**Solutions of equations**

Q. 1. Find all solutions of $z^5 = 1 + i\sqrt{3}$.
Q. 2. Find all the solutions of $z^4 = 3 + 4i$.
Q. 3. Find all solutions of $z^4 + z^3 + 3z^2 - 31z + 26$.
Q. 4. Find all the solutions of $z^4 + 5z^3 + 2z^2 - 22z - 20 = 0$.
Q. 5. Find all solutions of $z^2 = 4(\sqrt{3} - 2)i$.
Q. 6. Find all solutions of $z^2 - 2(1 + \sqrt{3})(1 + i)z + 4i = 0$.
Q. 7. Find all solutions of $\cos z = \sqrt{5}$.
Q. 8. Find all solutions of $z^2 + 2i = 0$.
Q. 9. Find all solutions of $z^2 - 3(1 + i)z + 5i = 0$.
Q. 10. Find all solutions of

$$\sinh z = \frac{1+i}{\sqrt{2}}.$$

Q. 11. Find the exact value of $(1 - \sqrt{3}i)^{314}$.
Q. 12. Find all the solutions of $\sin z = 5$.
Q. 13. Express in the form $x + iy$: $(\sqrt{3} + i)^{125}$.
Q. 14. Find all the distinct roots of the equation $z^4 = 1 + i$.

**Contour integration in the complex plane**

Q. 15. Find

$$\oint_C \frac{z(z^3 + 5z^2 + 4z - 1)\cos z}{(z^2 - 1)(z + 2)^2} dz$$

where (a) $C = \{|z| = 3/2\}$, (b) $C = \{|z| = 3\}$.
Q. 16. Find

$$\oint_C \frac{(z^3 + 2z^2 + 9z + 8)e^z \sinh z}{(z^2 + 1)(z + 2)^2} dz$$

where (a) $C = \{|z| = 3/2\}$, (b) $C = \{|z| = 3\}$.
Q. 17. Find $\int_C F(z)dz$ if

$$F(z) = \frac{(z^4 + 6z^3 + 56z - 48)\exp(-z^2)}{(z - 2)^2(z + 2)^3},$$

and when $C$ is (a) $|z+1| = 2$, (b) $|z-1| = 2$. (You may use: $z^4 + 4z^3 - 16z - 16 = (z^2 - 4)(z^2 + 4z + 4)$.)
Q. 18. Integrate

$$f(z) = \frac{(z^3 + 12z - 8)e^z \sin z}{z^2(z - 2)^3}$$

counterclockwise around $C$, where $C$ is given by (a) $|z-3| = 1/2$, (b) $|z-1/2| = 1$. Justify your answers.
Q. 19. Find

$$\int_C \frac{(z^3 - 2z^2 + 12z - 18)\cosh^2 z}{z^5 - 6z^4 + 9z^3} dz$$

where $C$ is (a) $|z - 1| = 3/2$, (b) $|z - 2| = 4$.
Q. 20. Integrate $f(z) = \dfrac{(1 + iz)\exp(z^2)}{z^2(z - i)^2}$ counterclockwise around $C$, where $C$ is given by (a) $|z-i| = 3$, (b) $|z| = 1$. Justify your answers.
Q. 21. Integrate

$$f(z) = \frac{(z^3 + 12z - 8)e^z \sin z}{z^2(z - 2)^3}$$

counterclockwise around $C$, where $C$ is given by (a) $|z - 3| = 1/2$, (b) $|z| = 5$. Justify your answers.
Q. 22. Find $A, B, C, D$ such that

$$\frac{3z^3 - 5z^2 - 2z - 8}{z^4 - 6z^3 + 8z^2} = \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z-2} + \frac{D}{z-4}.$$

Hence, evaluate $\int_C \frac{3z^3 - 5z^2 - 2z - 8}{z^4 - 6z^3 + 8z^2} dz$, where $C$ is the contour $|z| = 3$, traversed anticlockwise. (Hint: Factorize the denominator first.)

Q. 23. Find

$$\int_C \left\{ \frac{4\cos z}{(z+2i)^3} + \frac{2e^{z^2}}{(z+4i)^4} \right\} dz,$$

where $C$ is the contour $|z-1| = 2$ traversed in the anticlockwise direction.

**Harmonic functions, CR equations**

Q. 24. Find all $c$ such that $u = e^{3x}\cos cy$ is harmonic, and find the corresponding harmonic conjugates.

Q. 25. Find all the values of $c$ for which $u = \sin x \cosh ay$ is harmonic, and find the corresponding harmonic conjugates.

Q. 26. Find the harmonic conjugate of $u(x, y) = \exp(-2xy)\cos(x^2 - y^2)$. Otherwise, explain why it does not exist.

Q. 27. Find the harmonic conjugate of

$$u(x, y) = \exp(x^2 - y^2)\sin(2xy).$$

Otherwise, explain why it does not exist.

Q. 28. Let $e^z \cos z = u + iv$, where $u$, $v$ are real valued functions. Find $u$, $v$, $u_x$, $u_y$, $v_x$, $v_y$.

Q. 29. Using Cauchy-Riemann equations, verify that

$$\sin(x^2 - y^2)\cosh(2xy) + i\cos(x^2 - y^2)\sinh(2xy)$$

is analytic at all points $x + iy$.

Q. 30. Let $u = e^{x+y}\cos(x - y)$. Find $v$ such that $u + iv$ is analytic.

**Power series**

Q. 31. Find the power series expansion of $\cos^4 z$ around $z = \pi/2$.

Q. 32. Find the power series expansion of $e^z \sin z$ around $z = i$.

Q. 33. Find the Taylor expansion of the function $\frac{1}{\sqrt{1 - z + z^2}}$ around $z = 1/2$. What is the radius of convergence of this expansion?

Q. 34. Find the Taylor expansion of $\exp(z^2)\sin z$ around $z = i$.

Q. 35. Find the Taylor expansion of $e^z \sin^3 z$ around $z = -\pi$.

**Vector algebra**

Q. 36. Let $\mathbf{c} = [2, 3, 4]$, $\mathbf{d} = [6, -7, 2]$. Calculate $\mathbf{c} \bullet \mathbf{d}$ and $\mathbf{c} \times \mathbf{d}$.

Q. 37. Let $\mathbf{a} = [2, -5, 3]$, $\mathbf{c} = [-9, 4, 1]$. Calculate $\mathbf{a} \bullet \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$.

**Arclength, curvature, etc.**

Q. 38. Find the arclength of $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + (\sqrt{2}e^t)\mathbf{k}$, $0 \le t \le (1/2)\log 3$.

Q. 39. Find the arclength of

$$\mathbf{r}(t) = (1/2)e^{2t}\mathbf{i} + \frac{2\sqrt{2}}{3}e^{3t/2}\mathbf{j} + \sqrt{2}e^t\mathbf{k}, \qquad 0 \le t \le \log 2.$$

Q. 40. For the curve $\mathbf{r}(t) = [e^t, e^{-t}, (\sqrt{2})t]$, find the vectors $\mathbf{T}$, and the arclength from $t = 0$ to $t = \log 2$.

Q. 41. Find the length of the curve $\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}$, $0 \le t \le \pi/2$.

Q. 42. Find the length of the curve $\mathbf{r}(t) = [t + \sin t\cos t, t - \sin t\cos t, (1/\sqrt{2})\cos(2t)]$ from $t = 0$ to $t = 3$.

Q. 43. Find the arclength of the curve $\mathbf{r} = t^2\cos t\mathbf{i} + t^2\sin t\mathbf{j} + (t^3/\sqrt{3})\mathbf{k}$ from $t = 0$ to $t = 1$.

**Gradient, divergence, etc.**

Q. 44. Let $f(x, y, z) = e^x\cos(y^2 + z^2)$. Find $\text{div}(\text{grad}f)$.
Q. 45. Find the gradient of $f(x, y) = \log(xy + \sqrt{x^2 - y^2})$.
Q. 46. Find the divergence and curl of

$$F(x, y, z) = \left[\sec^{-1}(yz/x), \cosh(e^x + e^y + e^z), \log\left(\frac{(x^3 + 2)^{100}(y^2 + 43)^{3/4}}{(z^4 + 72)^7/8}\right)\right].$$

Q. 47. Find the gradient of

$$f(x, y, z) = \arctan\left(\frac{z}{\sqrt{x^2 + y^2}}\right).$$

Q. 48. Find the divergence and curl of

$$\mathbf{F}(x, y, z) = \left[e^x\cos y, \log(y^2 + z^2), \arctan(z/\sqrt{x^2 + y^2})\right].$$

Q. 49. Let $\mathbf{F}(x, y, z) = [e^x\cos y, \log(y^2 + z^2), \arctan(z/\sqrt{x^2 + y^2})]$. Evaluate div $\mathbf{F}$ and curl $\mathbf{F}$.
Q. 50. Let $\mathbf{F} = [\cos\pi y, \sin\pi x, \cos\pi x]$, $\mathbf{G} = [x^4, y^4, z^4]$. Find curl($\mathbf{F}$). Find div($\mathbf{G}$).
Q. 51. Let $\mathbf{F} = [y^2, 2xy\sin x, 0]$, $\mathbf{G} = [x^5, y^5, z^5]$. Find curl($\mathbf{F}$). Find div($\mathbf{G}$).
Q. 52. Find the divergence and curl of

$$F(x, y, z) = \left[\sec^{-1}(yz/x), \cosh(e^x + e^y + e^z), \log\left(\frac{(x^3 + 2)^{100}(y^2 + 43)^{3/4}}{(z^4 + 72)^7/8}\right)\right].$$

Q. 53. Find the divergence and curl of

$$\mathbf{F} = \left[\tan^{-1}(yz/x), \log(e^x + e^y - e^z), \cos^2(x^3 + y^3 + z^3)\right].$$

Q. 54. For $f(x, y, z) = \tan^{-1}\dfrac{z}{\sqrt{x^2 + y^2}}$, find $\text{div}(\text{grad}f)$.

**Surface and line integrals**

Q. 55. Let $\mathbf{F} = [x^3, y^3, z^3]$, $S$ be the surface of the sphere $x^2 + y^2 + z^2 = 4$. **Using Gauss divergence theorem**, evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}dA$. (Hint: Use spherical coordinates.)
Q. 56. Let $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + (1/2)z^3\mathbf{k}$, and $S$ be the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$. Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}dA$. You **cannot** use the divergence theorem here, but may wish to simplify first $\cos^4 t + \sin^4 t$.
Q. 57. Let $\mathbf{F}(x, y, z) = [x, y, -z]$, and S be the portion of the cone $z^2 = x^2 + y^2$ between $z = 0$ and $z = 1$. Evaluate **by a direct computation** $\iint_S \mathbf{F} \bullet \mathbf{n}dA$.
Q. 58. Let $\mathbf{F} = [3xy^2, yx^4 - y^3, 3zx^4]$, $S$ be the surface of the closed cylinder $x^2 + y^2 = 25$ between $z = 0$ and $z = 2$. **Using the Gauss divergence theorem,** evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}dA$.
In the following four questions, $\mathbf{F} = [\cos\pi y, \sin\pi x, \cos\pi x]$, $\mathbf{G} = [x^4, y^4, z^4]$, $C$ is the boundary of the rectangle $0 \le x \le 1/2$, $0 \le y \le 4$, $z = x$, and $S$ is the sphere $x^2 + y^2 + z^2 = 4$.
Q. 59. Find directly, without using Stoke's (or Green's) theorems, $\int_C \mathbf{F} \bullet d\mathbf{r}$.
Q. 60. Find $\int_C \mathbf{F} \bullet d\mathbf{r}$ using Stoke's theorem.
Q. 61. Find $\iint_S \mathbf{G} \bullet \mathbf{n}dA$ directly, without using Gauss divergence theorem.
Q. 62. Find $\iint_S \mathbf{G} \bullet \mathbf{n}dA$.

In the following four questions, $\mathbf{F} = [y^2, 2xy\sin x, 0]$, $\mathbf{G} = [x^5, y^5, z^5]$, $C$ is the boundary of the rectangle $0 \le x \le \pi/2$, $0 \le y \le 2$, $z = 0$, and $S$ is the sphere $x^2 + y^2 + z^2 = 9$.
Q. 63. Find directly, without using Stoke's (or Green's) theorems, $\int_C \mathbf{F} \bullet d\mathbf{r}$.
Q. 64. Find $\int_C \mathbf{F} \bullet d\mathbf{r}$ as a double integral, using Stoke's/Green's theorem.

Q. 65. Find $\iint_S \mathbf{G} \bullet \mathbf{n} dA$ directly, without using Gauss divergence theorem.
Q. 66. Find $\iint_S \mathbf{G} \bullet \mathbf{n} dA$ using Gauss divergence theorem.
Q. 67. Find $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = [2z, x, -y]$, $\mathbf{r} = [\cos t, t, 2\sin t]$, $0 \leq t \leq 2\pi$.
Q. 68. Find $f$ such that $\nabla f = \exp(x - y + z^2)[1, -1, 2z]$. Hence, calculate

$$\int_{(0,-1,1)}^{(2,4,0)} \exp(x - y + z^2)(dx - dy + 2zdz).$$

Q. 69. Find $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = [-y^2, x^2]$, and $\mathbf{r}(t) = [\cos t, \sin t]$, $0 \leq t \leq \pi/2$.
Q. 70. Find $f$ such that $\nabla f = (x^2 + y^2 z^2)^{-1}[-yz, xz, xy]$. Hence, calculate

$$\int_{(1,1,1)}^{(\sqrt{3},\sqrt{3},\sqrt{3})} (x^2 + y^2 z^2)^{-1}(-yzdx + xzdy + xydz).$$

Q. 71. Using Stoke's theorem, find $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = [xye^z, yze^x, zxe^y]$ and $C$ is the boundary of triangle with vertices $(0,0,1), (1,0,1), (0,1,1)$ traversed so that the outward normal to the triangle is given by $\mathbf{k}$.