quad \text{for } s > a.
\end{aligned}$$

Notice that if $s \le a$, then the integral diverges.

From the above example, it follows easily that

$$\mathcal{L}(1) = \mathcal{L}(e^{0t}) = \frac{1}{s},\ s > 0. \tag{8.2}$$

Example 8.3. *Compute the Laplace transform of the function*

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t \ge a, \end{cases}$$

where $a \ge 0$ is a constant. This function is called a unit step function (often also called the Heaviside function). We reserve the notation $u_a(t)$ to denote the unit step function with jump discontinuity at $t = a$.

Solution: Using the same definition, it easily follows that

$$
\begin{aligned}
\mathcal{L}\{u_a(t)\} &= \int_0^\infty e^{-st} u_a(t) dt \\
&= \int_a^\infty e^{-st} dt \\
&= \lim_{R\to\infty} \left. \frac{-e^{-st}}{s} \right|_a^R \\
&= \frac{e^{-as}}{s}, \; s > 0.
\end{aligned}
$$

$f(t)$	$\mathcal{L}\{f(t)\}$
c	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n, \; n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
u_a	$\frac{e^{-as}}{s}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Table 8.1: Brief list of Laplace transform of some elementary functions

8.2 Properties of Laplace transform

Not every function has a Laplace transform. The existence of transform is determined by whether or not the improper integral (8.1) converges. The following theorem states the criteria that a function must have to ensure the existence of the Laplace transform.

Theorem 8.1. ***(Existence and Uniqueness)*** *Let $f(t)$ be a function that is* ***piecewise continuous*** *on the interval $[0, \infty)$ and of* ***exponential order*** *c. Then its*

Laplace transform $\mathcal{L}\{f\}(s)$ exists for $s > c$. Moreover, suppose $g(t)$ is another function satisfying the same condition and that there exists another constant C for which $\mathcal{L}\{f\}(s) = \mathcal{L}\{g\}(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \in [0, \infty)$.

Recall that a piecewise continuous function is one that is continuous at every point in the interval, except possibly at a finite number of points where it has a jump discontinuity. A function is of exponential order c if it does not grow faster than the exponential function e^{ct}, that is,

$$|f(t)| \leq Ke^{ct}, \quad \text{for all } t \geq T,$$

where K and T are constants. It follows that an easy way to check whether a function is of exponential order is by computing the limit

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite, then $f(t)$ is of exponential order.

The function $f(t) = 3e^{2t} \cos t$ is continuous and of exponential order 2 with $K = 3$ since $|3e^{2t} \cos t| = 3\,|e^{2t}| \cdot |\cos t| \leq 3e^{2t}$. Hence, it has a Laplace transform. On the other hand, the Laplace transform for the function $f(t) = e^{t^2}$ does not exist since for any c,

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{ct}} = \lim_{t\to\infty} e^{t(t-c)} = \infty,$$

which shows that e^{t^2} is not of exponential order.

Fortunately, most functions that occur in applications involving linear differential equations with constant coefficients are usually (piecewise) continuous and of exponential order.

There are several important properties of Laplace transform that are crucial for our work from now on. The first one is ***linearity***, which is given by the following theorem:

Theorem 8.2. *Let f_1, f_2 be functions for which $\mathcal{L}\{f_1\}, \mathcal{L}\{f_2\}$ exist, and c_1, c_2 be arbitrary constants. Then*

$$\boxed{\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}}. \tag{8.3}$$

Proof: The proof is quite straightforward. By linearity properties of the integral, we get

$$\begin{aligned}\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]dt \\ &= c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt \\ &= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \quad \square\end{aligned}$$

Example 8.4. *Compute the Laplace transform $\mathcal{L}\{\sin^2 at\}$.*
Solution: Using the half-angle identity $\sin^2 at = \frac{1}{2} - \frac{1}{2}\cos 2at$ and the linearity property, we have

$$\begin{aligned}\mathcal{L}\{\sin^2 at\} &= \mathcal{L}\left\{\frac{1}{2}\right\} - \mathcal{L}\left\{\frac{1}{2}\cos 2at\right\} \\ &= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2at\} \\ &= \left(\frac{1}{2} \cdot \frac{1}{s}\right) - \left(\frac{1}{2} \cdot \frac{s}{s^2 + (2a)^2}\right) \\ &= \frac{2a^2}{s(s^2 + 4a^2)}.\end{aligned}$$

The second property comes from a quick observation that

$$\int_0^\infty e^{-st} e^{at} f(t)dt = \int_0^\infty e^{-(s-a)} f(t)dt,$$

which tells us

$$\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}. \tag{8.4}$$

This property is often called the ***translation*** property.

The next property deals with Laplace transform of the derivative $\mathcal{L}\{f'\}$, which exists if f' satisfies the existence condition given in Theorem 8.1 (with f replaced with f' in the statement of the theorem). If that is the case, then by Definition 8.1

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)dt = \lim_{R\to\infty} \int_0^R e^{-st} f'(t)dt.$$

Using integration by parts with $u = e^{-st}$ and $dv = f'(t)dt$, we have

$$\begin{aligned}
&\lim_{R\to\infty} \left[e^{-st} f(t)\Big|_0^R + \int_0^R s f(t) e^{-st} dt \right] \\
&= \lim_{R\to\infty} \left[\frac{f(R)}{e^{sR}} - f(0) + s \int_0^R f(t) e^{-st} dt \right].
\end{aligned}$$

Note that the first term goes to 0 since $f(t)$ is of exponential order, giving us the formula

$$\boxed{\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)}. \tag{8.5}$$

From the above formula, one can recursively obtain the formula for the second order derivatives, of course assuming that f, f' are continuous and f'' is piecewise continuous, and all of them are of exponential order.

$$\begin{aligned}
\mathcal{L}\{f''\} &= s\mathcal{L}\{f'\} - f'(0) \\
&= s[s\mathcal{L}\{f\} - f(0)] - f'(0) \\
&= s^2\mathcal{L}\{f\} - sf(0) - f'(0).
\end{aligned}$$

The Laplace transform for higher order derivatives can be derived in a similar fashion.

Example 8.5. *Find the Laplace transform of the equation*

$$f''(t) - 6f'(t) + 5f(t) = 0 \tag{8.6}$$

with initial conditions $f(0) = 3$ and $f'(0) = 7$.

Solution: Applying Laplace transform on both sides of the equation and its linearity property, we have

$$\begin{aligned}
&\mathcal{L}\{f''\} - 6\mathcal{L}\{f'\} + 5\mathcal{L}\{f\} = 0 \\
&\left[s^2\mathcal{L}\{f\} - sf(0) - f'(0)\right] - 6\left[s\mathcal{L}\{f\} - f(0)\right] + 5\mathcal{L}\{f\} = 0 \\
&(s^2 - 6s + 5)\mathcal{L}\{f\} - (s - 6)f(0) - f'(0) = 0.
\end{aligned} \tag{8.7}$$

Substituting the initial values $f(0) = 3, f'(0) = 7$ gives us

$$\mathcal{L}\{f\} = \frac{3s - 11}{s^2 - 6s + 5}. \tag{8.8}$$

8.3 The inverse transform

The last example in the previous section shows us how Laplace transform converts a differential equation (8.6) into an algebraic equation (8.7), which can be easily solved to get (8.8). The next question is, given $\mathcal{L}\{f\}$, how can we obtain the solution to the original differential equation, which is f? Obviously, we seek an *inverse mapping* for the Laplace transform. From Theorem 8.1, it follows that such inverse, denoted by $\mathcal{L}^{-1}$, exists and is unique, and

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t)$$

as is expected.

The inverse Laplace transform is formally defined in terms of complex integral (also known as *Fourier-Mellin integral*). However, it is beyond the scope of these notes and for problems involving inverse transform in this chapter, we will use Table 8.1, together with algebraic techniques, such as partial fractions. The linearity property of the inverse transform follows naturally from the linearity of the Laplace transform itself, and is often very useful.

Example 8.6. *Find the inverse Laplace transform of* $F(s) = \dfrac{1}{s+3}$.
Solution: From Table 8.1, it is clear that

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-(-3)}\right\} = e^{-3t}.$$

Example 8.7. *Compute the inverse Laplace transform*

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\}.$$

Solution: Note that

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2-9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\right\} \\
&= \cosh 3t + \mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2-9}\right\} \\
&= \cosh 3t + \frac{1}{3}\sinh 3t.
\end{aligned}$$

Example 8.8. *Find the inverse Laplace transform of*

$$F(s) = \frac{1}{s^2 + 6s + 13}.$$

Solution: We approach this problem by first completing the square.

$$\begin{aligned}\frac{1}{s^2 + 6s + 13} &= \frac{1}{(s^2 + 6s + 9) + 4} \\ &= \frac{1}{(s+3)^2 + 2^2} \\ &= \frac{1}{2}\frac{2}{(s-(-3))^2 + 2^2}.\end{aligned}$$

From Table 8.1 and by translation property (8.4),

$$\frac{1}{2}\frac{2}{(s-(-3))^2 + 2^2} = \frac{1}{2}F(s-(-3)) = \frac{1}{2}\mathcal{L}\Big\{e^{-3t}\sin 2t\Big\},$$

which follows that

$$\mathcal{L}^{-1}\Big\{\frac{1}{2}\frac{2}{(s-(-3))^2 + 2^2}\Big\} = \frac{1}{2}e^{-3t}\sin 2t.$$

8.4 Solving initial value problems

Our goal now is to use the Laplace transform and inverse transform to solve initial value problems for linear differential equations with constant coefficients. We have learned several techniques for this in Chapters 3 and 4. Those methods require us to first find the general solution to the equation, and then by using the given initial conditions, we find the desired solution. However, with Laplace transform, finding a general solution is not needed. We return to Example 8.5 to illustrate this method.

Example 8.9. *In Example 8.5, we found the Laplace transform*

$$F(s) = \mathcal{L}\{f\} = \frac{3s - 11}{s^2 - 6s + 5}$$

to the equation $f''(t) - 6f'(t) + 5f(t) = 0$. *Compute the inverse transform and verify that it is the solution to the given equation.*

Solution: We first rewrite $F(s)$ as partial fractions

$$\frac{3s-11}{s^2-6s+5} = \frac{A}{s-5} + \frac{B}{s-1},$$

which gives us the equation

$$3s - 11 = A(s-1) + B(s-5).$$

By substituting appropriate values for s, we find $A = 1$ and $B = 2$. Thus by linearity,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3s-11}{s^2-6s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{s-1}\right\} \\ &= e^{5t} + 2e^t.\end{aligned}$$

Verifying that $f(t) = e^{5t} + 2e^t$ is indeed the solution to the differential equation is easy. Since

$$f'(t) = 5e^{5t} + 2e^t \quad \text{and} \quad f''(t) = 25e^{5t} + 2e^t,$$

clearly

$$\begin{aligned}&f''(t) - 6f'(t) + 5f(t) = 25e^{5t} + 2e^t - 6(5e^{5t} + 2e^t) + 5(e^{5t} + 2e^t) = 0 \\ &f(0) = 3 \quad \text{and} \quad f'(0) = 7.\end{aligned}$$

In the next example, we will show how we solve a linear differential equation that involves discontinuous function using Laplace transform. Many physical problems is often modeled using discontinuous applications. Some examples are the on/off switching of electrical circuit, signaling pulse, or the sudden invasion of a species to an existing population. It may not be so convenient to solve a differential equation with discontinuity using methods in Chapters 3 and 4. Laplace transform provides a tool to deal with it.

Example 8.10. *Solve the initial value problem*

$$\frac{dy}{dt} = -3y + u_2(t), \; y(0) = -2,$$

where

$$u_2(t) = \begin{cases} 0 & t < 2 \\ 1 & t \geq 2 \end{cases}$$

is the unit step function with jump discontinuity at $t = 2$.

Solution: Taking Laplace transform on both sides of the equation we get

$$\begin{aligned}
\mathcal{L}\Big\{\frac{dy}{dt}\Big\} &= -3\mathcal{L}\{y\} + \mathcal{L}\{u_2\}\\
s\mathcal{L}\{y\} - y(0) &= -3\mathcal{L}\{y\} + \frac{e^{-2s}}{s}\\
(s+3)\mathcal{L}\{y\} &= -2 + \frac{e^{-2s}}{s}\\
\mathcal{L}\{y\} &= \frac{-2}{s+3} + \frac{e^{-2s}}{s(s+3)}.
\end{aligned}$$

Now we take the inverse transform to get

$$y = \mathcal{L}^{-1}\Big\{\frac{-2}{s+3}\Big\} + \mathcal{L}^{-1}\Big\{\frac{e^{-2s}}{s(s+3)}\Big\}. \tag{8.9}$$

The first term of (8.9) is easy as from the table we obtain

$$\mathcal{L}^{-1}\Big\{\frac{-2}{s+3}\Big\} = -2e^{-3t}. \tag{8.10}$$

To compute the second term, we use partial fraction to rewrite

$$\frac{1}{s(s+3)} = \frac{1/3}{s} - \frac{1/3}{s+3}.$$

Hence,

$$\begin{aligned}
\mathcal{L}^{-1}\Big\{\frac{e^{-2s}}{s(s+3)}\Big\} &= \frac{1}{3}\mathcal{L}^{-1}\Big\{\frac{e^{-2s}}{s}\Big\} - \frac{1}{3}\mathcal{L}^{-1}\Big\{\frac{e^{-2s}}{s+3}\Big\}\\
&= \frac{1}{3}u_2(t) - \frac{1}{3}\mathcal{L}^{-1}\Big\{\frac{e^{6}e^{-2(s+3)}}{s+3}\Big\}\\
&= \frac{1}{3}u_2(t) - \frac{1}{3}e^{6}e^{-3t}u_2(t)\\
&= \frac{1}{3}u_2(t) - \frac{1}{3}e^{(-3t+6)}u_2(t). \qquad (8.11)
\end{aligned}$$

Combining (8.10) and (8.11), the solution is given by

$$y(t) = -2e^{-3t} + \frac{1}{3}u_2(t) - \frac{1}{3}e^{-3t+6}u_2(t).$$

With Laplace transform, the first-order equation in the above example can be solved simultaneously even though there is discontinuity at $t = 2$. The problem can also be solved using the method from Chapter 2. However, one needs to solve two

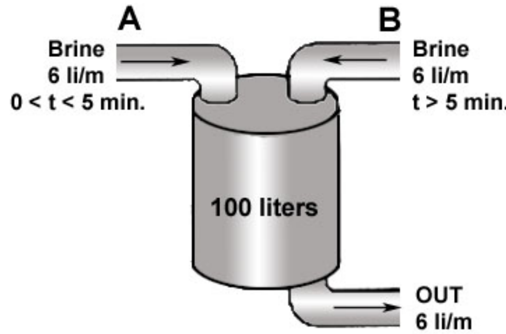


Figure 8.1: Mixing tank

different initial-value problems for different time period, namely

$$\frac{dy}{dt} = -3y \quad \text{for } 0 < t < 2 \tag{8.12}$$

$$\frac{dy}{dt} = -3y + 1 \quad \text{for } t \geq 2. \tag{8.13}$$

The first equation uses initial value $y(0) = -2$. Having found the solution to this problem on the time interval $(0, 2)$, we compute $y(2)$ and use it as the initial value for the second equation.

Example 8.11. *A storage tank with capacity 100 liters, shown in Figure 8.1, is full with brine. Connected to it are two input valves A and B, each of which delivers brine solution with different concentration. Initially, the tank contains 5 kg of salt (in 100 liters). At time $t = 0$, brine solution containing 0.3 kg of salt per liter flows into the tank from valve A at the rate 6 liters/min. At time $t = 5$ min., valve A is closed and valve B is opened, delivering brine solution, whose salt concentration is 0.6 kg/liter, at the rate 6 liters/min. The brine is pumped out of the tank also at 6 liters/min and thus maintaining the constant volume of the tank at all times. Determine the amount of salt in the tank at any given time t.*

Solution: Let $x(t)$ denote the amount of salt (in kg) in the tank at time t. Then the concentration of salt in the tank at time t is $x(t)/100$ kg/liter. The salt is added into the tank through the input valves at the rate $g(t)$, where

$$g(t) = \begin{cases} 0.3 \text{ kg/liter} \times 6 \text{ liters/min} = 1.8 \text{ kg/min}, & 0 < t < 5 \quad (\text{valve } A) \\ 0.6 \text{ kg/liter} \times 6 \text{ liters/min} = 3.6 \text{ kg/min}, & t \geq 5 \quad (\text{valve } B). \end{cases}$$

On the other hand, the amount of salt leaving the tank is $6x(t)/100$ kg/min. Hence, the rate of change of the amount of salt in the tank is given by

$$\frac{dx}{dt} = \text{ input rate } - \text{ output rate},$$

that is,

$$\frac{dx}{dt} = g(t) - \frac{6x}{100} \tag{8.14}$$

with the initial condition

$$x(0) = 5. \tag{8.15}$$

Before we proceed with Laplace transform, we notice that $g(t)$ is a step function with jump discontinuity at $t = 5$. Thus, we need to rewrite $g(t)$ in terms of $u_5(t)$, a *unit* step function whose Laplace transform is simply e^{-5s}/s. A quick observation tells us

$$g(t) = 1.8 + 1.8u_5(t)$$

and equation (8.14) now becomes

$$\frac{dx}{dt} = 1.8 + 1.8u_5 - 0.06x. \tag{8.16}$$

We take the Laplace transform of (8.16) and substitute the initial value (8.15) to get

$$\begin{aligned}
\mathcal{L}\{x'\} &= 1.8\mathcal{L}\{1\} + 1.8\mathcal{L}\{u_5\} - 0.06\mathcal{L}\{x\} \\
s\mathcal{L}\{x\} - x(0) &= \frac{1.8}{s} + 1.8\frac{e^{-5s}}{s} - 0.06\mathcal{L}\{x\} \\
(s+0.06)\mathcal{L}\{x\} &= 5 + \frac{1.8}{s} + 1.8\frac{e^{-5s}}{s} \\
\mathcal{L}\{x\} &= \frac{5}{s+0.06} + \frac{1.8}{s(s+0.06)} + \frac{1.8e^{-5s}}{s(s+0.06)}.
\end{aligned}$$

The solution is then given by

$$x = \mathcal{L}^{-1}\Big\{\frac{5}{s+0.06}\Big\} + \mathcal{L}^{-1}\Big\{\frac{1.8}{s(s+0.06)}\Big\} + \mathcal{L}^{-1}\Big\{\frac{1.8e^{-5s}}{s(s+0.06)}\Big\},$$

where the first term is

$$\mathcal{L}^{-1}\Big\{\frac{5}{s+0.06}\Big\} = 5e^{-0.06t} \tag{8.17}$$

and for the last two terms we again consider partial fraction

$$\frac{1}{s(s+0.06)} = \frac{1/0.06}{s} - \frac{1/0.06}{s+0.06},$$

from which it follows that

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\frac{1.8}{s(s+0.06)}\Big\} &= \mathcal{L}^{-1}\Big\{1.8\frac{1/0.06}{s}\Big\} - \mathcal{L}^{-1}\Big\{1.8\frac{1/0.06}{s+0.06}\Big\} \\ &= 30 - 30e^{-0.06t} \qquad (8.18)\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\frac{1.8e^{-5s}}{s(s+0.06)}\Big\} &= \mathcal{L}^{-1}\Big\{1.8(1/0.06)\frac{e^{-5s}}{s}\Big\} - \mathcal{L}^{-1}\Big\{1.8(1/0.06)\frac{e^{-5s}}{s+0.06}\Big\} \\ &= 30u_5(t) - 30\mathcal{L}^{-1}\Big\{\frac{e^{-5(s+0.06)}e^{0.3}}{s+0.06}\Big\} \\ &= 30u_5(t) - 30e^{0.3}e^{-0.06t}u_5(t) \\ &= 30u_5(t) - 30e^{-0.06(t-5)}u_5(t). \qquad (8.19)\end{aligned}$$

From (8.17), (8.18) and (8.19), we have the solution

$$x(t) = -25e^{-0.06t} + 30 + 30u_5(t) - 30e^{-0.06(t-5)}u_5(t),$$

or equivalently,

$$x(t) = -25e^{-0.06t} + 30 + 30 \cdot \begin{cases} 0, & 0 < t < 5 \\ 1 - e^{-0.06(t-5)}, & t \geq 5 \end{cases}.$$

8.5 Exercises

1. Use Definition 8.1 to determine the Laplace transform of the following functions:

   (a) $f(t) = 3t^2$

   (b) $f(t) = \sin t$

   (c) $f(t) = te^{-t}$

   (d) $f(t) = e^{-3t}\cos 2t$

(e) $f(t) = \begin{cases} 0, & 0 < t < 4 \\ 2, & t > 4 \end{cases}$

(f) $f(t) = \begin{cases} e^t, & 0 < t < 4 \\ 1, & t > 4 \end{cases}$

2. Find the Laplace transform $\mathcal{L}\{f(t)\}$ of the following functions:

   (a) $f(t) = t^2 + \sin 2t$

   (b) $f(t) = \cos^2 3t$

   (c) $f(t) = e^t \sin 3t - t^3 + 2$

   (d) $f(t) = \cos\sqrt{2}t - te^{2t} + e^{-t}$

3. Find the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ of the following functions:

   (a) $F(s) = \dfrac{1}{s^3}$

   (b) $F(s) = \dfrac{24}{(s-2)^5}$

   (c) $F(s) = \dfrac{s}{s^2-2}$

   (d) $F(s) = \dfrac{3s}{s^2+2}$

   (e) $F(s) = \dfrac{2s-8}{(s+2)(s-1)}$

   (f) $F(s) = \dfrac{2s-1}{2s^2-5s-3}$

4. Solve the following initial value problems using Laplace transform:

   (a) $y' - y = e^{-t},\ y(0) = 1$

   (b) $y' + 7y = -1,\ y(0) = -2$

   (c) $y'' - 5y' + 6y = e^{2t},\ y(0) = 0, y'(0) = -1$

   (d) $y'' - 3y' + 2y = \sin t,\ y(0) = 0, y'(0) = -1$

   (e) $y'' - y = g(t),\ y(0) = 2, y'(0) = 1$, where

   $$g(t) = \begin{cases} 1, & t < 3 \\ 0, & t > 3 \end{cases}$$

   (f) Consider the tank problem in Example 8.11. Suppose at initial time $t = 0$ valve B is opened for 10 minutes and then switched off and valve A is opened. Find the amount of salt in the tank at any given time t.

Appendix A

Review of Basic Linear Algebra

A.1 Introduction

A vector space $\mathcal{V}$ over the real numbers $\mathbb{R}$ (often denoted by $\mathcal{V}/\mathbb{R}$) is a set of elements with two operations: "addition" $+$ and "multiplication" $\cdot$ such that

$$\begin{aligned}
&\mathbf{u}, \mathbf{v} \in \mathcal{V} \Rightarrow \mathbf{u} + \mathbf{v} \in \mathcal{V} \\
&c \in \mathbb{R}, \mathbf{u} \in \mathcal{V} \Rightarrow c\,\mathbf{u} \in \mathcal{V} \\
&\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V} \Rightarrow \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\
&c \in \mathbb{R}, \mathbf{u}, \mathbf{v} \in \mathcal{V} \Rightarrow c\,(\mathbf{u} + \mathbf{v}) = (c\,\mathbf{u}) + (c\,\mathbf{v}) \\
&c, d \in \mathbb{R}, \mathbf{u} \in \mathcal{V} \Rightarrow c\,(d\,\mathbf{u}) = (cd)\,\mathbf{u} \\
&1\,\mathbf{u} = \mathbf{u} \\
&\exists\ \mathbf{0} \in \mathcal{V} \text{ such that } \forall\,\mathbf{u} \in \mathcal{V} \Rightarrow \mathbf{u} + \mathbf{0} = \mathbf{u}
\end{aligned}$$

Let us consider a couple of familiar examples.

The familiar 3-dimensional space $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$. Here vector addition is defined by $(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$ and vector multiplication by $c\,(x, y, z) = (cx, cy, cz)$ for any $c \in \mathbb{R}$. The "zero vector" is defined to be $\mathbf{0} = (0, 0, 0)$, i.e. the vector whose entries are all zero.

The space of 3×3 matrices with real entries,

$$M_{3\times3} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is a vector space. Vector addition here is the familiar matrix addition:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

For any scalar $c \in \mathbb{R}$,

$$c \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & ca_{13} \\ ca_{21} & ca_{22} & ca_{23} \\ ca_{31} & ca_{32} & ca_{33} \end{pmatrix}$$

and the "zero matrix" is simply the matrix all of whose entries are 0.

A.2 Linear independence

In a vector space $\mathcal{V}/\mathbb{R}$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are said to be ***linearly dependent*** if there are constants $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}.$$

This simply means that one of the $\mathbf{v}$'s can be written as a *non-trivial* linear combination of the others, that is, there is an i such that

$$\mathbf{v}_i = \sum_{\substack{j=1 \\ j \neq i}}^{k} d_j \mathbf{v}_j,$$

with at least one of the scalars d_js not equal to zero.

For instance, in $\mathbb{R}^2$, the vectors $\mathbf{v}_1 = (1,1), \mathbf{v}_2 = (0,3)$ and $\mathbf{v}_3 = (2,0)$ are linearly dependent since

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1(1,1) + c_2(0,3) + c_3(2,0) = (c_1 + 2c_3, c_1 + 3c_2) = (0,0)$$

if we simply choose $c_3 = -c_1/2, c_2 = -c_1/3$. One obvious choice is $c_1 = 6, c_2 = -2, c_3 = -3$ and indeed $6(1,1) - 2(0,3) - 3(2,0) = (0,0)$.

The vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are said to be ***linearly independent*** if they are ***not*** linearly dependent. Testing vectors for linear dependence is usually not easy! It often involves solving systems of equations. For example in $\mathbb{R}^3$, are the vectors

$(1,2,1), (-3,0,4)$ and $(6,-2,2)$ linearly independent? To check we try to find constants c_1, c_2, c_3 such that

$$c_1(1,2,1) + c_2(-3,0,4) + c_3(6,-2,2) = (0,0,0)$$

that is,

$$(c_1 - 3c_2 + 6c_3, 2c_1 - 2c_3, c_1 + 4c_2 + 2c_3) = (0,0,0),$$

which is equivalent to solving the system of equations

$$\begin{aligned} c_1 - 3c_2 + 6c_3 &= 0 \\ 2c_1 \qquad\quad - 2c_3 &= 0 \\ c_1 + 4c_2 + 2c_3 &= 0 \end{aligned}$$

It is an exercise for you to show that the only solution for this system is $c_1 = c_2 = c_3 = 0$ and it follows that the vectors $(1,2,1), (-3,0,4)$ and $(6,-2,2)$ are indeed linearly independent.

A.3 Bases

Given a vector space $\mathcal{V}/\mathbb{R}$, a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ is said to form a ***basis*** for $\mathcal{V}$ if for every vector $\mathbf{v} \in \mathcal{V}$, there exist *unique* constants $c_1, c_2, \cdots, c_k$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

The integer k is called the ***dimension*** of $\mathcal{V}$. In this case $\mathcal{V}$ is said to be ***finite dimensional***. It is a deep theorem in Linear Algebra, that in a vector space $\mathcal{V}$ of dimension k, there can be many sets of basis vectors! *But any basis will contain exactly k vectors only.* In fact, any k linearly independent vectors form a basis!

In $\mathbb{R}^3$, of dimension 3, any basis will contain three linear independent vectors. The *standard* basis is the familiar set of vectors $(1,0,0), (0,1,0)$ and $(0,0,1)$ (often referred as $\mathbf{i}, \mathbf{j}, \mathbf{k}$). But this is not the only basis. The set of vectors $\{(1,2,1), (2,0,1), (1,1,0)\}$ is also a basis. It is easy to verify that they are linearly independent. Note that there must still be three vectors in a basis.

In an n-dimensional vector space (e.g. $\mathbb{R}^n$), we require n constants $c_1, c_2, \cdots, c_n$ to determine a vector. For this reason we sometimes say we have n ***degrees of***

freedom. This point is worth noting for the rest of the course.

Appendix B

Operator Methods with Complex Coefficients

B.1 Introduction

In Chapter 4 we considered finding a particular solution for the equation

$$p(D)\,y = Q(x) \tag{B.1}$$

by using the method of undetermined coefficients and the operator method. However, while using the operator method we noticed it was easier to deal with some equations such as

$$y'' + a^2 y = \cos ax$$

by reverting to the UC method (Chapter 4 Remark 18). As you have seen, the UC method can become very tedious in some cases. If you wondered whether it was at all possible to avoid the method of undetermined coefficients and continue using the operator method while dealing with the trigonometric functions $\cos ax,\ \sin ax$, the answer is YES! This can be done elegantly by using complex numbers but one needs to be careful.

The operator method using complex coefficients is particularly useful when $Q(x)$ in (B.1) is of the form $x^k \cos ax, x^k \sin ax$ since the UC method may be very long and we cannot shift x^k. The method is also useful for those cases where $Q(x) = e^{kx}\cos bx$. Here one can simply shift e^{kx} and use the usual operator method. But the use of complex variables is equally well suited.

Recall Euler's formula,

$$e^{(a\pm ib)x} = e^{ax}(\cos bx \pm i \sin bx).$$

In particular $e^{iax} = \cos ax + i \sin ax$ and hence we can write

$$\cos ax = \Re(e^{iax}), \quad \sin ax = \Im(e^{iax}).$$

Here $\Re$ stands for the real part and $\Im$ for the imaginary part. It is useful to note that

$$1/i = -i \quad \text{and} \quad e^{iax}/i = \sin ax - i \cos ax.$$

These facts are used in problems involving complex numbers.

The following may seem a little heavy going at first. But if you study it carefully, it is no more difficult than the other methods. One just needs to keep track of real and imaginary parts at all times.

Consider Example 4.11 from Chapter 4 once again:

Example B.1. *Find y_p for the equation*

$$y'' + a^2 y = \cos ax$$

using the operator method.

Solution: We will find the particular solution η_p for the problem $y'' + a^2 y = e^{iax}$ and use the fact $y_p = \Re(\eta_p)$. We have

$$\begin{aligned}
\eta_p &= \frac{1}{D^2 + a^2} e^{iax} \\
&= e^{iax} \frac{1}{(D + ia)^2 + a^2} \cdot 1 \quad \textit{(using exponential shift)} \\
&= e^{iax} \frac{1}{(D^2 + 2iaD)} \cdot 1 \\
&= e^{iax} \frac{1}{D} \frac{1}{(D + 2ia)} \cdot e^{0x} \\
&= \frac{e^{iax}}{2ia} \frac{1}{D} \cdot 1 \qquad \textit{(from Remark 14)} \\
&= \frac{\sin ax - i \cos ax}{2a} \cdot x
\end{aligned}$$

Hence,

$$y_p = \Re(\eta_p) = \frac{x \sin ax}{2a},$$

as obtained by the method of undetermined coefficients.

Notice that as a bonus we have found y_p for the problem

$$y'' + a^2 y = \sin ax$$

since in this case

$$y_p = \Im(\eta_p) = \frac{-x\cos ax}{2a},$$

also as found by the method of undetermined coefficients.

This method is quite powerful! Consider the following example.

Example B.2. *Solve the equation*

$$(D^4 + 2a^2 D^2 + a^4)y = \cos ax.$$

Solution: First note that the left hand side can be factored as $(D^2 + a^2)^2 y$. Thus, the complementary solution is

$$y_c = (Ax + B)\cos ax + (Cx + D)\sin ax.$$

The method of undetermined coefficients is very involved. On the other hand, proceeding as in Example B.1,

$$\begin{aligned}
&\frac{1}{\left(D^2 + a^2\right)^2} e^{iax} \\
&= e^{iax} \frac{1}{\left((D + ia)^2 + a^2\right)^2} \cdot 1 \\
&= e^{iax} \frac{1}{\left(D^2 + 2iaD\right)^2} \cdot 1 \\
&= e^{iax} \frac{1}{D^2} \frac{1}{(D + 2ia)^2} \cdot e^{0x} \\
&= \frac{e^{iax}}{4i^2 a^2} \frac{1}{D^2} \cdot 1 \qquad \textit{(from Remark 14)} \\
&= -\frac{e^{iax}}{4a^2} \frac{x^2}{2} \\
&= -\frac{x^2}{8a^2}(\cos ax + i\sin ax).
\end{aligned}$$

It follows that

$$y_p = \Re\left\{\frac{1}{(D^2 + a^2)^2}\right\} e^{iax} = -\frac{x^2 \cos ax}{8a^2},$$

and the complete solution is

$$y = (Ax + B)\cos ax + (Cx + D)\sin ax - \frac{x^2 \cos ax}{8a^2}.$$

Now try doing that with the method of undetermined coefficients!

B.2 Exercises

Solve the following equations using the operator method.

1. $y'' + y = x^2 \cos x$.
2. $y'' + 4y = x \cos 2x$.
3. $(D - 2)^2 y = e^{2x} \sin x$.

Answers and Hints to Selected Exercises

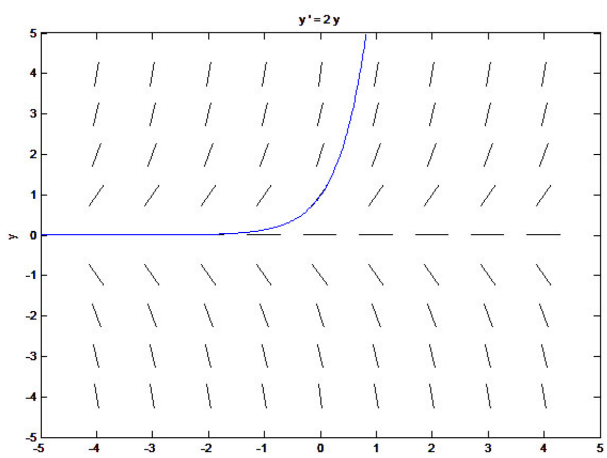
Exercises 1.4

1. (a) Trivial.

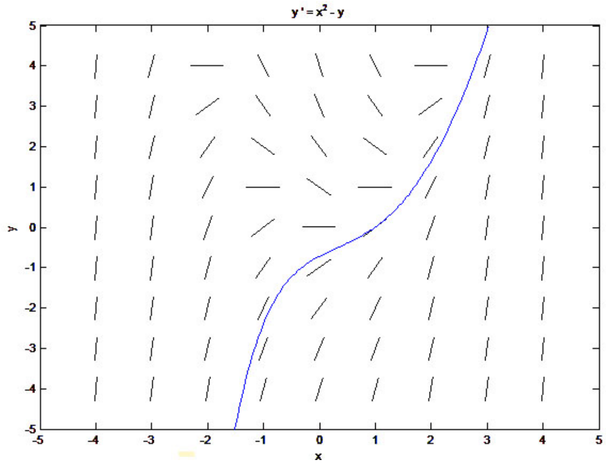
   (c) Differentiate the first equation to get $y' = (xy' + y)/\sqrt{1 - x^2y^2}$ and cross multiply.

   (e) Differentiate twice to get $y' = a\cos x - b\sin x$ and $y'' = -a\sin x - b\cos x = -y$ from which $y'' + y = 0$.

2. (a) The isoclines are the lines $2y = c$, where c is any constant. For any x_0, if $y(x_0) > 0$, then $y \to \infty$ as $x \to \infty$. If $y(x_0) < 0$, then $y \to -\infty$, and $y = 0$ if $y(x_0) = 0$.



   (c) The isoclines are the parabolas $y = x^2 - c$, where c is any constant.



Exercises 2.6

1. (a) $1/y^4 + 1/x^4 = c$.

   (b) Rewrite as $dy/y = \tan x dx$. Solution is $y = c \sec x$.

   (c) $y^2/2 + 2y = \arctan x + c$.

   (d) Solution is $2y^2 = 2x^2 + x^4 + c$. From the initial conditions $c = 15$, the unique solution is $2y^2 = 2x^2 + x^4 + 15$.

   (e) $-1/y = x^3/3 - 1/6$.

2. (a) $y = \frac{6}{7}x^3 + cx^{-4}$.

   (c) Dividing by x throughout, $\dfrac{dy}{dx} + \dfrac{2x+1}{x(x+1)}y = \dfrac{x-1}{x}$. This is linear with $P = (2x+1)/(x^2+x)$ so the integrating factor is (x^2+x). Hence the solution is $y(x^2+x) = x^3/3 - x + c$.

   (e) $xy = 1 + c\exp(-y^2/2)$.

   (g) $yx - x = cy$.

3. (a) Regroup as $(3x + x^2)dx + (y - \frac{1}{y^2})dy + 2(ydx + xdy) = 0$. Recall that $(ydx + xdy) = d(xy)$.

   (c) The given equation can be rewritten as $(6xydx + 3x^2dy) + (2y^2dx + 4xydy) = 0$ and the solution is $3x^2y + 2xy^2 = c$.

   (e) If you rewrite it as $(1/t)\{(3s^2 - 2)ds\} + (s^3 - 2s)\{\frac{-dt}{t^2}\}$ the solution is obvious!

(h) Rewrite as $-\{dx/x^2 + dy/y^2\} + (ydx - xdy)/x^2 = 0$ and remember the last term is $d(y/x)$.

(i) $f(x, y) = 3x^2y + 2xy^2 - 5x - 6y + c$.

4. (c) After you substitute $y = vx$, you should get $v + x(dv/dx) = v \ln v$ from which one gets $dv/(v(\ln v - 1)) = dx/x$. This can be integrated as $\ln v - 1 = cx$ and the solution is $y = x \exp(cx + 1)$.

5. $T_0 = 30$ degrees.

Exercises 3.4

2. The auxiliary equation is $4m^2 - 7m + 3 = 0$ whose roots are $m = 1, 3/4$ and the general solution is $y = c_1 \exp(x) + c_2 \exp(\frac{3}{4}x)$.

5. The auxiliary equation is $(m + 2)^2 = 0$ and has repeated roots -2, -2. The solution is $y = c_1 \exp(-2x) + c_2 x \exp(-2x)$.

6. The auxiliary equation can be factored as $(m^2 + 1)(m + 3) = 0$ whose roots are $m = \pm i, -3$. The solution is $c_1 \cos x + c_2 \sin x + \exp(-3x)$.

7. The general solution is $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$. Applying the initial conditions we get $c_1 = -3, c_2 = 2$ and the solution is $y = e^{3x}(-3 \cos 4x + 2 \sin 4x)$.

8. One root of the auxiliary equation is -1.

9. $y = e^{-x}(c_1 \cos \frac{3}{4}x + c_2 \sin \frac{3}{4}x)$.

10. The auxiliary equation is $8m^3 - 12m^2 + 6m + 1 = 0$ which is really $(2m+1)^3 = 0$ with repeated roots $-1/2, -1/2, -1/2$.

11. The equation of motion of the pendulum is $\theta(t) = c \cos(\omega t + \delta)$, where $\omega = \sqrt{g/\ell}$. The velocity is given by $v = -c\omega \sin(\omega t + \delta)$. At $t = 0, \theta = 1$ rad, $v = 0$. Hence, $\sin \delta = 0 \Rightarrow \delta = 0$. It follows that $c = 1$ and hence the equation of motion is $\theta(t) = \cos(\omega t)$. The amplitude is 1, period T is $2\pi/\omega = 2\pi/\sqrt{g}$, the frequency $f = \sqrt{g}/2\pi$. At the equilibrium position $\theta = 0 \Rightarrow \omega t = \pm\pi/2, v = \pm\omega = \pm\sqrt{g}$, which is the linear velocity. Then angular velocity is $v\ell = \pm\sqrt{g}$, same as v.

13. In equilibrium, the spring is stretched by six inches = 1/2 ft. The only forces on the object are the restorative force of the spring and its weight 4 lbs. If the mass of the object is m then $4 = mg, m = 4/32 = 1/8$. Assuming that

the constant of proportionality of Hooke's law is k, $(1/2)k = 4$, from which $k = 8$. At any time time t,since the object stretches it by 1/2 ft, the total extension of the spring length is $y + \frac{1}{2}$. The restorative force of the spring acting upward is $k(y + 1/2) = 8y + 4$ acting upwards with its weight mg acting downwards. The net downward force then is $mg - (8y + 4) = -8y$. By Newton's law, $m\dfrac{d^2y}{dt^2} = -8y$, that is, $\dfrac{d^2y}{dt^2} + 64y = 0$ and this shows the motion is simple harmonic. The equation of motion is $y = c\cos(8t + \delta)$. Hence, $v = -8c\sin(8t + \delta)$. Since $y(0) = 1/2, v(0) = 0$, $\delta = 0$, and $c = 1/2$. Thus, the equation of motion is $y(t) = \frac{1}{2}\cos(8t)$. The period is $2\pi/8 = \pi/4$ sec, the frequency $f = 4/pi$ cycels per second. The amplitude is 6 inches.

Exercises 4.5

1. The auxiliary equation is $m^2 + 3m - 10 = 0$ with roots $m = -5, 2$. Hence $y_c = c_1e^{-5x} + c_2e^{2x}$. This is Case I. To find y_p, let $y_p = Ae^{4x}$. Therefore, $y_p' = 4Ae^{4x}, y_p'' = 16Ae^{4x}$. Substituting in the original equation we get $18Ae^{4x} = 6e^{4x}$ that is $A = \frac{1}{3}$ and the complete solution is $y = c_1e^{-5x} + c_2e^{2x} + \frac{1}{3}e^{4x}$.

3. The auxiliary equation $m^2 - m - 6 = 0$ has roots $m = 3, -2$ and $y_c = c_1e^{3x} + c_2e^{-2x}$. This is Case II with $k = 0$ since e^{-2x} occurs on the right. We assume $y_p = Axe^{-2x}$ and get $y_p' = A(e^{-2x} - 2xe^{-2x})$, $\quad y_p'' = A(-4e^{-2x} + 4xe^{-2x})$. Substitute in the original equation to get $-5Ae^{-2x} = 20e^{-2x}$ that is $A = -4$. The complete solution is:$y = c_1e^{3x} + c_2e^{-2x} - 4xe^{-2x}$. Again note cancelation of the term xe^{-2x} when we substitute in the original equation.

5. From the auxiliary equation $m^2-2m+1 = 0$ we get $m = 1, 1,\ y_c = (c_1+c_2x)e^x$. This is Case III. The roots $m = 1$ has multiplicity 2, that is $r = 2$. The right side is simply e^x so that $k = 0$. Hence we assume $y_p = Ax^2e^x$ from which $y_p' = Ae^x(2x+x^2)$, $\quad y_p'' = Ae^x(2+4x+x^2)$. Substitute to get $Ae^x = 6e^x, A = 3$ and $y = c_1e^x + c_2xe^x + 3x^2e^x$.

7. The auxiliary equation being $m^2 + 4 = 0$, we have $m = \pm 2i, y_c = c_1\cos 2x + c_2\sin 2x,\ F = \{\cos 2x, \sin 2x\}$, $\quad G = \{\cos 2x, \cos x, x^2, x\}$. The first term in G is case II, the others are case I. To find y_p we break the problem into three subproblems: (a) $y''+4y = 4\cos 2x$, (b) $y''+4y = 6\cos x$, (c) $y''+4y = 8x^2-4x$. We will write $y_{p_1}, y_{p_2}, y_{p_3}$ for the particular solutions of these subproblems.

   (a) Here $G = \{\cos 2x\}$, and this is case II. Hence, $y_{p_1} = x(A\sin 2x + B\cos 2x)$.

This gives $y'_{p_1} = x(2A\cos 2x - 2B\cos 2x) + (A\sin 2x + B\cos 2x)$ and $y''_{p_1} = x(-4A\sin 2x - 4B\cos 2x) + (4A\cos 2x - 4B\sin 2x)$. Substitute in (a) to get $y''_{p_1} + 4y_{p_1} = 4A\cos 2x - 4B\sin 2x = 4\cos 2x$. Thus, $A = 1, B = 0$ and $y_{p_1} = x\sin 2x$.

(b) Here $G = \{\cos x\}$ so this is case I and $y_{p_2} = A\cos x + B\sin x$ so that $y'_{p_2} = -A\sin x + B\cos x, y''_{p_2} = -A\cos x - B\sin x$. Therefore, $y''_{p_2} + 4y_{p_2} = 3A\cos x + 3B\sin x = 6\cos x$, that is, $A = 2, B = 0, y_{p_2} = 2\cos x$.

(c) It is trivial to see this is also case I so $y_{p_3} = Ax^2 + Bx + C$ to get $y'_{p_3} = 2Ax + B, y''_{p_3} = 2A, y''_{p_3} + 4y_{p_3} = Ax^2 + 4Bx + (2A + 4C) = 8x^2 - 4x$. Hence, $A = 2, B = -1, C = -1$ so $y_{p_3} = 2x^2 - x - 1$.

From (i), (ii) and (iii)the particular solution of the original problem is $y_p = y_{p_1} + y_{p_2} + y_{p_3} = x\sin 2x + 2\cos x + 2x^2 - x - 1$. The general solution is $y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + x\sin 2x + 2\cos x + 2x^2 - x - 1$.

9. Here $y_p = \frac{10}{(D+2)(D+2)}x^3e^{-2x} = \frac{10e^{-2x}}{D^2}x^3 = 10e^{-2x}x^5/20 = \frac{1}{2}x^5e^{-2x}$.

11. Here $y_p = \frac{1}{D^2-D+1}(x^3 - 3x^2 + 1) = \frac{1}{1+D^2-D}(x^3 - 3x^2 + 1) = \{1 - (D^2 - D) + (D^2 - D)^2 - (D^2 - D)^3 + ...\}(x^3 - 3x^2 + 1) = \{1 - D^2 + D + D^4 - 2D^3 - (D^6 - 3D^5 + 3D^4 - D^3)\}(x^3 - 3x^2 + 1) = \{1 + D - D^3\}(x^3 - 3x^2 + 1) = \{(x^3 - 3x^2 + 1) + (3x^2 - 6x) - 6)\} = x^3 - 6x - 5$, where we have dropped powers of D higher than 3.

13. $y_p = \frac{1}{D^3-8}16x^2 = -\frac{16}{8}\frac{1}{1-D^3/8}x^2 = -2(1 + D^3/8 + D^6/64 + ..)x^2 = -2x^2$.

15. $(D-2)^2 = e^{2x}\sin x$. The auxiliary equation is $(m-2)^2 = 0$ with roots $m = 2, 2$. Hence, $y_c = c_1e^{2x} + c_2e^{2x}$ and $y_p = \frac{1}{(D-2)^2}e^{2x}\sin x = e^{2x}\cdot\frac{1}{D^2}\sin x = -e^{2x}\sin x$ and $y = c_1e^{2x} + c_2e^{2x} - e^{2x}\sin x$.

Exercises 5.5

1. $y = cx + dx^4$, where c, d are constants.

3. $y = ce^{2x} + d(x + 1)$, where c, d are constants.

5. If $y = e^x$ is a solution then substituting for y in $y'' + py' + qy = 0$ gives $e^x + pe^x + qe^x = 0$ which is possible only if $1 + p + q = 0$. Clearly the converse is also true. This condition applies in this problem since $(x-1) - x + 1 = 0$. Thus, as can easily be verified $y = e^x$ is a solution. To find another solution, in this

problem $p = -x/(x-1)$, $\int(-pdx) = x + \ln(x-1)$, $\exp(\int(-p)dx = (x-1)e^x$. Hence, $v = \int(x-1)e^x/(e^{2x})dx = \int(x-1)e^{-x}dx = -xe^{-x}$. Hence $y_2 = -x$ as can be verified. The complete solution is $y = c_1 x + c_2 e^x$.

7. $y = c_1 \cos x + c_2 \sin x + \cos x \ln \cos x + x \sin x$.

9. $y_c = c_1 \cdot 1 + c_2 = e^{2x}$ and $y = y_c + 2x^2 e^{2x} - 2xe^{2x}$.

11. $y = x^2 e^x \left(\frac{1}{2}\ln x - \frac{3}{4}\right)$.

13. $y = \cos x \ln \cos x + x \sin x$.

15. $y = \frac{x^4}{6} - \frac{x^2}{2}$.

17. The auxiliary equation is $\theta^2 - 4\theta + 6 = 0$ with roots $2 \pm i\sqrt{3}$. So the solution is $e^{2t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t) = x^2(c_1 \cos\sqrt{3}\ln x + c_2 \sin\sqrt{3}\ln x)$.

19. The auxiliary equation is $(\theta-2)(\theta-1)\theta + 2(\theta-1)\theta - 10\theta - 8 = 0$. By inspection, $\theta = -1$ is a root so $(\theta+1)$ is a factor. Dividing by $(\theta+1)$ we get $(\theta+1)(\theta^2 - 2\theta - 8) = (\theta+1)(\theta-4)(\theta+2) = 0 \Rightarrow y = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{4t} = c_1 x^{-1} + c_2 x^{-2} + c_3 x^4$.

21. $y = c_1 x^5 + c_2 x^{-1} - (5/9)x^2$.

Exercises 6.6

1. $\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)x^n$.

3. a_0 and a_1 are arbitrary, $a_2 = a_0$ and $a_{n+2} = -\frac{(n-2)a_n + 3a_{n-1}}{(n+2)(n+1)}$, $n \geq 1$.

5. $a_2 = -\frac{1}{4}\, a_0$, $\quad a_{n+2} = -\frac{1}{2(n+2)} a_n$, $\quad n \geq 1$. One quickly computes that

$$\begin{aligned} a_2 &= -(1/2^2)a_0, \quad a_4 = (1/(2^4 2!))a_0, \quad a_6 = -(1/2^6 3!)a_0, \ldots, \\ a_{2n} &= [(-1)^n/(2^{2n}n!)]a_0 \ldots \end{aligned}$$

and likewise we have $a_3 = -(1/2 \cdot 3)a_1$, $\quad a_5 = (-1)^2(1/(2^2 \cdot 3 \cdot 5)), \ldots, a_{2n+1} = (-1)^n/[2^n(1 \cdot 3 \cdot 5 \ldots (2n+1)]a_1$, both formulas valid for $n \geq 1$.

7. Using Leibniz's theorem we get

$$\left\{(1+x)^2 y_{n+2} + \binom{n}{1} 2x\, y_{n+1} + \binom{n}{2} 2\, y_n\right\} - \left\{x y_{n+1} + \binom{n}{1} y_n\right\} + \left\{x y_n + \binom{n}{1} y_{n-1}\right\} = 0,$$

which simplified becomes $(1+x^2)y_{n+2}+(2n-1)\,xy_{n+1}+(n^2-2n+x)y_n+ny_{n-1} = 0$.

9. $a_{n+2} = -\frac{1}{(n+1)(n+2)}\left\{(n-1)a_n + 2a_{n-2}\right\}, \quad n \geq 2$, where a_0, a_1 are arbitrary, $a_2 = \frac{1}{2}a_0$, and $a_3 = 0$.

11. $a_{n+2} = \frac{(n+1)a_{n+1}-2a_{n-1}}{(n+1)(n+2)}, \quad n \geq 1, \; a_2 = (1/2)a_1$ and a_0, a_1 are arbitrary.

13. $a_{n+2} = -\frac{1}{2}\frac{1}{(n+2)}a_n, \quad n \geq 0, \; a_0, a_1$ arbitrary.

14. By Leibniz's theorem, $y_{n+2} - xy_n - ny_n = 0$ and at $x = 0$ this becomes $y_{n+2} = ny_n$. After substituting $(n+2)!a_{n+2}$ for y_{n+2} and $(n_1)!a_{n-1}$ for y_{n_1} we get the recurrence relation $a_{n+2} = \frac{1}{(n+2)(n+1)}\,a_{n-1}, \quad n \geq 1$. Since $an+2$ is given in terms of a_{n-1}, the subscripts for a_n jump by three. From the given equation, $y_2 = 0 = a_2$. Hence $a_2 = a_5 = a_8 = a_{11} = \ldots = 0$. In general, $a_{3n+2} = 0, \; n \geq 0$. From the recurrence formula it is easy to compute that

$$a_3 = \frac{a_0}{3\cdot 2}, \; a_6 = \frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2}, a_9 = \frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}\ldots,$$

and likewise

$$a_4 = \frac{a_1}{4\cdot 3}, \; a_7 = \frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3}, \; a_{10} = \frac{a_7}{10\cdot 9} = \frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}, \ldots .$$

These results seem to show that

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\ldots(3n-1)(3n)}, \quad a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\ldots(3n)(3n+1)}, \quad n \geq 1.$$

Thus the general solution is

$$y = a_0\left\{1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdot 5\cdots(3n-1)(3n)}\right\}+a_1\left\{x+\sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}\right\}.$$

Exercises 7.5

1. $x = c_1e^{-2t} + \frac{1}{2}e^{4t}, \; y = -\frac{2}{3}c_1e^{-2t} - \frac{1}{3}e^{4t}$.

3. $x = c_1e^{-2t} + \frac{1}{6}e^t + \frac{1}{10}e^{3t}, \; y = -\frac{1}{2}c_1e^{-2t} - \frac{1}{3}e^t + \frac{1}{5}e^{3t}$.

5. $x = c_1e^{-3t} + c_2e^t - \frac{1}{3}t + \frac{1}{9}, \; y = \frac{5}{3}t - 5c_1e^{-3t} - c_2e^t - \frac{5}{9}$.

7. $x = c_1\cos t + c_2\sin t - t - 2, \; y = -(c_1+c_2)\cos t + (c_1-c_2)\sin t - 2$.

9. In determinant form this is

$$\begin{vmatrix} 2D-1 & D-1 \\ D+1 & D-1 \end{vmatrix} x = - \begin{vmatrix} D-1 & 2e^t \\ D-1 & e^t \end{vmatrix}.$$

In this example, $p_2(D) = p_4(D)$, that is the coefficients of y in both equations are the same and any attempt to eliminate y' will also eliminate y (this is the degenerate case). We cannot use the determinant method here since this will introduce an additional factor $(D-1)$ and hence the additional term c_1e^t in x_c. We eliminate y (and y') simply by subtraction to get $x' - 2x = e^t$ from which $x = c_1e^{2t} - e^t$. From the second equation, $y' - y = e^t - x' - x$ from which $y = c_2e^t - 3c_1e^{2t} + 3te^t$.

11. $x = c_1e^{4t} + c_2e^{-2t}, \quad y = c_1e^{4t} - c_2e^{-2t}$.

12. In determinant form, this problem becomes

$$\begin{vmatrix} D^2-1 & D+1 \\ D-1 & D^2+1 \end{vmatrix} x = - \begin{vmatrix} D+1 & \sin t \\ D^2+1 & \cos t \end{vmatrix} = \sin t - \cos t$$

or $(D^4 - D^2)x = \sin t - \cos t$ and whose solution is

$$\begin{aligned} x_c &= c_1 + c_2t + c_3e^t + c_4e^{-t}, \\ x_p &= \frac{1}{D^2(D^2-1)}(\sin t - \cos t) = \frac{1}{D^2-1}(\cos t - \sin t) \\ &= (\sin t - \cos t)/2. \end{aligned}$$

To find y_p, we subtract the second equation from the first to get

$$x_p'' - y_p'' - (x_p' - y_p') = \sin t - \cos t,$$

that is

$$(D^2 - D)(x-y)_p = (\sin t - \cos t), \quad (x-y)_p = \frac{1}{D^2 - D}(\sin t - \cos t)$$

from which

$$x_p - y_p = (x-y)_p = -\frac{1-D}{1-D^2}(\sin t - \cos t) = \cos t.$$

It follows that

$$y_p = \tfrac{1}{2}\big(\sin t - 3\cos t\big).$$

Finally to find y_c we have from the second equation,

$$(D^2+1)y_c + (D-1)x_c = 0 \quad \text{from which} \quad (D^2+1)y_c = x_c - x_c'.$$

Hence,

$$\begin{aligned}(D^2+1)y_c &= \big\{c_1 + c_2 t + c_3 e^t + c_4 e^{-t}\big\} \\ &\qquad - \big\{c_2 + c_3 e^t - c_4 e^{-t}\big\} \\ &= (c_1 - c_2) + c_2 t + 2c_4 e^{-t},\end{aligned}$$

from which $y_c = (c_1 - c_2) + c_2 t + c_4 e^{-t}$. It follows that

$$\begin{aligned}x &= c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + \tfrac{1}{2}(\sin t - \cos t) \\ y &= (c_1 - c_2) + c_2 t + c_4 e^{-t} + \tfrac{1}{2}(\sin t - 3\cos t).\end{aligned}$$

Exercises 8.5

1. (a) $2/s^3$.
   (c) $1/(s+1)^2$.
   (e) The integral becomes $\int_4^\infty 2e^{-st}dt$, which after taking the limit becomes $(2/s)e^{-4s}$.

2. (a) $2/s^3 + 2/(s^2+4)$.
   (c) $\frac{3}{(s-1)^2+9} - \frac{6}{s^4} + \frac{2}{s}$.

3. (a) $\frac{1}{2}t^2$.
   (c) $\cosh \sqrt{2}t$.
   (e) $4e^{-2t} - 2e^t$.

4. (a) $y = \frac{-1}{2}e^{-t} + \frac{3}{2}e^t$.
   (c) $y = -te^{2t}$.
   (e) Write $g(t) = 1 - u_3(t)$, where u_3 is a unit step function with jump discontinuity at $t = 3$. Taking Laplace transform on both sides and simplifying gives $\mathcal{L}\{y\} = \dfrac{2s^2+s+1-e^{-3s}}{s(s+1)(s-1)}$. Using partial fractions, $\frac{2s^2+s+1}{s(s+1)(s-1)} =$

$\frac{-1}{s} + \frac{1}{s+1} + \frac{2}{s-1}$ and $\frac{e^{-3s}}{s(s+1)(s-1)} = \frac{-e^{-3s}}{s} + \frac{1/2e^{-3s}}{s+1} + \frac{1/2e^{-3s}}{s-1}$. Taking the inverse transform we have $y = e^{-t} + 2e^{t} - 1 + (1 - \frac{1}{2}e^{-t+3} - \frac{1}{2}e^{t-3})u_3(t)$.

(f) Write the input rate $g(t)$ as $g(t) = 3.6 - 1.8u_{10}(t)$.

Exercises B.2

1. $(D^2+1)y = x^2\cos x$. You should not dream of shifting x. Instead we will use complex numbers. Let $z = \{1/(D^2+1)\}x^2e^{ix}$. Then $y_p = \Re(z)$ and we have $z = \frac{1}{(D^2+1)}x^2e^{ix} = e^{ix}\frac{1}{(D+i)^2+1}x^2 = e^{ix}\frac{1}{(D^2+2iD)}x^2 = e^{ix}\frac{1}{D(D+2i)}x^2 = \frac{e^{ix}}{2i}\left\{\frac{1}{D}\frac{1}{(1+D/2i)}\right\}x^2 = \frac{e^{ix}}{2i}\left\{\frac{1}{D}\left(1 - \frac{D}{2i} + \frac{D^2}{4i^2}\right)\right\}x^2 = \frac{e^{ix}}{2i}\left\{\frac{1}{D}\left(x^2 - \frac{2x}{2i} + \frac{2}{4i^2}\right)\right\} = \frac{-ie^{ix}}{2}\left\{\frac{1}{D}\left(x^2+ix-\frac{1}{2}\right)\right\} = \frac{-ie^{ix}}{2}\left(\frac{x^3}{3}+\frac{ix^2}{2}-\frac{x}{2}\right) = \frac{(\sin x - i\cos x)}{2}\left(\frac{x^3}{3}+\frac{ix^2}{2}-\frac{x}{2}\right)$, where we have used $1/i = -i$. Since $y_p = \Re(z)$, it follows that $y_p = (\sin x)/2\left\{x^3/3 - x/2\right\} + (x^2\cos x)/4$.

3. The auxiliary equation is $(m-2)^2 = 0$ with roots $m = 2, 2$. Hence, $y_c = c_1e^{2x} + c_2e^{2x}$ and $y_p = \frac{1}{(D-2)^2}e^{2x}\sin x = e^{2x}\cdot\frac{1}{D^2}\sin x = -e^{2x}\sin x$ and $y = c_1e^{2x} + c_2e^{2x} - e^{2x}\sin x$.