

# On bounded interpolatory and quasi–interpolatory polynomial operators

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*Dedicated to the memory of Ambikeshwarji Sharma*

## Abstract

We discuss our recent work in the theory of approximation of functions using values of the function at scattered sites on the circle, the real line, the unit interval, and the unit sphere. As an alternative to interpolation, we present quasi–interpolatory operators for this purpose. We also prove the existence of bounded operators, yielding entire functions of finite exponential type, that interpolate a Birkhoff data for a function on a Euclidean space, where a finite number of derivatives, of order not exceeding a fixed number, are prescribed at each point.

## 1 Introduction

In most applications of approximation theory, one wishes to approximate a function given its values at finitely many points. Typically, the approximation is desired in the space  $C(\Omega)$  comprising of uniformly continuous, bounded, real functions on a subset  $\Omega$  of a Euclidean space; the space being endowed with the supremum norm:  $\|f\|_\Omega := \sup_{\mathbf{x} \in \Omega} |f(\mathbf{x})|$ ,  $f \in C(\Omega)$ . In the theoretical set up, one has a nested sequence of subspaces  $V_n \subset V_{n+1} \subset C(\Omega)$ , with the dimension of  $V_n$  being  $n$ . Given a data of the form  $\{(\mathbf{x}_j, f(\mathbf{x}_j))\}_{j=1}^N$ ,  $\mathbf{x}_j \in \Omega$ ,  $j = 1, \dots, N$ , the problem of interpolation consists of finding a function  $\mathcal{I}_N(f) \in V_N$ , such that  $\mathcal{I}_N(f)(\mathbf{x}_j) = f(\mathbf{x}_j)$ ,  $j = 1, \dots, N$ . The subject is studied in great detail in a variety of situations (cf. for example, [24, 25, 3, 6]).

In the case of multivariate approximation, it is often not guaranteed that the interpolation problem from a given space will have a solution. Even if a solution exists, its numerical computation involves a matrix inversion, and is therefore, costly. Moreover, the sequence of interpolants,  $\{\mathcal{I}_N(f)\}$  does not converge for *every* continuous function  $f$ . There are two ways to remedy the last problem. One approach is to seek bounded

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operators taking values in  $V_M$  for some  $M > N$  which interpolate at the given data. This approach has been explored in great detail in the univariate context in the book [24, Chapter II] of Szabados and Vértési. The other approach is to abandon the requirement that the operators should interpolate the data, and seek bounded operators taking values in  $V_M$  for some  $M < N$ , and constructed from the data in some other way.

The purpose of this paper is to survey some of our recent work in both the directions.

## 2 Bounded interpolatory operators

The motivation behind the work in this section is provided by the following theorem (cf. [24, Chapter II, Theorem 2.7]), where we denote the class of all univariate algebraic polynomials of degree at most  $n$  by  $\Pi_n$ . Throughout this paper, the symbols  $c, c_1, \dots$  will denote generic constants, depending only on the fixed parameters in the discussion, and any other explicitly indicated parameters.

**Theorem 2.1** *Let  $x_{k,n} = \cos \theta_{k,n} \in [-1, 1]$  be an arbitrary system of nodes ( $0 \leq \theta_{1,n} < \dots < \theta_{n,n} \leq \pi$ ) and let*

$$d_n := \min_{1 \leq k \leq n-1} \theta_{k+1,n} - \theta_{k,n}.$$

*Then for any  $\epsilon > 0$ , there exist linear polynomial operators  $P_n$  on  $C([-1, 1])$  with the following properties: (a) If  $m = \lfloor \pi(1 + \epsilon)/d_n \rfloor$  then  $P_n(P) = P$  for all  $P \in \Pi_m$ , (b) for  $f \in C([-1, 1])$ ,  $P_n(f) \in \Pi_N$  where  $N = (\pi/d_n + 1)(1 + 3\epsilon)$ , (c)  $P_n(f, x_{k,n}) = f(x_{k,n})$  for  $k = 1, \dots, n$ , and (d)  $\|P_n(f)\|_{[-1,1]} \leq c(\epsilon)\|f\|_{[-1,1]}$ .*

We note that the conditions (a) and (d) imply that

$$\|f - P_n(f)\|_{[-1,1]} \leq c(\epsilon) \min_{P \in \Pi_m} \|f - P\|_{[-1,1]}.$$

Giving up the requirement that the operator should be linear, we obtained a far reaching generalization of this result in [15]. The following theorem of Narcowich and Ward [20, Proposition 3.1] is a further generalization of the result in [15]. In the sequel, if  $Y$  is a Banach space,  $V \subset Y$ , we will write  $\|\cdot\|_Y$  to denote the norm on  $Y$ , and write

$$\text{dist}(Y; f, V) := \text{dist}(f, V) := \inf_{g \in V} \|f - g\|_Y, \quad f \in Y. \quad (2.1)$$

We note that our notation  $\|f\|_\Omega$  for the supremum norm on  $C(\Omega)$  may be thought of as an abbreviation for  $\|f\|_{C(\Omega)}$ .

**Theorem 2.2** *Let  $\mathcal{Y}$  be a (possibly complex) Banach space,  $\mathcal{V}$  be a subspace of  $\mathcal{Y}$ , and  $Z^*$  be a finite dimensional subspace of  $\mathcal{Y}^*$ , the dual of  $\mathcal{Y}$ . If there exists  $\beta > 0$  such that*

$$\|z^*\|_{\mathcal{Y}^*} \leq \beta \|z^*|_{\mathcal{V}}\|_{\mathcal{V}^*}, \quad z^* \in Z^*, \quad (2.2)$$

*then for every  $y \in \mathcal{Y}$ , there exists  $v \in \mathcal{V}$ , such that  $z^*(v) = z^*(y)$  for every  $z^* \in Z^*$ , and in addition,  $\|y - v\|_{\mathcal{Y}} \leq (1 + 2\beta) \text{dist}(y, \mathcal{V})$ .*

The main difference between this theorem and the corresponding theorem in [15] is that the space  $\mathcal{V}$  is not required to be finite dimensional. This is accomplished by an appeal to the notion of local reflexivity principle, rather than the fact that finite dimensional spaces are reflexive. This principle is formulated as follows (cf., for example, [7]), where the space  $Y$  is identified with a subspace of the dual  $Y^{**}$  of the space  $Y^*$  in the standard way.

**Proposition 2.1** *Let  $Y$  be a Banach space,  $V \subset Y^{**}$  and  $W \subset Y^*$  be finite dimensional spaces. Given  $\epsilon > 0$ , there exists a linear operator  $T : V \rightarrow Y$  such that  $T(y) = y$  if  $y \in V \cap Y$ ,  $y^*(T(v)) = v(y^*)$  for all  $v \in V$ ,  $y^* \in W$ , and  $(1 - \epsilon)\|v\|_{Y^{**}} \leq \|T(v)\|_Y \leq (1 + \epsilon)\|v\|_{Y^{**}}$ , for all  $v \in V$ .*

In [15], we showed that an analogue of Theorem 2.1 always holds if a Jackson-type theorem holds. In particular, we obtained the analogues in the case when the approximating spaces consist of spherical polynomials and zonal function networks. Theorem 2.2 was used in [20] to obtain the analogue of Theorem 2.1 in the case of approximation by entire functions of finite exponential type.

One restriction of all of the above theorems is that they require interpolation at distinct points. It seems reasonable to expect that a similar theorem will hold in the case of Birkhoff data; i.e., where the values of certain derivatives of the function are prescribed as well, provided there are only finitely many such conditions at each point, the number being independent of the number of points. In this section, we explore this question further. The following theorem gives a recipe for applying Theorem 2.2.

**Theorem 2.3** *Let  $\mathcal{Y}$  be a (possibly complex) Banach space,  $\mathcal{V}$  be a subspace of  $\mathcal{Y}$ , and  $Z^*$  be a finite dimensional subspace of  $\mathcal{Y}^*$ . Suppose there exists a compact set  $K \subset \mathcal{Y}$  such that*

$$\kappa := \sup_{y \in K} \text{dist}(y, \mathcal{V}) < \inf_{y^* \in Z^*, \|y^*\|_{\mathcal{Y}^*} = 1} \sup_{y \in K} |y^*(y)| =: m. \quad (2.3)$$

*Let  $B := \max_{y \in K} \|y\|_{\mathcal{Y}}$ , and  $\beta := 2(2B + m + \kappa)/(m - \kappa)$ . Then for every  $y \in \mathcal{Y}$ , there exists  $v \in \mathcal{V}$ , such that  $z^*(v) = z^*(y)$  for every  $z^* \in Z^*$ , and in addition,  $\|y - v\|_{\mathcal{Y}} \leq (1 + 2\beta) \text{dist}(y, \mathcal{V})$ .*

PROOF. We observe that (2.3) implies that

$$m = \inf_{y^* \in Z^*, \|y^*\|_{\mathcal{Y}^*} = 1} \sup_{y \in K} |y^*(y)| > 0. \quad (2.4)$$

Now, let  $y^* \in Z^*$ ,  $\|y^*\|_{\mathcal{Y}^*} = 1$ . In view of the fact that  $\kappa < m$ , we find  $y \in K$  such that  $|y^*(y)| \geq (3/4)m + (1/4)\kappa$  (cf. (2.4)). The estimate (2.3) implies that there exists  $v \in \mathcal{V}$  such that  $\|y - v\|_{\mathcal{Y}} \leq (m + \kappa)/2$ . Then

$$|y^*(v)| \geq |y^*(y)| - |y^*(y - v)| \geq (3/4)m + (1/4)\kappa - (m + \kappa)/2 = (m - \kappa)/4.$$

Also,  $\|v\|_{\mathcal{Y}} \leq \|y\|_{\mathcal{Y}} + (m + \kappa)/2 \leq (2B + m + \kappa)/2$ . Thus,

$$\|y^*\|_{\mathcal{Y}^*} = 1 \leq \frac{2(2B + m + \kappa)}{m - \kappa} \frac{|y^*(v)|}{\|v\|_{\mathcal{Y}}} \leq \frac{2(2B + m + \kappa)}{m - \kappa} \|y^*|_{\mathcal{V}}\|_{\mathcal{Y}^*}.$$

We may now apply Theorem 2.2 to complete the proof.  $\square$

One way to construct a compact set as in Theorem 2.3 is the following. Let  $\{y_1^*, \dots, y_N^*\}$  be a basis for  $Z^*$ , with each  $\|y_\ell^*\|_{\mathcal{Y}^*} = 1$ . The Hahn–Banach theorem yields a set  $\{x_1^{**}, \dots, x_N^{**}\}$  in the dual  $\mathcal{Y}^{**}$  of  $\mathcal{Y}^*$  such that each  $\|x_\ell^{**}\|_{\mathcal{Y}^{**}} = 1$  and  $\mathbf{x}_\ell^{**}(y_j^*) = \delta_{\ell,j}$ ,  $\ell, j = 1, \dots, N$ . Let  $\epsilon > 0$ . The principle of local reflexivity then implies the existence of  $\{x_1, \dots, x_N\} \in \mathcal{Y}$  such that  $1 - \epsilon \leq \|x_\ell\|_{\mathcal{Y}} \leq 1 + \epsilon$ ,  $1 \leq \ell \leq N$  and  $y_j^*(x_\ell) = \delta_{\ell,j}$ ,  $\ell, j = 1, \dots, N$ . We may choose  $K$  to be the set  $\{\sum_{\ell=1}^N a_\ell x_\ell : \max_{1 \leq \ell \leq N} |a_\ell| = 1\}$ . If  $y^* = \sum_{j=1}^N b_j y_j^*$ , and  $\|y^*\|_{\mathcal{Y}^*} = 1$ , then

$$1 = \|y^*\|_{\mathcal{Y}^*} \leq \sum_{j=1}^N |b_j| = y^* \left( \sum_{j=1}^N (\operatorname{sgn} b_j) x_j \right) \leq \sup_{y \in K} |y^*(y)|.$$

Therefore, the conclusion of Theorem 2.3 holds if  $\sup_{y \in K} \operatorname{dist}(y, V) < 1$ .

We illustrate this process by giving a qualitative generalization of [20, Theorem 3.5]. Thus, we will not have explicit constants as in [20, Theorem 3.5], but our theorem will be valid when Birkhoff interpolatory conditions are required on the approximation.

Let  $s \geq 1$  be a fixed integer. We will write  $\mathbf{x} = (x_1, \dots, x_s)$ ,  $|\mathbf{x}| = \sum_{j=1}^s |x_j|$ ,  $|\mathbf{x}|_\infty = \max_{1 \leq j \leq s} |x_j|$ . The symbols  $\mathbf{k}, \mathbf{m}, \mathbf{n}$  will denote multiintegers with nonnegative components. We write  $D_j$  for the derivative with respect to  $x_j$ ,  $f^{(\mathbf{k})} := (\prod_{j=1}^s D_j^{k_j})f$ . For an integer  $r \geq 0$ ,  $C^r(\mathbb{R}^s)$  is defined to be the space of functions  $f$  such that  $f^{(\mathbf{k})} \in C(\mathbb{R}^s)$  for all  $\mathbf{k}$  with  $|\mathbf{k}| \leq r$ . For  $\tau \geq 0$ , we denote by  $\mathcal{M}_\tau = \mathcal{M}_{\tau,s}$  the class of all entire functions  $g$  of  $s$  complex variables, such that the restriction of  $g$  to  $\mathbb{R}^s$  is real valued, and for some  $A = A(g) > 0$ ,

$$|g(x_1 + iy_1, \dots, x_s + iy_s)| \leq A \exp \left( \tau \sum_{j=1}^s |y_j| \right), \quad (x_1 + iy_1, \dots, x_s + iy_s) \in \mathbb{C}^s.$$

We note that the class  $\mathcal{M}_\tau$  consists of entire functions of finite exponential type at most  $\tau$ , that are bounded and real valued on  $\mathbb{R}^s$ . The Bernstein inequality [22, Section 3.2.2, eqn. (8)],

$$\|g^{(\mathbf{k})}\|_{\mathbb{R}^s} \leq \tau^{|\mathbf{k}|} \|g\|_{\mathbb{R}^s}, \quad g \in \mathcal{M}_\tau, \quad (2.5)$$

shows that  $\mathcal{M}_\tau \subset C^r(\mathbb{R}^s)$  for all  $\tau \geq 0$  and integer  $r \geq 0$ .

**Theorem 2.4** *Let  $R \geq 0$  be an integer,  $\mathbf{x}_j$ ,  $j = 1, \dots, N$ , be a set of distinct points in  $\mathbb{R}^s$ , and*

$$\eta := \min_{j \neq \ell} |\mathbf{x}_j - \mathbf{x}_\ell|_\infty, \quad (2.6)$$

*and  $S_j \subseteq \{\mathbf{m} : |\mathbf{m}| \leq R\}$ ,  $j = 1, \dots, N$ . Then there exists a constant  $\alpha$  with the following property. For any  $f \in C^R(\mathbb{R}^s)$ , there exists  $g \in \mathcal{M}_{\alpha/\eta}$  such that*

$$g^{(\mathbf{k})}(\mathbf{x}_j) = f^{(\mathbf{k})}(\mathbf{x}_j), \quad \mathbf{k} \in S_j, \quad j = 1, \dots, N, \quad (2.7)$$

*and*

$$\max_{|\mathbf{k}| \leq R} \eta^{|\mathbf{k}|} \|f^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \leq c(R) \max_{|\mathbf{m}| \leq R} \eta^{|\mathbf{m}|} \inf_{P \in \mathcal{M}_{\alpha/(4\eta)}} \|f^{(\mathbf{m})} - P\|_{\mathbb{R}^s}. \quad (2.8)$$

In the rest of this section,  $R$  will be treated as a fixed parameter, and the dependence of the various constants on  $R$  will not be indicated explicitly.

In order to prove the theorem, we recall from [22, Section 8.6] some facts regarding approximation by entire functions of finite exponential type. For  $\tau > 0$ ,  $\mathbf{x} \in \mathbb{R}^s$ , and  $f \in C(\mathbb{R}^s)$ , let

$$V_\tau(\mathbf{x}) := \frac{1}{\tau^s} \prod_{j=1}^s \frac{\cos \tau x_j - \cos 2\tau x_j}{x_j^2}, \quad (2.9)$$

and

$$\mathcal{F}_\tau(f, \mathbf{x}) := \frac{1}{\pi^s} \int_{\mathbb{R}^s} V_\tau(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}. \quad (2.10)$$

**Lemma 2.1** *If  $g \in \mathcal{M}_\tau$  then  $\mathcal{F}_\tau(g) = g$ . For any  $f \in C(\mathbb{R}^s)$ ,  $\mathcal{F}_\tau(f) \in \mathcal{M}_{2\tau}$ , and*

$$\|\mathcal{F}_\tau(f)\|_{\mathbb{R}^s} \leq c \|f\|_{\mathbb{R}^s}, \quad \|f - \mathcal{F}_\tau(f)\|_{\mathbb{R}^s} \leq c \operatorname{dist}(f, \mathcal{M}_\tau). \quad (2.11)$$

Moreover, if  $r \geq 1$  is an integer,  $f \in C^r(\mathbb{R}^s)$ ,  $g \in \mathcal{M}_{2\tau}$ , and  $\|f - g\|_{\mathbb{R}^s} \leq c \operatorname{dist}(f, \mathcal{M}_\tau)$ , then for any multiinteger  $\mathbf{k}$  with  $|\mathbf{k}| \leq r$ ,

$$\tau^{-|\mathbf{k}|} \|f^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \leq c(r) \tau^{-r} \max_{|\mathbf{m}|=r} \operatorname{dist}(f^{(\mathbf{m})}, \mathcal{M}_{\tau/2}). \quad (2.12)$$

In particular,

$$\max_{|\mathbf{k}| \leq r} \tau^{-|\mathbf{k}|} \|f^{(\mathbf{k})} - \mathcal{F}_\tau(f)^{(\mathbf{k})}\|_{\mathbb{R}^s} \leq c(r) \tau^{-r} \max_{|\mathbf{m}|=r} \|f^{(\mathbf{m})}\|_{\mathbb{R}^s}, \quad (2.13)$$

and

$$\inf_{P \in \mathcal{M}_{2\tau}} \max_{|\mathbf{m}| \leq R} \tau^{-|\mathbf{m}|} \|f^{(\mathbf{m})} - P^{(\mathbf{m})}\|_{\mathbb{R}^s} \leq c \max_{|\mathbf{m}| \leq R} \tau^{-|\mathbf{m}|} \inf_{P \in \mathcal{M}_{\tau/2}} \|f^{(\mathbf{m})} - P\|_{\mathbb{R}^s}. \quad (2.14)$$

PROOF. The first two statements in the lemma and the estimates (2.11) are proved in [22, Section 8.6]. Let  $f \in C^r(\mathbb{R}^s)$ ,  $g \in \mathcal{M}_{2\tau}$ ,  $\|f - g\|_{\mathbb{R}^s} \leq c \operatorname{dist}(f, \mathcal{M}_\tau)$ , and  $\mathbf{k}$  be a multiinteger with  $|\mathbf{k}| \leq r$ . The second estimate in (2.11) implies that  $\|g - \mathcal{F}_\tau(f)\|_{\mathbb{R}^s} \leq c \operatorname{dist}(f, \mathcal{M}_\tau)$ . Since  $\mathcal{F}_\tau(f) - g \in \mathcal{M}_{2\tau}$ , the Bernstein inequality (2.5) shows that

$$\|\mathcal{F}_\tau(f)^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} = \|\mathcal{F}_\tau(f)^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \leq c(r) \tau^{|\mathbf{k}|} \|\mathcal{F}_\tau(f) - g\|_{\mathbb{R}^s} \leq c(r) \tau^{|\mathbf{k}|} \operatorname{dist}(f, \mathcal{M}_\tau). \quad (2.15)$$

In view of the direct theorem for approximation from  $\mathcal{M}_\tau$  (cf. [22, Section 5.2.2, eqn. (4), Section 4.2, eqn. (15)]), it follows that

$$\tau^{|\mathbf{k}|} \operatorname{dist}(f, \mathcal{M}_\tau) \leq c(r) \tau^{|\mathbf{k}|-r} \sum_{|\mathbf{m}|=r} \|f^{(\mathbf{m})}\|_{\mathbb{R}^s} \leq c(r) \tau^{|\mathbf{k}|-r} \max_{|\mathbf{m}|=r} \|f^{(\mathbf{m})}\|_{\mathbb{R}^s}.$$

Therefore, (2.11) leads to

$$\begin{aligned} \operatorname{dist}(f, \mathcal{M}_\tau) &= \operatorname{dist}(f - \mathcal{F}_{\tau/2}(f), \mathcal{M}_\tau) \\ &\leq c(r) \tau^{-r} \max_{|\mathbf{m}|=r} \|f^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(f)^{(\mathbf{m})}\|_{\mathbb{R}^s} \\ &= c(r) \tau^{-r} \max_{|\mathbf{m}|=r} \|f^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(f^{(\mathbf{m})})\|_{\mathbb{R}^s} \\ &\leq c(r) \tau^{-r} \max_{|\mathbf{m}|=r} \operatorname{dist}(f^{(\mathbf{m})}, \mathcal{M}_{\tau/2}). \end{aligned} \quad (2.16)$$

Using this estimate with  $f^{(\mathbf{k})}$  in place of  $f$ , and  $r - |\mathbf{k}|$  in place of  $r$ , we obtain in view of (2.11) that

$$\|f^{(\mathbf{k})} - \mathcal{F}_\tau(f^{(\mathbf{k})})\|_{\mathbb{R}^s} \leq c(r) \operatorname{dist}(f^{(\mathbf{k})}, \mathcal{M}_\tau) \leq c(r)\tau^{|\mathbf{k}|-r} \max_{|\mathbf{m}|=r} \operatorname{dist}(f^{(\mathbf{m})}, \mathcal{M}_{\tau/2}). \quad (2.17)$$

The estimates (2.17), (2.15), and (2.16) lead to

$$\begin{aligned} \|f^{(\mathbf{k})} - g^{(\mathbf{k})}\|_{\mathbb{R}^s} &\leq \|f^{(\mathbf{k})} - \mathcal{F}_\tau(f^{(\mathbf{k})})\|_{\mathbb{R}^s} + \|\mathcal{F}_\tau(f^{(\mathbf{k})}) - g^{(\mathbf{k})}\|_{\mathbb{R}^s} \\ &\leq c(r)\tau^{|\mathbf{k}|-r} \max_{|\mathbf{m}|=r} \operatorname{dist}(f^{(\mathbf{m})}, \mathcal{M}_{\tau/2}). \end{aligned}$$

This completes the proof of (2.12). The estimates (2.13) and (2.14) follow from (2.11) and (2.12).  $\square$

PROOF OF THEOREM 2.4. In this proof, for any integer  $r \geq 0$  and a function  $f \in C^r(\mathbb{R}^s)$ , we write

$$\|f\|_r := \max_{|\mathbf{k}|=r} \|f^{(\mathbf{k})}\|_{\mathbb{R}^s}. \quad (2.18)$$

We will apply Theorem 2.3 with  $Y = C^R(\mathbb{R}^s)$ , and for  $f \in Y$ , define

$$\|f\|_Y := \max_{0 \leq r \leq R} \eta^r \|f\|_r. \quad (2.19)$$

We will write  $\tau = \alpha/\eta$  for a constant  $\alpha$  to be chosen later, and take  $\mathcal{M}_\tau$  for the subspace  $V$  of  $Y$ . In the remainder of this proof,  $\operatorname{dist}(f, V)$  is defined as in (2.1) with respect to the norm  $\|\cdot\|_Y$ . For multiinteger  $\mathbf{m}$  and  $\ell = 1, \dots, N$ , let  $y_{\mathbf{m}, \ell}^*$  denote the functional on  $Y$  defined by  $y_{\mathbf{m}, \ell}^*(f) = f^{(\mathbf{m})}(\mathbf{x}_\ell)$ . Let  $Z^*$  be the subspace of  $Y^*$  spanned by  $\{y_{\mathbf{m}, \ell}^* : \mathbf{m} \in S_\ell, \ell = 1, \dots, N\}$ . We now construct the compact set  $K$  as required in Theorem 2.3, following the ideas outlined after the proof of that theorem. In this proof only, we find it useful to retain the values of the constants  $c, c_1, \dots$ .

Let  $\psi : \mathbb{R} \rightarrow [0, \infty)$  be an infinitely often differentiable function such that  $\psi(t) = 1$  if  $|t| \leq 1/6$  and  $\psi(t) = 0$  if  $|t| \geq 1/3$ . For any multiinteger  $\mathbf{k} \geq 0$ , let

$$\phi_{\mathbf{k}}(\mathbf{x}) := \prod_{\ell=1}^s \psi(x_\ell/\eta) \frac{x_\ell^{k_\ell}}{k_\ell!},$$

and

$$\Phi_{\mathbf{k}, j}(\mathbf{x}) = \phi_{\mathbf{k}}(\mathbf{x} - \mathbf{x}_j), \quad j = 1, \dots, N. \quad (2.20)$$

We note that if  $|\mathbf{x} - \mathbf{x}_j|_\infty \geq \eta/3$ , then  $\Phi_{\mathbf{k}, j}(\mathbf{x}) = 0$  for all  $\mathbf{k} \geq 0$ . In particular, for any multiinteger  $\mathbf{m} \geq 0$ ,

$$\|\Phi_{\mathbf{k}, j}^{(\mathbf{m})}\|_{\mathbb{R}^s} \leq c_1 \eta^{|\mathbf{k}|-|\mathbf{m}|}. \quad (2.21)$$

Since  $\Phi_{\mathbf{k}, j}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_j)^{\mathbf{k}}/\mathbf{k}!$  for all  $\mathbf{x}$  with  $|\mathbf{x} - \mathbf{x}_j|_\infty \leq \eta/6$ , it follows that

$$y_{\mathbf{m}, \ell}^*(\Phi_{\mathbf{k}, j}) = \Phi_{\mathbf{k}, j}^{(\mathbf{m})}(\mathbf{x}_\ell) = \begin{cases} 1, & \text{if } \mathbf{k} = \mathbf{m}, j = \ell, \\ 0, & \text{otherwise.} \end{cases} \quad (2.22)$$

We define

$$K := \left\{ \sum_{j=1}^N \sum_{\mathbf{k} \in S_j} b_{\mathbf{k}, j} \eta^{-|\mathbf{k}|} \Phi_{\mathbf{k}, j} : \max_{1 \leq j \leq N, \mathbf{k} \in S_j} |b_{\mathbf{k}, j}| \leq 1 \right\}.$$

We now estimate the quantities  $B$ ,  $m$ ,  $\kappa$  appearing in (2.3).

Let  $g = \sum_{j=1}^N \sum_{\mathbf{k} \in S_j} b_{\mathbf{k},j} \eta^{-|\mathbf{k}|} \Phi_{\mathbf{k},j} \in K$ . Let  $r$  be any integer,  $0 \leq r \leq \mathcal{R} + 1$ ,  $\mathbf{n}$  be any multiinteger with  $|\mathbf{n}| = r$ , and  $\mathbf{x} \in \mathbb{R}^s$ . If  $|\mathbf{x} - \mathbf{x}_j|_\infty > \eta/3$  for every  $j$ , then  $g^{(\mathbf{n})}(\mathbf{x}) = 0$ . Otherwise, there is a unique  $j$  such that  $|\mathbf{x} - \mathbf{x}_j|_\infty \leq \eta/3$ , and

$$g^{(\mathbf{n})}(\mathbf{x}) = \sum_{\mathbf{k} \in S_j} b_{\mathbf{k},j} \eta^{-|\mathbf{k}|} \Phi_{\mathbf{k},j}^{(\mathbf{n})}(\mathbf{x}).$$

Since the number of elements in  $S_j$  is bounded independently of  $N$ , and  $\max_{\mathbf{k} \in S_j} |b_{\mathbf{k},j}| \leq 1$ , (2.21) implies that

$$|g^{(\mathbf{n})}(\mathbf{x})| \leq \sum_{\mathbf{k} \in S_j} \eta^{-|\mathbf{k}|} |\Phi_{\mathbf{k},j}^{(\mathbf{n})}(\mathbf{x})| \leq c_2 \eta^{-|\mathbf{n}|}.$$

Thus,  $\|g^{(\mathbf{n})}\|_{\mathbb{R}^s} \leq c_2 \eta^{-r}$  for all  $\mathbf{n}$  with  $|\mathbf{n}| = r$ ; i.e.,

$$\|g\|_r \leq c_2 \eta^{-r}, \quad g \in K, \quad r \geq 0. \quad (2.23)$$

It follows that

$$B = \sup_{g \in K} \|g\|_Y \leq c_2. \quad (2.24)$$

Let  $y^* = \sum_{\ell=1}^N \sum_{\mathbf{m} \in S_\ell} a_{\mathbf{m},\ell} y_{\mathbf{m},\ell}^* \in Z^*$ , and  $g \in Y$ . Then

$$|y^*(g)| = \left| \sum_{\ell=1}^N \sum_{\mathbf{m} \in S_\ell} a_{\mathbf{m},\ell} g^{(\mathbf{m})}(\mathbf{x}_\ell) \right| \leq \|g\|_Y \sum_{\ell=1}^N \sum_{\mathbf{m} \in S_\ell} |a_{\mathbf{m},\ell}| \eta^{-|\mathbf{m}|}.$$

Therefore,

$$\|y^*\|_{Y^*} \leq \sum_{\ell=1}^N \sum_{\mathbf{m} \in S_\ell} |a_{\mathbf{m},\ell}| \eta^{-|\mathbf{m}|}. \quad (2.25)$$

Let  $g_{y^*} = \sum_{j=1}^N \sum_{\mathbf{k} \in S_j} (\text{sgn } a_{\mathbf{k},j}) \eta^{-|\mathbf{k}|} \Phi_{\mathbf{k},j} \in K$ . It is easy to verify using (2.22) that

$$y^*(g_{y^*}) = \sum_{\ell=1}^N \sum_{\mathbf{m} \in S_\ell} |a_{\mathbf{m},\ell}| \eta^{-|\mathbf{m}|} \geq \|y^*\|_{Y^*}.$$

Thus,

$$m = \inf_{y^* \in Z^*, \|y^*\|_{Y^*} = 1} \sup_{g \in K} |y^*(g)| \geq 1. \quad (2.26)$$

Next, let  $g \in K$ . For any multiinteger  $\mathbf{m} \geq 0$ ,  $|\mathbf{m}| \leq R$ , (2.13) implies that

$$\|g^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(g)^{(\mathbf{m})}\|_{\mathbb{R}^s} \leq \frac{c_3}{\tau^{R+1-|\mathbf{m}|}} \|g\|_{R+1}. \quad (2.27)$$

Now, (2.23) with  $R + 1$  in place of  $r$  implies that

$$\eta^{|\mathbf{m}|} \|g^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(g)^{(\mathbf{m})}\|_{\mathbb{R}^s} \leq \frac{c_4}{(\tau\eta)^{R+1-|\mathbf{m}|}}.$$

Since this estimate holds for all  $\mathbf{m}$  with  $0 \leq |\mathbf{m}| \leq R$ , we have

$$\text{dist}(g, \mathcal{M}_\tau) \leq \|g - \mathcal{F}_{\tau/2}(g)\|_Y = \max_{0 \leq |\mathbf{m}| \leq R} \eta^{|\mathbf{m}|} \|g^{(\mathbf{m})} - \mathcal{F}_{\tau/2}(g)^{(\mathbf{m})}\|_{\mathbb{R}^s} \leq \frac{c_4}{\tau\eta}. \quad (2.28)$$

With the choice  $\alpha = 2c_4$ , we conclude from (2.28) and (2.26) that (2.3) is satisfied for all  $\tau \geq \alpha/\eta$ . In view of Theorem 2.3 and (2.14), this completes the proof.  $\square$

Similar theorems can be obtained in a variety of other situations, where a sequence of simultaneously approximating operators is known; for example, in approximation by trigonometric polynomials [2], approximation by spherical polynomials [5], approximation by periodic basis functions [12], and approximation by Gaussian networks [8, Chapter 11.2].

### 3 Quasi–interpolatory operators

In many practical applications, although one needs to construct an approximation using point evaluations, interpolation is not necessarily desirable. For example, the data may be noisy, or too large to allow for an efficient computation of the interpolant. While the theorems in the previous section assert the existence of a bounded interpolatory operator, it is desirable to obtain explicit, computationally efficient constructions for approximations, whether they interpolate or not.

Given a sequence of subspaces  $\{\mathcal{V}_n\}$  with  $\mathcal{V}_n \subset \mathcal{V}_{n+1} \subset C(\Omega)$ ,  $n = 0, 1, \dots$ , a quasi–interpolatory operator  $\mathcal{T}_{n,N}$  is a linear operator that is constructed from the  $N$  data points, but has properties similar to the operator  $\mathcal{F}_\tau$  as in Lemma 2.1; i.e., we require that  $\mathcal{T}_{n,N} : C(\Omega) \rightarrow \mathcal{V}_n$ ,  $\|\mathcal{T}_{n,N}(f)\|_\Omega \leq c\|f\|_\Omega$  for some  $c > 0$  independent of  $n$  and  $N$ , and for some  $\alpha > 0$ ,  $\mathcal{V}_{\alpha n}$  is invariant under  $\mathcal{T}_{n,N}$ . Here, and in the sequel, the symbol  $\mathcal{V}_x$  denotes the space  $\mathcal{V}_{\lfloor x \rfloor}$ . These assumptions imply that for any  $P \in \mathcal{V}_{\alpha n}$ ,

$$\|f - \mathcal{T}_{n,N}(f)\|_\Omega = \|(f - P) - \mathcal{T}_{n,N}(f - P)\|_\Omega \leq c\|f - P\|_\Omega,$$

and hence,

$$\|f - \mathcal{T}_{n,N}(f)\|_\Omega \leq c \text{dist}(f, \mathcal{V}_{\alpha n}). \quad (3.1)$$

Thus, if  $\bigcup_{n=0}^\infty \mathcal{V}_n$  is dense in  $C(\Omega)$ , the sequence  $\mathcal{T}_{n,N}(f)$  always converges to  $f$  for every  $f \in C(\Omega)$ , and at a near optimal rate in the sense of (3.1).

In [18], we have described a very general construction for quasi–linear operators. Let  $(\Omega, \Sigma)$  be *any* measure space. We will assume that all measures on  $\Omega$  to be discussed below will be defined for all the subsets  $A \in \Sigma$ . Let  $\mu^*$  be a sigma–finite measure on  $\Omega$ ,  $\nu$  be a signed measure (necessarily, with bounded variation) or a positive, sigma–finite measure on  $\Omega$ ,  $|\nu|$  denote  $\nu$  if  $\nu$  is a positive measure, and its total variation measure if it is a signed measure. If  $A \subseteq \Omega$  is  $|\nu|$ -measurable, and  $f : A \rightarrow \mathbb{R}$  is  $|\nu|$ -measurable, we write

$$\|f\|_{\nu;p,A} := \begin{cases} \left\{ \int_A |f(t)|^p d|\nu|(t) \right\}^{1/p}, & \text{if } 1 \leq p < \infty, \\ |\nu| - \text{ess sup}_{t \in A} |f(t)|, & \text{if } p = \infty. \end{cases}$$

The class of measurable functions  $f$  for which  $\|f\|_{\nu; p, A} < \infty$  is denoted by  $L^p(\nu; A)$ , with the standard convention that two functions are considered equal if they are equal  $|\nu|$ -almost everywhere on  $A$ . We will work with a fixed orthonormal set  $\{\phi_k\}_{k=0}^\infty \subset L^2(\mu^*; \Omega)$ ; i.e.,

$$\int_{\Omega} \phi_k(t)\phi_j(t)d\mu^*(t) = \begin{cases} 0, & \text{if } k \neq j, \\ 1, & \text{if } k = j, \end{cases} \quad (3.2)$$

and assume that it is complete in the space  $L^2(\mu^*; \Omega)$ . We assume also that each  $\phi_k \in L^1(\mu^*; \Omega) \cap L^\infty(\mu^*; \Omega)$ . We will write  $\mathbb{P}_n := \text{span} \{\phi_0, \dots, \phi_n\}$ , and the symbol  $X^p(\Omega)$  will denote the  $L^p(\mu^*; \Omega)$ -closure of  $\bigcup_{n=0}^\infty \mathbb{P}_n$ .

We pause in the general discussion to give a few examples.

1.  $\Omega = [-1, 1]$ ,  $\mu^*$  is the Jacobi measure  $(1-x)^\alpha(1+x)^\beta dx$ ,  $\alpha, \beta \geq -1/2$ ,  $\phi_k$  is the orthonormalized Jacobi polynomial  $p_k^{(\alpha, \beta)}$  of degree  $k$  with respect to  $\mu^*$ .
2.  $\Omega = \mathbb{R}$ ,  $\mu^*$  is the Lebesgue measure on  $\Omega$ ,  $\phi_k$  is the orthonormalized weighted Freud polynomial  $\phi_k^F = w_Q p_k$ , where  $w_Q$  is a Freud weight (cf. [8, Definition 3.1.1]), and  $p_k$  is the degree  $k$  orthonormal polynomial such that for all integers  $k, j = 0, 1, \dots$ ,

$$\int_{\mathbb{R}} p_k(x)p_j(x)w_Q^2(x)dx = \begin{cases} 0, & \text{if } k \neq j, \\ 1, & \text{if } k = j. \end{cases}$$

3.  $\Omega = [-\pi, \pi]$ ,  $\mu^*$  is the measure  $dx/\pi$  on  $\Omega$ ,

$$\{\phi_k\} = \{1/\sqrt{2}, \cos x, \sin x, \dots, \cos kx, \sin kx, \dots\}.$$

4.  $\Omega$  is the unit sphere  $\mathbb{S}^q$  embedded in the Euclidean space  $\mathbb{R}^{q+1}$ ,  $\mu^*$  is the area measure on  $\Omega$ ,  $\{\phi_k\}$  is the set of all orthonormal spherical harmonics [19, 23].

A convenient way to define a quasi-interpolatory operator is the following. Let  $h : [0, \infty) \rightarrow [0, \infty)$ ,  $h(x) = 1$  if  $x \leq 1/2$ , and  $h(x) = 0$  if  $x > 1$ . We define the kernels

$$\Phi_n(h; x, t) := \sum_{k=0}^n h(k/n)\phi_k(x)\phi_k(t), \quad (3.3)$$

and the operators

$$\sigma_n(\nu; h, f, x) := \int_{\Omega} f(t)\Phi_n(h; x, t)d\nu(t). \quad (3.4)$$

We note that in the case when  $\nu$  is a discrete measure, the operator  $\sigma_n(\nu; h, f)$  is defined in terms of the values of  $f$  at the points in the support of  $\nu$ . In this case, we may view  $\sigma_n(\nu; h, f)$  as a discretization of the ‘‘continuous’’ operator  $\sigma_n(\mu^*; h, f)$ . It is clear that

$$\|\sigma_n(\nu; h, f)\|_{\mu^*; \infty, \Omega} \leq \|f\|_{\nu; \infty, \Omega} \sup_{x \in \Omega} \int_{\Omega} |\Phi_n(h, x, t)|d|\nu|(t). \quad (3.5)$$

The quantity on the right hand side above is often uniformly bounded for suitable choices of  $h$  for the case when  $\nu = \mu^*$ . The bounds in the discrete case are then obtained using the

Marcinkiewicz-Zygmund inequalities (3.6). Formally, we say that  $\nu$  is an M–Z measure of order  $n$  if

$$\|P\|_{\nu;p,\Omega} \leq c\|P\|_{\mu^*;p,\Omega}, \quad P \in \mathbb{P}_n, \quad 1 \leq p \leq \infty. \quad (3.6)$$

The property that  $\sigma_n(\nu; h, P) = P$  for  $P \in \mathbb{P}_{n/2}$  is ensured if

$$\int_{\Omega} P_1 P_2 d\nu = \int_{\Omega} P_1 P_2 d\mu^*, \quad P_1, P_2 \in \mathbb{P}_n. \quad (3.7)$$

We say that  $\nu$  is a quadrature measure of order  $n$  if (3.7) holds, and that  $\nu$  is an M–Z quadrature measure of order  $n$  if it is both an M–Z measure of order  $n$  and a quadrature measure of order  $n$ .

M–Z quadrature measures based on equidistant data for the case of trigonometric polynomials are given in [26, Chapter X, Theorems 7.5, 7.28]. Nevai [21, Theorem 25, p. 168] has given an example of M–Z quadrature measures for the Jacobi weights. For  $m \geq 1$ , let  $\{x_{k,m}\}_{k=1}^m$  be the zeros of  $p_m^{(\alpha,\beta)}$ , and  $\lambda_{k,m}$  be the corresponding Cotes' numbers. Nevai has proved that for  $m \geq cn$ , the measure  $\nu_m^*$  that associates the mass  $\lambda_{k,m}$  with each  $x_{k,m}$  is an M–Z quadrature measure of order  $n$ . Similar results in the context of Freud polynomials are given in [16]. The analogues in the case of the sphere are obtained in [13] based on an arbitrary set of points. In [11], we have given an intrinsic characterization of M–Z measures on the sphere  $\mathbb{S}^q$ . Given a sequence of measures  $\{\nu_n\}$  on  $\mathbb{S}^q$ , we have shown that each  $\nu_n$  is an M–Z measure of order at least  $n$  if and only if for each spherical cap  $C$ ,

$$|\nu_n|(C) \leq c(\mu_q^*(C) + 1/n^q),$$

where  $\mu_q^*$  is the surface area measure for  $\mathbb{S}^q$ .

The following theorem explains the construction of quasi-interpolatory operators from their “continuous” analogues using M–Z quadrature measures.

**Theorem 3.1** *Let  $\{\nu_n\}$  be a sequence of signed or positive sigma-finite measures, such that each  $\nu_n$  is an M–Z quadrature measure of order at least  $n$ . Then  $\sigma_n(\nu_n; h, P) = P$  for all  $P \in \mathbb{P}_{n/2}$ . For  $f \in L^1(\mu^*; \Omega)$ ,  $\sigma_n(\nu_n; h, f) \in \mathbb{P}_n$ . Let  $1 \leq p \leq \infty$ . If*

$$\sup_{n \geq 0, x \in \Omega} \int_{\Omega} |\Phi_n(h, x, t)| d\mu^*(t) =: B < \infty, \quad (3.8)$$

then

$$\|\sigma_n(\nu_n; h, f)\|_{\mu^*;p,\Omega} \leq B\|f\|_{\nu_n;p,\Omega}. \quad (3.9)$$

In particular, if  $\|f\|_{\nu_n;p,\Omega} \leq c\|f\|_{\mu^*;p,\Omega}$  for all  $f \in L^p(\mu^*; \Omega)$ , then

$$\|\sigma_n(\nu_n; h, f)\|_{\mu^*;p,\Omega} \leq B\|f\|_{\mu^*;p,\Omega}, \quad f \in L^p(\mu^*; \Omega). \quad (3.10)$$

In [18], we have surveyed various conditions on the function  $h$  to ensure that (3.8) is satisfied in each of the four examples listed above. In each of these cases, a sufficient condition is that  $h$  should have sufficiently many derivatives having bounded variation. In general, the greater the smoothness of  $h$ , the more localized is the kernel. To illustrate

this phenomenon, we recall that for  $q \geq 1$ , the cardinal  $B$ -spline of order  $q$  is the function defined by (cf. [1, p. 131])

$$\begin{aligned} M_1(x) &:= \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\ M_q(x) &:= \frac{1}{q-1} \{xM_{q-1}(x) + (q-x)M_{q-1}(x-1)\}, \quad q \geq 2. \end{aligned} \quad (3.11)$$

We consider the function  $h$  defined by

$$h_q(x) = \sum_{j=-q}^q M_q(2qx - j), \quad x \geq 0.$$

The function  $h_q$  is  $q-2$  times continuously differentiable on  $\mathbb{R}$ ,  $h_q^{(q)}$  is a piecewise constant function,  $h_q(x) = 1$  if  $|x| < 1/2$ , and  $h_q(x) = 0$  if  $|x| > 1$ . The kernel

$$F_{q,n}(x) := 1 + \sum_{k=1}^n h_q(k/n) \cos kx \quad (3.12)$$

is the well known de la Vallée Poussin kernel when  $q = 2$ . In Figure 1, we demonstrate the graphs of the kernels  $F_{q,64}$  for the whole interval  $[-\pi, \pi]$  for  $q = 2$  and  $q = 5$ . The graphs of the same kernels on  $[\pi/2, \pi]$  in Figure 2 clearly illustrate the effect of the smoothness of  $h$  on the localization. The mathematical details for the localization properties depend upon the special function properties of the orthogonal systems involved; we refer to [18] for further references and details.

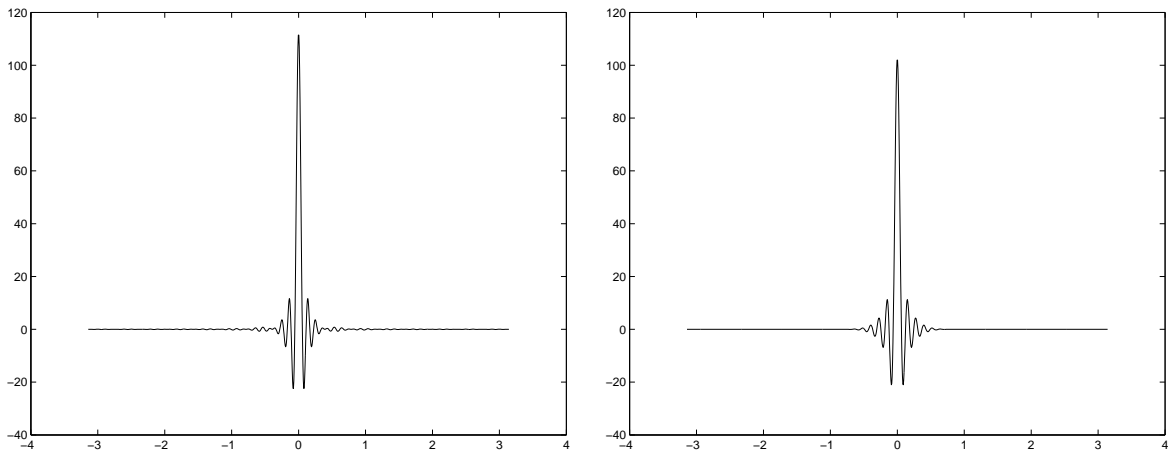


Figure 1: The graph of the de la Vallée Poussin kernel  $F_{2,64}$  on the left, the graph of  $F_{5,64}$  on the right.

We note that the most difficult part of these constructions is the construction of quadrature measures of high orders. It is much simpler to construct M-Z measures. For example, let  $N \geq 2$  be an integer, and  $-\pi \leq \theta_1 < \dots < \theta_N \leq \pi$  be points such that each subinterval of  $[-\pi, \pi]$  of length  $2\pi/N$  contains exactly one point  $\theta_k$ . It is not

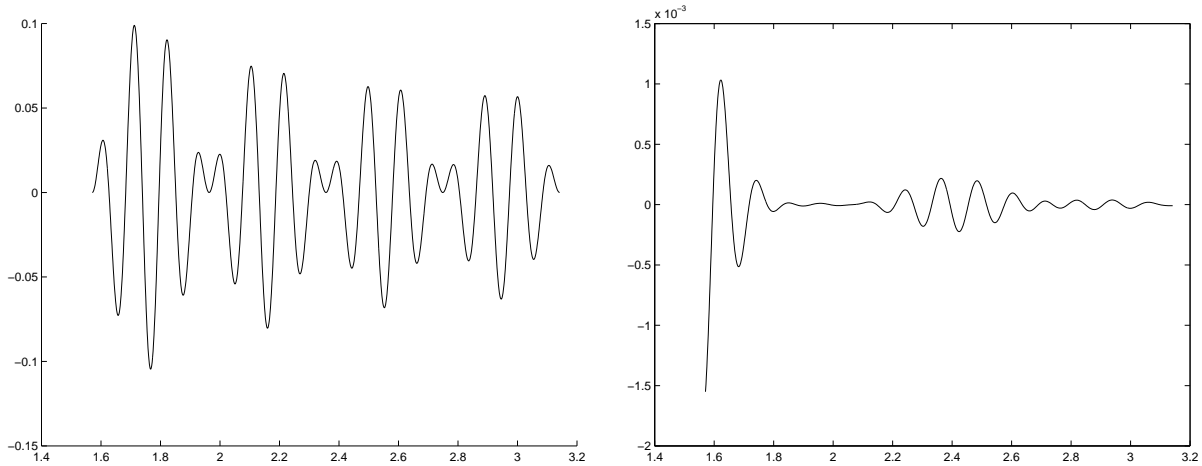


Figure 2: The graph of the de la Vallée Poussin kernel  $F_{2,64}$  on  $[\pi/2, \pi]$  on the left, the graph of  $F_{5,64}$  on  $[\pi/2, \pi]$  on the right. We note that the maximum absolute value of the graph on the right is nearly  $10^{-3}$  times that for the graph on the left.

difficult to check using the Bernstein inequality (cf. [17, Lemma 3.1]) that the measure  $\nu_N$  that associates the mass  $2\pi/N$  with each of the points  $\theta_k$  satisfies for all trigonometric polynomials  $T$  of order at most  $n$ :

$$\left| \|T\|_{\nu_N;1,[-\pi,\pi]} - \|T\|_{\mu^*;1,[-\pi,\pi]} \right| \leq \frac{2\pi n}{N} \|T\|_{\mu^*;1,[-\pi,\pi]} . \quad (3.13)$$

In [4], we have studied the construction of quasi-interpolatory operators based on M–Z measures satisfying an inequality analogous to (3.13). The measures are constructed using scattered data on  $[0, \pi]$ , extended symmetrically to  $[-\pi, \pi]$ , but are not necessarily quadrature measures. Instead, we use orthogonal polynomials with respect to the measures projected to  $[-1, 1]$  in the standard way, and prove a perturbation theorem to estimate the norms of the resulting quasi-interpolatory operators. The perturbation theorem is proved in a very general setting. We apply our operators for approximation of functions on the sphere using scattered, tensor-product data.

Finally, we note that quasi-interpolatory operators have been used to prove theorems about approximation by neural networks [9], zonal function networks [14], detection of singularities ([18] and references therein), solution of pseudo-differential equations on the sphere [5], and representation of smooth functions on the sphere using finitely many bits [10].

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