Sketch of solutions to Midterm 2, Math 402A W112

Q1: $\tan z = \tan\frac{\pi}{3}$

$$\tan z = \frac{e^{iz}-e^{-iz}}{2i}\cdot\frac{2}{e^{iz}+e^{-iz}} = \frac{e^{2iz}-1}{i(e^{2iz}+1)}.$$

$$\therefore e^{2iz}-1 = i\tan\frac{\pi}{3}(e^{2iz}+1) \quad \therefore e^{2iz}\left(1-i\tan\frac{\pi}{3}\right) = 1+i\tan\frac{\pi}{3}$$

$$\therefore e^{2iz}\,\frac{\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}}{\cos\pi/3} = \frac{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}{\cos\pi/3}$$

$$\therefore e^{2iz}e^{-i\pi/3} = e^{i\pi/3} \quad \therefore e^{2iz} = e^{2i\pi/3} \quad \therefore z = \frac{\pi}{3}+k\pi,\ k\in\mathbb{Z}.$$

Q2: $\nabla f = \frac{\partial f}{\partial x}\underline{i} + \frac{\partial f}{\partial y}\underline{j} + \frac{\partial f}{\partial z}\underline{k}$

$$= \frac{1}{1+x^2+yz}\,\frac{1}{2}(x^2+yz)^{-1/2}\{2x\underline{i}+z\underline{j}+y\underline{k}\}.$$

Q3: $\operatorname{div}\underline{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}.$

$$\frac{\partial F_1}{\partial x} = \frac{1}{\frac{yz}{x^2}\sqrt{\frac{y^2z^2}{x^4}-1}}\,(-2yzx^{-3}) = -\frac{2xyz}{yz\sqrt{y^2z^2-x^4}}.$$

$$\frac{\partial F_2}{\partial y} = \frac{1}{1+\frac{z^2}{x^2y^2}}\cdot\left(-\frac{z}{x}\right)y^{-2} = -\frac{xz}{x^2y^2+z^2}.$$

$$\frac{\partial F_3}{\partial z} = \frac{\partial}{\partial z}\left\{\frac{-5}{2}\log(z^3+7)\right\} = -\frac{5}{2(z^3+7)}\cdot 3z^2 = -\frac{15z^2}{z^3+7}.$$

Q4: From Q2, we may choose $f = 2\arctan\sqrt{x^2+yz}$

$\therefore$ The integral $= 2\arctan\sqrt{2} - 2\arctan 1$

$= 2\arctan(3-2\sqrt{2})$

Q5: $\frac{d\underline{r}}{dt} = [\sinh t, \cosh t, 1].$

$$\left|\frac{d\underline{r}}{dt}\right|^2 = \left(\frac{ds}{dt}\right)^2 = \sinh^2 t+\cosh^2 t+1 = 2\cosh^2 t$$

$$\therefore \frac{ds}{dt} = \sqrt{2}\cosh t.$$

$$\therefore \underline{T} = \frac{1}{\sqrt{2}}[\tanh t,\ 1,\ \operatorname{sech} t]$$

$$\text{Arclength} = \int_0^{\log 2}\sqrt{2}\cosh t\,dt = \sqrt{2}[\sinh(\log 2)-\sinh(0)] = \frac{e^2-e^{-2}}{\sqrt{2}}$$

$$\frac{dT}{dt} = \frac{1}{\sqrt{2}}[\operatorname{sech}^2 t, 0, -\operatorname{sech} t \tanh t].$$

$$\frac{dT}{ds} = \frac{1}{2}[\operatorname{sech}^3 t, 0, -\operatorname{sech}^2 t \tanh t].$$

$$\kappa^2 = \left|\frac{dT}{ds}\right|^2 = \frac{1}{4}[\operatorname{sech}^6 t + \operatorname{sech}^4 t \tanh^2 t]$$
$$= \frac{1}{4}\operatorname{sech}^4 t[\operatorname{sech}^2 t + \tanh^2 t] = \frac{1}{4}\operatorname{sech}^4 t$$

$$\therefore \kappa = \frac{1}{2}\operatorname{sech}^2 t, \quad \underline{N} = [\operatorname{sech} t, 0, \tanh t].$$

Q6: $\underline{F}\cdot\underline{n} = x^4 + y^4 + \frac{1}{2}z^4$. With

$x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$,

$$\underline{F}\cdot\underline{n} = \sin^4\theta(\cos^4\varphi + \sin^4\varphi) + \frac{1}{2}\cos^4\theta$$
$$\cos^4\varphi + \sin^4\varphi = (\cos^2\varphi + \sin^2\varphi)^2 - 2\cos^2\varphi\sin^2\varphi$$
$$= 1 - \frac{1}{2}\sin^2(2\varphi) = 1 - \frac{1}{4}(1-\cos 4\varphi) = \frac{3}{4} + \frac{1}{4}\cos 4\varphi$$

$dA = \sin\theta \, d\varphi \, d\theta$.

$$\therefore \iint_S \underline{F}\cdot\underline{n}\, dA = \int_0^{\pi/2}\int_{-\pi}^{\pi}\left[\sin^4\theta\left(\frac{3}{4} + \frac{1}{4}\cos 4\varphi\right) + \frac{1}{2}\cos^4\theta\right]\sin\theta \, d\varphi\, d\theta$$

$$= \int_0^{\pi/2} \frac{3\pi}{2}\sin^4\theta\sin\theta\, d\theta + \pi\int_0^{\pi/2}\cos^4\theta\sin\theta\, d\theta$$

$$= \frac{3\pi}{2}I + \pi\left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2} = \frac{3\pi}{2}I - \frac{\pi}{5}.$$

$$I = \int_0^{\pi/2}(1-\cos^2\theta)^2\sin\theta\, d\theta = \int_0^1 (1-u^2)^2\, du$$

$$= \int_0^1 (1 + u^4 - 2u^2)\, du = 1 + \frac{1}{5} - \frac{2}{3} = \frac{15+3-10}{15} = \frac{8}{15}.$$

$$\therefore \iint_S \underline{F}\cdot\underline{n}\, dA = \frac{4\pi}{5} - \frac{\pi}{5} = \frac{3\pi}{5}.$$