Local quadrature formulas on the sphere, II

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Abstract. Let $q \geq 1$ be an integer, K be a compact subset of the unit sphere $\mathbb{S}^q$ embedded in the Euclidean space $I\!R^{q+1}$, and K have a nonempty interior relative to $\mathbb{S}^q$. Let $\mathcal{C}$ be a set of points in a cap containing K, and $m \geq 1$ be an integer. We explore conditions on $\mathcal{C}$ and K which guarantee the existence of nonnegative numbers w_ξ, $\xi \in \mathcal{C}$, such that

$$\int_K P(\boldsymbol{x}) d\mu_q(\boldsymbol{x}) = \sum_{\xi \in \mathcal{C}} w_\xi P(\xi)$$

for all spherical polynomials P of degree at most m, where μ_q is the area measure on $\mathbb{S}^q$.

§1. Introduction

Many applications involving function approximation on the sphere require an approximate evaluation of an integral over a subset of the sphere. Many authors [3, 10, 9, 2] have developed formulas that are exact for polynomials of different degrees, assuming that the data can be chosen at specific sites on the sphere. In such applications as approximation by neural networks, one cannot choose the sites at which the data may be collected, and therefore, must deal with *scattered data*. Quadrature formulas based on scattered data, exact for integrating high degree polynomials on the whole sphere are developed in [4, 6].

In search of local quadrature formulas, we have established in [5] the existence of such formulas, exact for integration of a fixed degree polynomial on spherical caps. Since spherical caps cannot provide a partition of the sphere, we find it worthwhile to develop such formulas for spherical simplices, and for general compact subsets of the sphere. The purpose of

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this paper is to record the conditions on the data set and the compact set that are sufficient for the existence of such formulas.

We state our main result in Section 2. The idea behind the proof of this result is the same as that behind the results in [6,5]; namely, we prove a Marcinkiewicz–Zygmund inequality, and apply an argument involving norming sets. These concepts are reviewed in Section 3. The novelty of the paper consists of the proof of the necessary Marcinkiewicz–Zygmund inequality. This proof, and the subsequent proof of the main result in Section 2, is given in Section 4.

§2. Main result

Throughout this paper, m, q will denote fixed positive integers. We adopt the following convention regarding constants. The symbols $c, c_1, \cdots$ will denote positive constants depending only on q and m, and other explicitly mentioned quantities. The dependence on m will be polynomial, and the values of these constants may be different at different occurences, even within a single formula. The fact that $A \le c_1 B \le c_2 A$ will be denoted by $A \sim B$.

The symbol $\mathbb{S}^q$ will denote the unit sphere embedded in the Euclidean space $\mathbb{R}^{q+1}$, and μ_q will denote its volume (surface area) measure. The set of all spherical polynomials on $\mathbb{S}^q$ of degree at most m will be denoted by Π_m^q. If $\boldsymbol{x}_0 \in \mathbb{S}^q$ and $0 \le \alpha \le \pi$, the spherical cap with center $\boldsymbol{x}_0$ and radius α is defined by

$$\mathbb{S}^q_\alpha(\boldsymbol{x}_0) := \{\boldsymbol{y} \in \mathbb{S}^q \ : \ \boldsymbol{y} \cdot \boldsymbol{x}_0 \ge \cos\alpha\}.$$

Let K be a compact subset of $\mathbb{S}^q$ with a nonempty interior (with respect to $\mathbb{S}^q$). For any interior point $\boldsymbol{x}_0$ of K, let $\alpha_I(K, \boldsymbol{x}_0)$ (respectively, $\alpha_O(K, \boldsymbol{x}_0)$) be the supremum (respectively, infimum) of all α such that $\mathbb{S}^q_\alpha(\boldsymbol{x}_0) \subseteq K$ (respectively, $\mathbb{S}^q_\alpha(\boldsymbol{x}_0) \supseteq K$). We write $\tau(K, \boldsymbol{x}_0) := \alpha_O(K, \boldsymbol{x}_0)/\alpha_I(K, \boldsymbol{x}_0)$.

If $\mathcal{C}$ is a finite set of distinct points on $\mathbb{S}^q$, and $A \subseteq \mathbb{S}^q$, we define the mesh norm of $\mathcal{C}$ relative to A by

$$\delta(\mathcal{C}, A) := \sup_{\boldsymbol{x} \in A} \min_{\xi \in \mathcal{C}} \mathrm{dist}(\boldsymbol{x}, \xi), \tag{1}$$

where dist denotes the geodesic distance on $\mathbb{S}^q$. We observe that the mesh norm only measures how dense the set $\mathcal{C}$ is for elements of A and not its distribution in any sense; in particular, it is always possible to add many points to $\mathcal{C}$, thus ruining any previous distribution, and yet preserving the mesh norm.

Our main theorem is the following.

Theorem 1. *Let $q, m \ge 1$, K be a compact subset of $\mathbb{S}^q$, $\boldsymbol{x}_0$ be an interior point of K, and $\mathcal{C}$ be a set of distinct points on $\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}$. There exists a positive constant $c = c(q,m)$ with the following property. Let*

$$\delta(\mathcal{C},\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}) \le \begin{cases} c\left(\pi(2\tau(K,\boldsymbol{x}_0)-1)\right)^{-2mq/(q-1)} \tau(K,\boldsymbol{x}_0)^{-q^2/(q-1)}\alpha_O(K,\boldsymbol{x}_0), & \text{if } q \ge 2, \\ c\left(\pi(2\tau(K,\boldsymbol{x}_0)-1)\right)^{-2m} \tau(K,\boldsymbol{x}_0)^{-1}\alpha_O(K,\boldsymbol{x}_0), & \text{if } q = 1. \end{cases} \tag{2}$$

Then there exist nonnegative weights w_ξ, $\xi \in \mathcal{C}$, such that

$$\sum_{\xi\in\mathcal{C}} w_\xi P(\xi) = \int_K P(\boldsymbol{x})d\mu_q(\boldsymbol{x}), \qquad P \in \Pi^q_m. \tag{3}$$

Moreover,

$$|\{\xi \in \mathcal{C} \;:\; w_\xi \ne 0\}| \le C_1(K,\boldsymbol{x}_0), \tag{4}$$

where $C_1(K,\boldsymbol{x}_0)$ is a positive constant depending on m, q, K, and $\boldsymbol{x}_0$. In particular,

$$\max_{\xi\in\mathcal{C}} w_\xi \le \sum_{\xi\in\mathcal{C}} w_\xi = \mu_q(K). \tag{5}$$

We give an example to illustrate the condition (2).

Example 2. The new notation introduced in this example will be valid only within this example. Let $q = 2$, $L \ge 3$ be an integer. The projection of a set $A \subseteq \mathbb{R}^3$ (on $\mathbb{S}^2$) is the set $\{\boldsymbol{x}/|\boldsymbol{x}| \;:\; \boldsymbol{x} \in A\}$. Let K_L be a regular L-gon on the sphere, with the Eucliden distance ρ between consecutive nodes less than $2\sin(\pi/L)$. Then there exists a ϕ, $0 < \phi < \pi/2$, such that $\rho = 2\sin(\pi/L)\sin\phi$. Let G_L be the convex hull of the points $(\sin\phi\cos(2k\pi/L), \sin\phi\sin(2k\pi/L), \cos\phi)$, $k = 0, 1, \cdots, L-1$; *i.e.*, a regular L-gon inscribed in the circle at the intersection of $\mathbb{S}^2$ and the plane $x_3 = \cos\phi$. With appropriate rotations, we may assume that K_L is the projection of G_L, so that, in particular, the center of K_L is $\mathbf{n} := (0,0,1)$. It is clear that $\alpha_O(K_L,\mathbf{n}) = \phi$. The radius of the circle inscribed in G_L is given by $\sin\phi\cos(\pi/L)$. The projection of the point $(\sin\phi\cos(\pi/L), 0, \cos\phi)$ is given by $\mathbf{y} = (1-\sin^2\phi\sin^2(\pi/L))^{-1/2}(\sin\phi\cos(\pi/L), 0, \cos\phi)$. Since the largest circle with center at $\mathbf{n}$, inscribed in K_L, passes through $\mathbf{y}$, we see that

$$\begin{aligned}\sin\alpha_I(K_L,\mathbf{n}) &= \frac{\cos(\pi/L)}{(1-\sin^2\phi\sin^2(\pi/L))^{1/2}}\sin\phi \\ &= \frac{\cos(\pi/L)}{(1-\sin^2\phi\sin^2(\pi/L))^{1/2}}\sin\alpha_O(K_L,\mathbf{n}).\end{aligned}$$

Recalling that $2\theta/\pi \le \sin\theta \le \theta$ for $\theta \in (0, \pi/2)$, we conclude that

$$\begin{aligned}\tau(K_L, \mathbf{n}) &= \frac{\alpha_O(K_L, \mathbf{n})}{\alpha_I(K_L, \mathbf{n})} \\ &\le \frac{\pi}{2}\frac{\sin\alpha_O(K_L, \mathbf{n})}{\sin\alpha_I(K_L, \mathbf{n})} = \frac{\pi}{2}\frac{(1-\sin^2\phi\sin^2(\pi/L))^{1/2}}{\cos(\pi/L)} \\ &\le \frac{\pi}{2}\sec(\pi/L).\end{aligned}$$

Thus, for regular L-gon's on the 2-sphere, the mesh norm condition (2) is satisfied if $\delta(\mathcal{C}, \mathbb{S}^q_{\alpha_O(K_L, \boldsymbol{x}_0)}) \le A(m, L)\alpha_O(K_L, \boldsymbol{x}_0)$, where $\boldsymbol{x}_0$ is the center of the L-gon, and $A(m, L)$ is a positive constant dependent only on m and L, and not on the size of the L-gon. □

Our numerical experiments with quadrature formulas on the cap as well as the whole sphere suggest that the condition (2) is too conservative. In [8], we solved a number of quadratic programming problems to obtain the quadrature formulas exact for polynomials of degree 18, corresponding to a data set consisting of 400 points on $\mathbb{S}^2$. It appears that the problem is very difficult to solve numerically if the number of data points is too large. The following strategy looks attractive in the case when one has a large number of data points, but the degree of the polynomials for which exact quadrature formulas are required is not too high. Suppose one defines a partition $\mathcal{T}$ of the sphere $\mathbb{S}^q$ consisting of spherical simplices or regular polygons with different number of vertices each, and correspondingly, subdivide the set $\mathcal{C}$ of data points, so that for every $K \in \mathcal{T}$, the subset $\mathcal{C}_K \subset \mathcal{C}$ consists of points in vicinity of K. We then solve the quadratic programming problems for each K to obtain a quadrature formula of the form (3) (with $\mathcal{C}_K$ in place of $\mathcal{C}$). It is expected that the data set $\mathcal{C}_K$ being relatively small, each of these quadratic programming problems would be manageable. The desired quadratic formula on the whole sphere can be obtained by adding the individual quadratic formulas above. We note that it is not necessary that $\{\mathcal{C}_K\}$ be a partition of $\mathcal{C}$, merely that $\{K\}$ should be a partition of $\mathbb{S}^q$. The same strategy should work also when a quadrature formula on an "arbitrary" compact subset of $\mathbb{S}^q$ is desired. However, we don't yet have any numerical experiments to report in connection with obtaining global quadrature formulas based on these.

In Theorem 1, we observe that the set $\mathcal{C}$ is not assumed to be a subset of K, but of a cap containing K. Nevertheless, our proof will show that the weights w_ξ corresponding to points outside K which are not within a distance of $c\delta(\mathcal{C}, \mathbb{S}^q_{\alpha_O(K, \boldsymbol{x}_0)})$ may be chosen to be zero. Thus, the quadrature formula (3) depends only on points in a neighborhood of K.

§3. Norming sets

Let $(X, \|\cdot\|)$ be a finite dimensional normed linear space, X^* be its dual space, $Z \subseteq X^*$ be a finite set of functionals, and $|||\cdot|||$ be a norm on $\mathbb{R}^{|Z|}$. We say that Z is a norming set for X if

$$\|f\| \le M|||(x^*(f))_{x^* \in Z}|||, \qquad f \in X, \tag{6}$$

for some constant $M > 0$. A functional $x^* \in X^*$ is said to be positive with respect to Z if $x^*(x) \ge 0$ whenever $y^*(x) \ge 0$ for all $y^* \in Z$.

In [6, Proposition 4.1] (cf. [5, Theorem 3.1]), we proved the following general theorem regarding the representation of functionals positive with respect to Z.

Proposition 3. *Let X be a finite dimensional normed linear space, X^* be its dual, $Z \subseteq X^*$ be a finite, norming set for X, and $x^* \in X^*$ be positive with respect to Z. Suppose further that there exists a $x_0 \in X$ such that $y^*(x_0) > 0$ for all $y^* \in Z$. Then there exist nonnegative numbers w_{y^*}, $y^* \in Z$, such that*

$$x^*(x) = \sum_{y^* \in Z} w_{y^*} y^*(x), \qquad x \in X. \tag{7}$$

In applications to quadrature formulas, the various hypothesis of Proposition 3 are verified using Marcinkiewicz–Zygmund inequalities; *i.e.*, inequalities of the form (8) below.

Definition 4. *Let $\mathcal{C}$ be a set of distinct points on $\mathbb{S}^q$, $A \subseteq \mathbb{S}^q$, $\mu_q(A) > 0$, $\eta > 0$, and $m \ge 0$ be an integer. The notation $\mathcal{C} \in \mathcal{MZ}(A, m, \eta)$ will mean the following. There exists a subset $\mathcal{C}' \subseteq \mathcal{C}$, with $\delta(\mathcal{C}', A) \le c\delta(\mathcal{C}, A)$, and a partition of A, each member of which contains a unique ξ in the reduced set $\mathcal{C}'$, and hence, can be denoted by R_ξ, with the following property.*

$$\sum_{\xi \in \mathcal{C}'} \int_{R_\xi} |P(\boldsymbol{x}) - P(\xi)| d\mu_q(\xi) \le \eta \int_A |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}) \tag{8}$$

for all $P \in \Pi_m^q$.

The following proposition demonstrates the connection between Marcinkiewicz-Zygmund inequalities and quadrature formulas. It was proved and used in [6] in the case when $A = \mathbb{S}^q$ and in [5] in the case when A is a spherical cap. The proof below is the same; we only reproduce the important estimates.

Propostion 5. *Let $\mathcal{C}$ be a set of distinct points on $\mathbb{S}^q$, $A \subseteq \mathbb{S}^q$, $\mu_q(A) > 0$, $0 < \eta \le 1/2$, and $m \ge 0$ be an integer, and $\mathcal{C} \in \mathcal{MZ}(A, m, \eta)$. Then there exist nonnegative weights w_ξ, $\xi \in \mathcal{C}$, such that*

$$\int_A P(\boldsymbol{x}) d\mu_q(\boldsymbol{x}) = \sum_{\xi \in \mathcal{C}} w_\xi P(\xi), \qquad P \in \Pi_m^q. \tag{9}$$

Proof: We apply Proposition 3 with the following choices. The space X is taken to be Π_m^q, with $\|P\| := \int_A |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x})$, the set Z is the set of point evaluation functionals at the points of $\mathcal{C}'$, and the norm $|||\cdot|||$ is defined by $|||\mathbf{w}||| := \sum_{\xi \in \mathcal{C}'} \mu_q(R_\xi)|w_\xi|$. The inequality (8) implies that

$$\int_A |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}) \le \frac{1}{1-\eta} \sum_{\xi \in \mathcal{C}'} \mu_q(R_\xi)|P(\xi)| \le \frac{1+\eta}{1-\eta} \int_A |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}). \tag{10}$$

Thus, (6) is satisfied with $M = 1/(1-\eta)$.

Next, we define the functional x^* by $P \mapsto \int_A P(\boldsymbol{x}) d\mu_q(\boldsymbol{x})$. Let $P(\xi) \ge 0$, $\xi \in \mathcal{C}'$. Then (10) and (8) together imply that

$$\begin{aligned} &\left| \int_A P(\boldsymbol{x}) d\mu_q(\boldsymbol{x}) - \sum_{\xi \in \mathcal{C}'} \mu_q(R_\xi) P(\xi) \right| \\ &\qquad \le \eta \int_A |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}) \le \frac{\eta}{1-\eta} \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) P(\xi). \end{aligned}$$

Therefore,

$$\int_A P(\boldsymbol{x}) d\mu_q(\boldsymbol{x}) \ge \frac{1-2\eta}{1-\eta} \sum_{\xi \in \mathcal{C}} \mu_q(R_\xi) P(\xi) \ge 0.$$

Thus, x^* is positive with respect to Z.

Finally, the polynomial P_I, identically equal to 1, satisfies $P_I(\xi) > 0$, $\xi \in \mathcal{C}'$. Therefore, the proposition follows from Proposition 3. □

§4. Proof of the main result

In this section, we will assume, without loss of generality, that the point $\boldsymbol{x}_0$ in Theorem 1 is the north pole, and write $\mathbb{S}_\alpha^q$ in place of $\mathbb{S}_\alpha^q(\boldsymbol{x}_0)$. Our goal is to prove a Marcinkiewicz-Zygmund inequality for $\mathcal{C}$ and K. This will be based upon the following result from [5, Theorem 4.1 and Theorem 4.2].

Proposition 6. *Let $q \ge 2$, $0 < \alpha < \pi$, $\mathcal{C}$ be a set of distinct points in $\mathbb{S}^q_\alpha$, and $m \ge 1$ be an integer. Then there exists a constant $c := c(q,m) > 0$ with the following property. If $\eta > 0$ and $\delta(\mathcal{C}, \mathbb{S}^q_\alpha) \le c\eta^{q/(q-1)}\alpha$, then $\mathcal{C} \in \mathcal{MZ}(\mathbb{S}^q_\alpha, m, \eta)$. The same conclusion holds for $q = 1$ if $\delta(\mathcal{C}, \mathbb{S}^1_\alpha) \le c\eta\alpha$.*

In this paper, we will apply Proposition 6 with the cap $\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}$. The set $\{R_\xi \cap K\}$ provides a partition of K as desired. The main difficulty in the proof is to estimate $\int_{\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}} |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x})$ in terms of the integral $\int_{\mathbb{S}^q_{\alpha_I(K,\boldsymbol{x}_0)}} |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x})$ over a smaller cap. As in [5], we first prove a similar inequality for integrals over arcs of a circle. The starting point of our proof is to obtain a conformal mapping of the exterior of the unit circle in $\mathbb{C}$ to the exterior of an arc of a circle. The result is stated in an equivalent form by Andrievskii [1], without its simple proof.

Lemma 7. *Let $0 < \gamma < \pi/2$, and I° be the arc of the unit circle from $\exp(-2i\gamma)$ to $\exp(2i\gamma)$ passing through 1. The mapping*

$$\Psi(w) = \frac{w(1 - w\sin\gamma)}{w - \sin\gamma} \tag{11}$$

maps $\{w \in \mathbb{C} \ : \ |w| > 1\}$ conformally onto the exterior of I° with $\Psi(\infty) = \infty$.

Proof: The mapping $z_1 = i\dfrac{w + ie^{i\gamma}}{1 + ie^{i\gamma}w}$ maps $\{w \in \mathbb{C} \ : \ |w| > 1\}$ conformally onto the right half of the z_1 plane, the mapping $z_2 = z_1^2$ maps this further onto the plane, cut by the negative real axis, and the mapping $z = \dfrac{z_2 - e^{2i\gamma}}{e^{2i\gamma}z_2 - 1}$ maps this region onto the exterior of I°. We eliminate the intermediate variables, and simplify. □

Let $\mathbb{H}_m$ denote the class of all trigonometric polynomials of order at most m. The conformal mapping immediately yields the following Bernstein-Walsh inequality.

Lemma 8. *Let $m \ge 1$ be an integer, $R \in \mathbb{H}_m$, $0 < \gamma < \gamma_1 < \pi/2$, and $\max_{t\in[-2\gamma,2\gamma]} |R(t)| = 1$. Then*

$$\max_{t\in[-2\gamma_1,2\gamma_1]} |R(t)| \le \left(\frac{2\sin\gamma_1}{\sin\gamma}\right)^{2m} \le \left(\frac{\pi\gamma_1}{\gamma}\right)^{2m}. \tag{12}$$

Proof: Let $I^\circ = \{e^{i\theta} \ : \ |\theta| \le 2\gamma\}$, and P be the polynomial of degree at most $2m$, defined by $P(e^{it}) = e^{mit}R(e^{it})$. Then $\max_{z\in I^\circ} |P(z)| = 1$. Hence, the Bernstein-Walsh lemma [11, p. 77] implies that $|P(z)| \le |\Phi(z)|^{2m}$, where (with the principle branch of the square root),

$$\Phi(z) := \frac{1}{2\sin\gamma}\left\{1 - z - \sqrt{(1-z)^2 + 4z\sin^2\gamma}\right\},$$

is the inverse function of the conformal mapping Ψ defined in (11). It is then easy to deduce that for $t \notin [-\gamma, \gamma]$,

$$|P(e^{2it})| \le |\Phi(e^{2it})|^{2m} = \left|\frac{\sin t}{\sin\gamma} + \sqrt{\left(\frac{\sin t}{\sin\gamma}\right)^2 - 1}\right|^{2m} \le \left(\frac{2\sin t}{\sin\gamma}\right)^{2m}.$$

The estimate (12) follows. □

Corollary 9. *Let $m, q \ge 1$ be integers, $T \in \mathbb{H}_m$, $0 < \alpha < \beta < \pi$. Then*

$$\max_{t\in[0,\beta]} |T(t)| \le \left(\frac{\pi(2\beta-\alpha)}{\alpha}\right)^{2m} \max_{t\in[0,\alpha]} |T(t)|. \tag{13}$$

Proof: We use Lemma 8 with $\alpha/4$ in place of γ, $(2\beta - \alpha)/4$ in place of γ_1, and $R(t) = T(t + \alpha/2)/\max_{t\in[0,\alpha]} |T(t)|$. □

In order to convert (13) into the desired inequality for spherical polynomials, we need the following weighted Nikolskii inequality.

Lemma 10. *Let $m, q \ge 1$ be integers, $T \in \mathbb{H}_m$, $0 < \gamma < \pi$. Then*

$$\max_{0\le x\le\gamma} |T(t)| \sim \gamma^{-q} \int_0^\gamma |T(t)| \sin^{q-1} t dt. \tag{14}$$

Proof: In view of [5, Proposition 3.2],

$$\max_{0\le x\le\gamma} |T(t)| \sim \gamma^{-1} \int_0^\gamma |T(t)| dt. \tag{15}$$

A repeated application of [5, Proposition 3.3] yields

$$\int_0^\gamma |T(t)| dt \le c\gamma^{1-q} \int_0^\gamma |T(t)| \sin^{q-1} t dt.$$

Along with (15), this implies

$$\max_{0\le x\le\gamma} |T(t)| \le c\gamma^{-q} \int_0^\gamma |T(t)| \sin^{q-1} t dt,$$

and hence, (14). □

We are now in a position to state the estimate of integrals of spherical polynomials over a cap by those on a smaller cap.

Proposition 11. *Let $m, q \ge 1$ be integers, $P \in \Pi_m^q$, $0 < \alpha < \beta < \pi$. Then*

$$\max_{\boldsymbol{x}\in\mathbb{S}_\alpha^q} |P(\boldsymbol{x})| \sim \frac{1}{\mu_q(\mathbb{S}_\alpha^q)} \int_{\mathbb{S}_\alpha^q} |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}), \tag{16}$$

and

$$\int_{\mathbb{S}_\beta^q} |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}) \le c\left(\frac{\pi(2\beta-\alpha)}{\alpha}\right)^{2m} (\beta/\alpha)^q \int_{\mathbb{S}_\alpha^q} |P(\boldsymbol{x})| d\mu_q(\boldsymbol{x}). \tag{17}$$

Proof: In this proof only, we recall the standard parameterization of $\mathbb{S}^q$ embedded in $\mathbb{R}^{q+1}$ in terms of the angles $\theta_1, \ldots, \theta_q$, where $-\pi \le \theta_1 \le \pi$ and $0 \le \theta_k \le \pi$ for $k = 2, \ldots, q$. If $\boldsymbol{x} \in \mathbb{S}^q$, then the kth component of $\boldsymbol{x}$ is given by

$$x_k = \begin{cases} \prod_{j=1}^q \sin\theta_j, & k = 1, \\ \cos\theta_{k-1} \prod_{j=k}^q \sin\theta_j, & 2 \le k \le q, \\ \cos\theta_q, & k = q+1, \end{cases} \tag{18}$$

The measure μ_q on $\mathbb{S}^q$ can be expressed in these coordinates as

$$d\mu_q(\boldsymbol{x}) = \prod_{k=1}^q \sin^{k-1}(\theta_k) d\theta_k. \tag{19}$$

Note that

$$d\mu_q = \sin^{q-1}(\theta_q) d\theta_q d\mu_{q-1}. \tag{20}$$

As in [5], we observe that any $\boldsymbol{x} \in \mathbb{S}^q$ can be written in the form $\sin\theta_q(\boldsymbol{x}', 0) + \cos\theta_q \mathbf{e}_{q+1}$, where $\boldsymbol{x}' \in \mathbb{S}^{q-1}$ and $\mathbf{e}_{q+1}$ is the unit vector $(0, \cdots, 0, 1)$. Accordingly, for any $\boldsymbol{y} \in \mathbb{S}^{q-1}$ and $\phi \in [0, \pi]$, we write

$$[\boldsymbol{y}, \phi] := \sin\phi(\boldsymbol{y}, 0) + \cos\phi \mathbf{e}_{q+1} \in \mathbb{S}^q,$$

and for $\boldsymbol{x} \in \mathbb{S}^q$, $\boldsymbol{x} = [\boldsymbol{x}', \theta_q(\boldsymbol{x})]$. We define $\mathbf{e}'_{q+1} := \mathbf{e}_q \in \mathbb{S}^{q-1}$ and $(-\mathbf{e}_{q+1})' := -\mathbf{e}_q \in \mathbb{S}^{q-1}$.

Now, for any $\phi \in [0, \alpha]$, we have from [7, Proposition 2.1] that

$$\max_{\boldsymbol{y}\in\mathbb{S}^{q-1}} |P([\boldsymbol{y}, \phi])| \le c \int_{\mathbb{S}^{q-1}} |P([\xi, \phi])| d\mu_{q-1}(\xi). \tag{21}$$

Using (14) with $P([\xi, \cdot])$ in place of T, we see that

$$|P([\xi, \phi])| \le c\alpha^{-q} \int_0^\alpha |P([\xi, t])| \sin^{q-1}(t) dt.$$

Along with (21) and (20), this shows that

$$\begin{aligned}\max_{\boldsymbol{x}\in\mathbb{S}^q_\alpha}|P(\boldsymbol{x})| &\le c\alpha^{-q}\int_{\mathbb{S}^{q-1}}\int_0^\alpha |P([\xi,t])|\sin^{q-1}(t)dtd\mu_{q-1}(\xi)\\ &= \frac{c}{\mu_q(\mathbb{S}^q_\alpha)}\int_{\mathbb{S}^q_\alpha}|P(\zeta)|d\mu_q(\zeta).\end{aligned}$$

This implies (16).

Now, for any $\boldsymbol{y}\in\mathbb{S}^{q-1}$, the estimate (13) with $P([\boldsymbol{y},\cdot])$ in place of T shows that

$$\max_{0\le t\le\beta}|P([\boldsymbol{y},t])| \le \left(\frac{\pi(2\beta-\alpha)}{\alpha}\right)^{2m}\max_{0\le t\le\alpha}|P([\boldsymbol{y},t])|;$$

i.e.,

$$\max_{\boldsymbol{x}\in\mathbb{S}^q_\beta}|P(\boldsymbol{x})| \le \left(\frac{\pi(2\beta-\alpha)}{\alpha}\right)^{2m}\max_{\boldsymbol{x}\in\mathbb{S}^q_\alpha}|P(\boldsymbol{x})|. \tag{22}$$

In view of (16), this implies (17). □

The following analogue of Proposition 6 is now easy to prove.

Proposition 12. *If K is a compact subset of $\mathbb{S}^q$, $\boldsymbol{x}_0$ be an interior point of K, $\mathcal{C}$ is a set of distinct points on $\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}$, $\eta>0$, and*

$$\delta(\mathcal{C},\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}) \le \begin{cases} c\Big(\pi(2\tau(K,\boldsymbol{x}_0)-1)\Big)^{-2mq/(q-1)}\times \\ \quad \tau(K,\boldsymbol{x}_0)^{-q^2/(q-1)}\eta^{q/(q-1)}\alpha_O(K,\boldsymbol{x}_0), & \text{if } q\ge 2,\\ c\Big(\pi(2\tau(K,\boldsymbol{x}_0)-1)\Big)^{-2m}\tau(K,\boldsymbol{x}_0)^{-1}\eta\alpha_O(K,\boldsymbol{x}_0), & \text{if } q=1.\end{cases} \tag{23}$$

Then $\mathcal{C}\in\mathcal{MZ}(K,m,\eta)$.

Proof: In this proof, we will write $\alpha:=\alpha_I(K,\boldsymbol{x}_0)$ and $\beta:=\alpha_O(K,\boldsymbol{x}_0)$. Clearly, $\mathbb{S}^q_\alpha\subseteq K\subseteq\mathbb{S}^q_\beta$. In view of Proposition 6, there exists a constant c such that the condition (23) implies the existence of a subset $\mathcal{C}'$ of $\mathcal{C}$ and a partition $\{R_\xi\}_{\xi\in\mathcal{C}'}$ of $\mathbb{S}^q_\beta$, each member R_ξ of which contains exactly one element $\xi\in\mathcal{C}'$, such that

$$\begin{aligned}&\sum_{\xi\in\mathcal{C}'}\int_{R_\xi}|P(\boldsymbol{x})-P(\xi)|d\mu_q(\xi)\\ &\qquad\le\Big(\pi(2\tau(K,\boldsymbol{x}_0)-1)\Big)^{-2m}\tau(K,\boldsymbol{x}_0)^{-q}\eta\int_{\mathbb{S}^q_\beta}|P(\boldsymbol{x})|d\mu_q(\boldsymbol{x}).\end{aligned} \tag{24}$$

Using Proposition 11, we conclude that

$$\begin{aligned}\sum_{\xi\in\mathcal{C}'}\int_{R_\xi\cap K}|P(\boldsymbol{x})-P(\xi)|d\mu_q(\xi) &\le \eta\int_{\mathbb{S}^q_\alpha}|P(\boldsymbol{x})|d\mu_q(\boldsymbol{x})\\ &\le \eta\int_K |P(\boldsymbol{x})|d\mu_q(\boldsymbol{x}).\end{aligned} \tag{25}$$

We note that $|\mathcal{C}'|\le B := B(K,\boldsymbol{x}_0)$, where B is a positive constant, depending only on K, $\boldsymbol{x}_0$, m, and q. $\square$

Proof of Theorem 1: By a judicious choice of the constant c, Proposition 12 implies that $\mathcal{C}\in\mathcal{MZ}(K,m,1/3)$. We may then choose $\mathcal{C}'\subset\mathcal{C}$, with $|\mathcal{C}'|\le B$, that satisfies (25) for a partition $\{R_\xi\}$ of $\mathbb{S}^q_{\alpha_O(K,\boldsymbol{x}_0)}$. In fact, in (25), we may replace $\mathcal{C}'$ by another subset $\mathcal{C}'' := \{\xi\in\mathcal{C}' \,:\, \mu_q(R_\xi\cap K)\neq 0\}$. Then $\mathcal{C}''\in\mathcal{MZ}(K,m,1/3)$. The theorem now follows from Proposition 5 used with $\mathcal{C}''$ in place of $\mathcal{C}$, and by setting $w_\xi = 0$ if $\xi\in\mathcal{C}\setminus\mathcal{C}''$. $\square$

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