On Marcinkiewicz-Zygmund-Type Inequalities

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January 1997

Abstract

We investigate the relationships between the Marcinkiewicz-Zygmund-type inequalities and certain shifted average operators. Applications to the mean boundedness of a quasi-interpolatory operator in the case of trigonometric polynomials, Jacobi polynomials, and Freud polynomials are presented.

1 Introduction

The Marcinkiewicz-Zygmund inequalities assert the following ([26], Theorem X.7.5). If $n \geq 1$ is an integer, $1 < p < \infty$, and S is a trigonometric polynomial of order at most n, then

$$\int_{-\pi}^{\pi} |S(t)|^p dt \leq \frac{c_1}{2n+1} \sum_{j=1}^{2n+1} \left| S\left(\frac{2\pi j}{2n+1}\right)\right|^p \leq c_2 \int_{-\pi}^{\pi} |S(t)|^p dt, \qquad (1)$$

where c_1 and c_2 are positive constants depending only on p, but not on S or n. An important application of these inequalities is to deduce the mean boundedness of the operator of trigonometric polynomial interpolation from that of the partial sum operator of the trigonometric Fourier series ([26],

*This research was supported, in part, by National Science Foundation Grant DMS 9404513, the Air Force Office of Scientific Research, and the Alexander von Humboldt Foundation.

Theorem X.7.14). There is a very close connection between the existence of such inequalities, the boundedness of interpolatory operators, and that of the partial sum operators (cf. [3], [4].) Similar inequalities have been studied by many authors in connection with orthogonal polynomial expansions and Lagrange interpolation at the zeros of orthogonal polynomials, both in the case of generalized Jacobi weights on $[-1,1]$ and the Freud-type and more general weights on the whole real axis ([10], [24], [25]). In the context of spline spaces, analogous questions have been studied in [18]. The mean convergence of interpolatory processes have been studied also by Varma and his collaborators. For example, Erdös-Feldheim-type results are obtained in [22] and [23].

This theory is usually not applicable in the "extreme cases"; i.e., in the case when one is interested in either the uniform convergence or convergence in the L^1 norm. It is well known that Lagrange interpolation polynomials, and in general also the Fourier operators are not convergent either uniformly or in the L^1 sense. In the case of trigonometric series, a useful substitute for the partial sums of the Fourier series is the shifted arithmetic mean $(1/n)\sum_{k=n+1}^{2n} s_k$ where s_k is the operator of taking the k-th partial sum of the Fourier expansion. These operators are linear, and provide a "near best" order of approximation both in the uniform and L^1 sense. They have many other interesting properties as well (cf. [8]). These operators have been studied also in more general situations ([14] and references therein). In the context of interpolation at the zeros of generalized Jacobi and Freud-type weight functions, we have studied certain quasi-interpolatory operators in [15].

In this paper, our main purpose is to investigate the connections between these operators and inequalities similar to (1). We formulate a few abstract results in the next section. These abstract results will be illustrated in the context of trigonometric polynomials in Section 3, generalized Jacobi polynomials in Section 4, and Freud polynomials in Section 5.

2 Abstract results

Let μ be a (positive) measure on a measure space Σ. For a μ-measurable function f, we write

$$\|f\|_{p,\mu} := \begin{cases} \left(\int |f|^p d\mu\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \mu - \text{ess sup}\,|f|, & \text{if } p = \infty. \end{cases}$$

The class L^p_μ consists of all functions f for which $\|f\|_{p,\mu} < \infty$, where two functions are considered equal if they are equal μ-almost everywhere.

We may interprete the inequalities (1) as the statement

$$\|S\|_{p,\mu} \le c_1 \|S\|_{p,\nu} \le c_2 \|S\|_{p,\mu},$$

where μ is the Lebesgue measure and ν is a discrete measure. The partial sum operator for the Fourier series and the Lagrange interpolation operator at the jump points of ν can be expressed in the form $\int f(t)D_n(\cdot - t)d\mu(t)$ and $\int f(t)D_n(\cdot - t)d\nu(t)$ respectively, where D_n is the Dirichlet kernel. Similar expressions hold also for orthogonal expansions and Lagrange interpolation polynomials at the zeros of orthogonal polynomials.

Motivated by these examples, we consider two σ-finite measures μ and ν on a measure space Σ, and corresponding operators (when defined)

$$T(\tau; f, x) := \int f(t)K(x,t)d\tau(t), \qquad \tau = \mu, \nu,$$

where K is a symmetric function. We assume that K is essentially bounded as well as integrable with respect to all the product measures $\mu \times \mu$, $\nu \times \nu$, and $\mu \times \nu$. We also consider two weight functions, w and W, which are both measurable and positive almost everywhere with respect to both μ and ν. We will adopt the convention that the symbols $c, c_1, c_2, \ldots$ will denote positive constants depending only on μ and other fixed constants of the problem under consideration, such as the norms and other explicitly indicated quantities. Their values may be different at different occurrences, even within a single formula. In this context, ν is not considered to be a fixed parameter. We are intentionally vague regarding the weight functions and the norms. In applications, it will be clearly specified if the constants depend upon these paramters or not.

Let $1 \le p \le \infty$. By condition $I(p)$, we denote the statement

$$\|w(wW)^{-1/p} T(\nu; f)\|_{p,\mu} \le c \|W(wW)^{-1/p} f\|_{p,\nu}, \quad f \in L^1_\nu.$$

By condition $O(p)$, we denote the statement

$$\|w(wW)^{-1/p} T(\mu; f)\|_{p,\nu} \le c \|W(wW)^{-1/p} f\|_{p,\mu}, \quad f \in L^1_\mu,$$

where we note that the constants in both the above inequalities depend upon μ, and not upon ν. We are also interested in the following variation of the condition $O(p)$, to be denoted by $C(p)$:

$$\|w(wW)^{-1/p} T(\mu; f)\|_{p,\mu} \le c \|W(wW)^{-1/p} f\|_{p,\mu}, \quad f \in L^1_\mu.$$

In typical applications, ν will be a discrete measure, and μ will be a continuous measure. The condition $O(p)$ then appears to be weaker than both $C(p)$ and $I(p)$, as we are estimating only a few values of the continuous operator in terms of the integrals over the whole domain of μ. Nevertheless, the following theorem shows a close connection between these conditions.

Theorem 1 *Let μ, ν be σ-finite measures on a measure space Σ, $1 \le p \le q \le r \le \infty$, and p' be the conjugate index for p. Further suppose that w and W^{-1} are in both L^1_μ and L^1_ν.*
(a) *Let $f, g : \Sigma \to I\!R$, and $f(x)g(t)K(x,t)$ be integrable with respect to the product measures $\mu(t) \times \nu(x)$. Then the following* reciprocity law *holds.*

$$\int fT(\mu; g)d\nu = \int T(\nu; f)gd\mu.$$

(b) *The conditions $I(p)$ and $O(p')$ are equivalent. The conditions $C(p)$ and $C(p')$ are equivalent.*
(c) *The conditions $I(p)$ and $I(r)$ together imply $I(q)$. The conditions $O(p)$ and $O(r)$ together imply $O(q)$. The conditions $C(p)$ and $C(r)$ together imply $C(q)$.*

Proof. By Fubini's theorem and the fact that $K(x,t) = K(t,x)$, we have

$$\begin{aligned} \int f(x)T(\mu; g, x)d\nu(x) &= \int f(x) \int g(t)K(x,t)d\mu(t)d\nu(x) \\ &= \int g(t) \int f(x)K(x,t)d\nu(x)d\mu(t) \\ &= \int g(t)T(\nu; f, t)d\mu(t). \end{aligned}$$

This proves part (a).

To verify part (b), we first prove that $O(p')$ implies $I(p)$. Let $g : \Sigma \to I\!R$ be an arbitrary, μ-measurable, simple function such that $\|g\|_{p',\mu} \le 1$. An application of Hölder's inequality,

$$\|w(wW)^{-1/p}\|_{1,\mu} \le \|w^{1/p'}\|_{p',\mu}\|W^{-1/p}\|_{p,\mu} = \|w\|_{1,\mu}^{1/p'}\|W^{-1}\|_{1,\mu}^{1/p},$$

shows that $w(wW)^{-1/p} \in L^1_\mu$. Since g is μ-essentially bounded, the function $T(\mu; w(wW)^{-1/p}g)$ is well defined, and the conditions of part (a) are also satisfied. Then using the reciprocity law and the facts that $W^{-1}(wW)^{1/p} = w(wW)^{-1/p'}$, $O(p')$, and $w(wW)^{-1/p} = W^{-1}(wW)^{1/p'}$, we obtain

$$\begin{aligned} \left|\int w(wW)^{-1/p}T(\nu; f)gd\mu\right| &= \left|\int fT(\mu; w(wW)^{-1/p}g)d\nu\right| \\ &\le \|W(wW)^{-1/p}f\|_{p,\nu}\|w(wW)^{-1/p'}T(\mu; w(wW)^{-1/p}g)\|_{p',\nu} \\ &\le c\|W(wW)^{-1/p}f\|_{p,\nu}\|W(wW)^{-1/p'}w(wW)^{-1/p}g\|_{p',\mu} \\ &\le c\|W(wW)^{-1/p}f\|_{p,\nu}. \end{aligned}$$

Since

$$\|w(wW)^{-1/p}T(\nu;f)\|_{p,\mu} = \sup\left|\int w(wW)^{-1/p}T(\nu;f)g\,d\mu\right|,$$

where the supremum is taken over all μ-measurable simple functions g with $\|g\|_{p',\mu} \le 1$, we have proved $I(p)$. Similarly, the condition $I(p)$ implies $O(p')$. Finally, we observe that when $\mu = \nu$, the conditions $I(p)$, $C(p)$ and $O(p)$ are the same. This completes the proof of part (b).

If $I(p)$ and $I(r)$ both hold, then the Stein-Weiss interpolation theorem ([2], Corollary 5.5.2) implies $I(q)$. The other assertions are proved in the same way.

In order to relate Theorem 1 with the Marcinkiewicz-Zygmund-type (M-Z) inequalities, we restrict ourselves to the case when $w = W$. We consider an increasing sequence of sets $\{\Pi_k\}$, $\Pi_k \subseteq \Pi_{k+1}$, $k = 0, 1, \ldots,$ which may be thought of as subsets of each of the spaces $L^1_\mu \cap L^\infty_\mu$ and $L^1_\nu \cap L^\infty_\nu$. We next consider a sequence of symmetric kernel functions K_k, and operators T_k defined (if possible) by

$$T_k(\tau;f,x) := \int f(t)K_k(x,t)d\tau(t), \qquad \tau = \mu,\nu, \quad k = 1,2,\ldots.$$

In applications, the measure ν will also depend upon n, but we do not need this fact in the current abstract formulation of our results. We assume that there exist integers $a \ge 1$ and b such that

$$T_k(\tau;f) \in \Pi_{ak+b}, \quad \tau = \mu,\nu, \quad k = 1,2,\ldots.$$

Sometimes, we also assume the following *reproducing property,* referred to as R_n:

$$T_k(\tau;P) = P, \quad P \in \Pi_k, \quad \tau = \mu,\nu, \quad k = 1,2,\ldots,n.$$

In applications, we will choose ν, depending upon n, so that the condition R_n will be satisfied. The condition $I_n(p)$ (respectively $O_n(p)$, $C_n(p)$) denotes the fact that each of the operators T_k, $1 \le k \le n$, satisfies the condition $I(p)$ (respectively $O(p)$, $C(p)$). Clearly, the conditions $O_n(p)$ (with $w = W$) and the reproducing property R_n imply the "simpler" M-Z inequality

$$\|w^{(p-2)/p}P\|_{p,\nu} \le c\|w^{(p-2)/p}P\|_{p,\mu}, \qquad P \in \Pi_n\ .$$

This condition will be referred to as $SMZ_n(p)$. By the full M-Z inequality, to be denoted by $MZ_n(p)$, we mean the estimates

$$\|w^{(p-2)/p}P\|_{p,\mu} \le c_1\|w^{(p-2)/p}P\|_{p,\nu} \le c_2\|w^{(p-2)/p}P\|_{p,\mu}, \qquad P \in \Pi_n\ .$$

The following theorem gives several connections between the various conditions defined above.

Theorem 2 *Let μ, ν be as in Theorem 1, $n \ge 1$ be an integer, $w = W$, and w, w^{-1} be in both L^1_μ and L^1_ν.*
(a) *Let R_n hold and let $1 \le p \le 2 \le r \le \infty$. The conditions $O_n(p)$, $O_n(r)$ (or $I_n(p)$, $I_n(r)$) together imply the full M-Z inequalities $MZ_n(q)$ for each q with $\max(p, r') \le q \le \min(p', r)$ where p' be the conjugate index of p.*
(b) *Let $1 \le p \le \infty$, and $SMZ_{an+b}(p')$ hold. The condition $C_n(p)$ implies $I_n(p)$.*
(c) *Let p, p' be as above, and $MZ_{an+b}(p')$ hold. The condition $I_n(p)$ implies $C_n(p)$.*

Proof. In view of Theorem 1(c), the conditions $O_n(p)$ and $O_n(r)$ together imply each of the conditions $O_n(s)$, $p \le s \le r$. Theorem 1(b) then yields the conditions $I_n(s')$, $r' \le s' \le p'$. Thus, each of the conditions $O_n(q)$ and $I_n(q)$ hold for $\max(p, r') \le q \le \min(p', r)$. The part (a) then follows easily from the reproduction property R_n.

Next, we prove part (b). Let $1 \le k \le n$ be an integer, and $C_n(p)$ hold. Theorem 1(b) then implies $C_n(p')$. Let $w_{p'} := w^{(p'-2)/p'}$. Since $T_k(\mu; f) \in \Pi_{an+b}$ for all f for which it is defined, the conditions $SMZ_{an+b}(p')$ and $C_n(p')$ together imply

$$\|w_{p'} T_k(\mu; f)\|_{p',\nu} \le c\|w_{p'} T_k(\mu; f)\|_{p',\mu} \le c\|w_{p'} f\|_{p',\mu},$$

which is the condition $O_n(p')$. In turn, this implies $I_n(p)$.

Finally, we prove part (c). The condition $I_n(p)$ implies $O_n(p')$. Together with $MZ_{an+b}(p')$, this gives for integer $1 \le k \le n$

$$\|w_{p'} T_k(\mu; f)\|_{p',\mu} \le c\|w_{p'} T_k(\mu; f)\|_{p',\nu} \le c\|w_{p'} f\|_{p',\mu}.$$

Thus, $C_n(p')$ holds, and therefore, $C_n(p)$ holds.

3 Trigonometric approximation.

In this section, let μ be the Lebesgue measure on $[-\pi, \pi]$, normalized to be 1. All functions will be assumed to be 2π-periodic. If $f \in L^1_\mu$, and $n \ge 1$ is an integer, the n-th partial sum of its Fourier series is given by

$$s_n^*(f, x) = \int_{-\pi}^{\pi} f(t) D_n(x - t) d\mu(t),$$

where the *Dirichlet kernel* is defined by

$$D_n(t) := \frac{\sin((n + 1/2)t)}{\sin(t/2)}.$$

Let $\ell \geq 0$ be an integer. The *de la Vallée Poussin* operators are defined by

$$P^*_{n,\ell}(f,x) := \int_{-\pi}^{\pi} f(t) V^*_{n,\ell}(x,t) d\mu(t),$$

where the kernel function $V^*_{n,\ell}$ is defined by

$$V^*_{n,\ell}(x,t) := \frac{1}{\ell+1} \sum_{k=n}^{n+\ell} D_k(x-t).$$

Let $m \geq 1$ be an integer, $-\pi < x_{1,m} < \cdots < x_{2m+1,m} \leq \pi$ be any distinct points, and ν_m be the measure that associates the mass $1/(2m+1)$ with each of these points. As an application of our results in Section 2, we study the convergence behavior of the discretized de la Vallée Poussin operators, defined by

$$L^*_{n,\ell,m}(f,x) = \int_{-\pi}^{\pi} f(t) V^*_{n,\ell}(x,t) d\nu_m(t).$$

The class of all trigonometric polynomials of order at most n will be denoted by $I\!H_n$. In the notation of Section 2, $\Pi_n = I\!H_n$, $K_n = V^*_{n,\ell}$, $T_n(\mu; f) = P^*_{n,\ell}(f)$, $T_n(\nu_m; f) = L^*_{n,\ell,m}(f)$, $a = 1$, $b = \ell$, and $w = W = 1$. For $1 \leq p < \infty$, let $\mathcal{R}_p$ denote the class of all 2π-periodic, Riemann integrable functions, and let $\mathcal{R}_\infty$ denote the class of all continuous 2π-periodic functions. We wish to estimate $\|f - L^*_{n,\ell,m}(f)\|_{p,\mu}$ for all $f \in \mathcal{R}_p$. This estimate will be given in terms of the degree of best one-sided trigonometric approximation, defined for $f \in \mathcal{R}_p$ by

$$\tilde{E}_{n,p}(f) := \inf\{\|Q - R\|_{p,\mu}\}$$

where the infimum is over all $Q, R \in I\!H_n$ for which $Q(x) \leq f(x) \leq R(x)$ for all $x \in [-\pi, \pi]$. There are several known Jackson-type estimates for $\tilde{E}_{n,p}(f)$ (cf. [19]). These imply that $\tilde{E}_{n,p}(f) \to 0$ as $n \to \infty$ for every $f \in \mathcal{R}_p$.

Theorem 3 *Let $n \leq \ell \leq \kappa n$, $m \geq n + \ell$, and*

$$\min_{1 \leq k \leq 2m+1} |x_{k+1,m} - x_{k,m}| \geq c/m, \tag{2}$$

where $x_{2m+2,m} := x_{1,m}$. Then for any $f \in \mathcal{R}_p$,

$$\|L^*_{n,\ell,m}(f)\|_{p,\mu} \leq \begin{cases} c(\kappa)\|f\|_{p,\nu_m}, & \text{if } \kappa > 1,\ 1 \leq p \leq \infty, \\ c(p)\|f\|_{p,\nu_m}, & \text{if } 1 < p < \infty. \end{cases} \tag{3}$$

Further, suppose that the nodes $\{x_{k,m}\}$ are chosen so that $L^*_{n,\ell,m}(T) = T$ for every $T \in I\!H_n$. If $f \in \mathcal{R}_p$, we have

$$\|f - L^*_{n,\ell,m}(f)\|_{p,\mu} \leq \begin{cases} c(\kappa)\tilde{E}_{n,p}(f), & \text{if } \kappa > 1,\ 1 \leq p \leq \infty, \\ c(p)\tilde{E}_{n,p}(f), & \text{if } 1 < p < \infty. \end{cases}$$

Proof. It is known [10] that the condition (2) implies $SMZ_{n+\ell}(p)$ for every p, $1 \le p \le \infty$. It is also known ([8], [21]) that for any $f \in L^p_\mu$,

$$\|P^*_{n,\ell}(f)\|_{p,\mu} \le \begin{cases} c(\kappa)\|f\|_{p,\mu}, & \text{if } \kappa > 1,\ 1 \le p \le \infty, \\ c(p)\|f\|_{p,\mu}, & \text{if } 1 < p < \infty. \end{cases} \tag{4}$$

Thus, the condition $C_n(p)$ is satisfied for all p, $1 < p < \infty$, and even for $p = 1, \infty$ if $\kappa > 1$. Since $SMZ_{n+\ell}(p')$ holds as well, Theorem 2(b) implies $I_n(p)$ for the same values of p. This is (3).

Next, let $1 < p < \infty$, $f \in \mathcal{R}_p$, $Q, R \in I\!H_n$ be chosen so that $Q(x) \le f(x) \le R(x)$ for all $x \in [-\pi, \pi]$ and $\|Q - R\|_{p,\mu} \le 2\tilde{E}_{n,p}(f)$. Using the reproduction property, the bounds (3), and the condition $SMZ_{n+\ell}(p)$, we deduce that

$$\begin{aligned} \|f - L^*_{n,\ell,m}(f)\|_{p,\mu} &= \|f - Q - L^*_{n,\ell,m}(f - Q)\|_{p,\mu} \\ &\le \|f - Q\|_{p,\mu} + c(p)\|f - Q\|_{p,\nu_m} \\ &\le \|R - Q\|_{p,\mu} + c(p)\|R - Q\|_{p,\nu_m} \\ &\le c_1(p)\|R - Q\|_{p,\mu} \le c_2(p)\tilde{E}_{n,p}(f). \end{aligned}$$

The remaining assertion is proved in the same way.

We end this section with a few remarks comparing the quantity $\tilde{E}_{n,p}(f)$ with the degree of best approximation of f, defined by

$$E_{n,p}(f) := \inf_{Q \in I\!H_n} \|Q - f\|_{p,\mu}.$$

In view of (4), one easily concludes that

$$\|f - P^*_{n,\ell}(f)\|_{p,\mu} \le cE_{n,p}(f).$$

The integral operator $P^*_{n,\ell}$ neglects the behavior of its argument f on a set of measure zero, whereas the estimates on the discrete operator $L^*_{n,\ell,m}$ depend upon countably many points through the best **one-sided** approximation (cf. also [6]). However, if f is of bounded variation on $[-\pi, \pi]$, and possesses at least one jump, then it is known [19] that

$$cn^{-1/p} \le E_{n,p}(f) \le \tilde{E}_{n,p}(f) \le c_1 n^{-1/p}.$$

Our results imply in this case that

$$cn^{-1/p} \le \|f - L^*_{n,\ell,m}(f)\|_{p,\mu} \le cn^{-1/p}.$$

If $f \notin \mathcal{R}_p$, then the sequence $L^*_{n,\ell,m}(f)$, even if well defined, may not converge.

4 Generalized Jacobi polynomials

A *generalized Jacobi weight* is a function of the form

$$w(x) := \begin{cases} \prod_{k=1}^{\rho}(x-\xi_k)^{\beta_k}, & x \in [-1,1], \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $\rho \geq 2$ is an integer, $-1 =: \xi_\rho < \cdots < \xi_1 := 1$, and $\beta_k > -1$ for $k = 1, \ldots, \rho$. The class of generalized Jacobi weights will be denoted by GJ. In this and the next section, $\mathcal{P}_n$ denotes the class of all polynomials of degree at most n. Associated with a weight $w \in GJ$, there is a system of orthonormalized polynomials $\{p_k \in \mathcal{P}_k\}$, called generalized Jacobi polynomials. For each integer $m \geq 1$, the polynomial p_m has m distinct zeros $\{x_{k,m}\}$ on the interval $[-1,1]$. Further, there exist positive numbers $\lambda_{k,m}$ such that

$$\int_{-1}^{1} P(t)w(t)dt = \sum_{k=1}^{m} \lambda_{k,m} P(x_{k,m}), \qquad P \in \mathcal{P}_{2m-1}.$$

Generalized Jacobi polynomials have been studied extensively by Nevai [17]. Following [17], if $w \in GJ$ is of the form (5), and $m \geq 1$ is an integer, we write for $x \in [-1,1]$,

$$\overline{w}_m(x) := \left(\sqrt{1-x}+\frac{1}{m}\right)^{2\beta_1+1} \prod_{k=2}^{\rho-1}\left(|x-\xi_k|+\frac{1}{m}\right)^{\beta_k}\left(\sqrt{1+x}+\frac{1}{m}\right)^{2\beta_\rho+1}.$$

If w is the Legendre weight, i.e., $\beta_k = 0$, $k = 1, \ldots, \rho$, then it is easy to see that

$$c_1\overline{w}_m(x) \leq \Delta_m(x) := \sqrt{1-x^2} + \frac{1}{m} \leq c_2\overline{w}_m(x).$$

In the rest of this section, the constants $c, c_1, \ldots$ may depend upon w.

We let μ the measure $w(t)dt$, and ν_m be the measure that associates the mass $\lambda_{k,m}$ with the zero $x_{k,m}$. The de la Vallée Poussin kernel is defined by

$$V_n(x,t) := \sum_{k=0}^{n} p_k(x)p_k(t) + \sum_{k=n+1}^{2n-1}\left(2-\frac{k}{n}\right)p_k(x)p_k(t).$$

The de la Vallée Poussin-type operator is defined by

$$v_n(f,x) := \int_{-1}^{1} f(t)V_n(x,t)w(t)dt,$$

and its discrete analogue is defined for $f : [-1,1] \to I\!R$ by

$$\tau_{n,m}(f,x) := \int_{-1}^{1} f(t) V_n(x,t) d\nu_m(t).$$

If $m \geq 2n$, the operators satisfy $\tau_{n,m}(P) = v_n(P) = P$ for every $P \in \mathcal{P}_n$.

In the language of Section 2, $T(\mu; f) = v_n(f)$, $T(\nu_m; f) = \tau_{n,m}(f)$, $a = 2$, $b = -1$. We let $\gamma \in (-1,1)$, and take $g_n := \Delta^{\gamma}_{\sqrt{n}} \sqrt{w_n}$ in place of the weight w of Section 2. In place of the weight W of Section 2, we take $G(x) := (1-x^2)^{\gamma/2}\sqrt{w(x)}$, where $w \in GJ$ is the fixed weight function considered in this section. It is easy to verify that g_n and G^{-1} are both in L^1_μ. In [15], we have proved that if $2n \leq m \leq Ln$ for some constant L, then

$$\begin{aligned} \|g_n v_n(f)\|_{\infty,\mu} &\leq c\|Gf\|_{\infty,\mu}, \\ \|g_n \tau_{n,m}(f)\|_{\infty,\mu} &\leq c\|Gf\|_{\infty,\nu_m}, \end{aligned}$$

where the constants may depend upon w, γ, and L only. Thus, we have proved $C(\infty)$ (and hence, $O(\infty)$) and $I(\infty)$ in the language of Section 2. Since the weight functions are not the same on both sides of these inequalities, we may not apply Theorem 2 directly in this context. We may apply Theorem 1. Together with the reproducing property R_n, this gives the following theorem.

Theorem 4 *Let $w \in GJ$, $-1 < \gamma < 1$, $1 \leq p \leq \infty$, $L \geq 2$, $n \geq 1$, $2n \leq m \leq Ln$ be integers. Let*

$$g_n := \Delta^{\gamma}_{\sqrt{n}} \sqrt{w_n}, \qquad G(x) := (1-x^2)^{\gamma/2}\sqrt{w(x)}.$$

Then

$$\begin{aligned} \|g_n (g_n G)^{-1/p} v_n(f)\|_{p,\mu} &\leq c\|G(g_n G)^{-1/p} f\|_{p,\mu}, \\ \|g_n (g_n G)^{-1/p} \tau_{n,m}(f)\|_{p,\mu} &\leq c\|G(g_n G)^{-1/p} f\|_{p,\nu_m}, \\ \|g_n (g_n G)^{-1/p} v_n(f)\|_{p,\nu_m} &\leq c\|G(g_n G)^{-1/p} f\|_{p,\mu}. \end{aligned}$$

For any polynomial $P \in \mathcal{P}_n$,

$$\|g_n (g_n G)^{-1/p} P\|_{p,\nu_m} \leq c\|G(g_n G)^{-1/p} P\|_{p,\mu}\ , \qquad (6)$$

$$\|g_n (g_n G)^{-1/p} P\|_{p,\mu} \leq c\|G(g_n G)^{-1/p} P\|_{p,\nu_m}\ . \qquad (7)$$

In the same manner as described for the trigonometric polynomial case, the approximation error for $v_n(f)$ and $\tau_{n,m}(f)$ could be bounded by a weighted best one-sided approximation. The convergence order of such weighted best one-sided approximation has been investigated in e.g. [5], [16], [20], [13].

In (6), (7), the number of nodes at which the polynomials are evaluated is greater than the degree of the polynomials, and we do not have a complete equivalence required by the condition $MZ_n(p)$. The conditions $SMZ_n(p)$ are proved in [17] and [10] with different discrete measures ν_n, where the number of nodes is less than the degree of the polynomials involved. In light of the results of Y. Xu [24] regarding the mean boundedness of the Fourier sums and of Kallaev [7] regarding the uniform boundedness of the de la Vallée Poussin operators for certain ultraspherical polynomials, results analogous to Theorem 3 can also be obtained. However, since this does not add any new insight to the problem, we do not present the details. In conclusion, we note that Y. Xu [24], [25] has studied the conditions $MZ_n(p)$, again with different discrete measures. These results require more stringent conditions on the weights involved than those in Theorem 4. Moreover, these do not apply in the extreme cases $p = 1, \infty$.

5 Freud-type weight functions

Let $w : I\!\!R \to (0, \infty)$, and $Q := \log(1/w)$. The function w is called a Freud-type weight function if each of the following conditions is satisfied. The function Q is an even, convex function on $I\!\!R$, Q is twice continuously differentiable on $(0, \infty)$ and there are constants c_1 and c_2 such that

$$0 < c_1 \le \frac{xQ''(x)}{Q'(x)} \le c_2 < \infty, \qquad 0 < x < \infty \ .$$

The most commonly discussed examples include $\exp(-|x|^\alpha)$, $\alpha > 1$. Associated with the Freud-type weight function w, there is a system of numbers $\{a_x\}$, called MRS-numbers, defined for $x > 0$ by the equations

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x t Q'(a_x t)}{\sqrt{1-t^2}} dt \ .$$

One of the most important properties of a_x is the following: For every integer $n \ge 1$ and $P \in \mathcal{P}_n$,

$$\max_{x \in I\!\!R} |P(x)w(x)| = \max_{|x| \le a_n} |P(x)w(x)| \ .$$

For a detailed discussion of approximation theory involving these weight functions, we refer the reader to [14].

In the remainder of this section, w will denote a fixed Freud-type weight function, the measure μ will be the measure $w^2(x)dx$ on $I\!\!R$, $\{p_k\}$ will denote the sequence of polynomials orthonormal on $I\!\!R$ with respect to the measure

μ. For each integer $n \geq 1$, the polynomial p_n has n real and simple zeros, $\{x_{k,n}\}$. We have the following quadrature formula for all $P \in \mathcal{P}_{2n-1}$:

$$\int_{I\!R} P(t)d\mu(t) = \sum_{k=1}^{n} \lambda_{k,n} P(x_{k,n}),$$

where the *Cotes' numbers* $\lambda_{k,n}$ are all positive. The measure ν_n associates the mass $\lambda_{k,n}$ with the point $x_{k,n}$. In the remainder of this section, all constants $c, c_1, \ldots$ will depend upon w.

The de la Vallée Poussin kernel for the Freud-type weight function is defined by

$$V_n(x,t) := \sum_{k=0}^{n} p_k(x)p_k(t) + \sum_{k=n+1}^{2n-1} \left(2 - \frac{k}{n}\right) p_k(x)p_k(t).$$

If $1 \leq p \leq \infty$, the de la Vallée Poussin operator is defined for $w^{(p-2)/p} f \in L^p_\mu$ by

$$v_n(f,x) := \int_{I\!R} f(t)V_n(x,t)d\mu(t),$$

and its discretized version is defined for all $f : I\!R \to I\!R$ by

$$\tau_{n,m}(f,x) := \int_{I\!R} f(t)V_n(x,t)d\nu_m(t).$$

In the notation of Section 2, $\Pi_n = \mathcal{P}_n$, $K_n = V_n$, $T_n(\mu; f) = v_n(f)$, $T_n(\nu_m; f) = \tau_{n,m}(f)$, the weights w and W are both equal to the fixed Freud-type weight function w, $a = 2$, and $b = -1$. It is not difficult to verify that both w and w^{-1} are in L^1_μ. It is known [14] that the conditions $C_n(p)$ are satisfied for all p, $1 \leq p \leq \infty$. Moreover, if $m \geq 2n$ then the reproducing property R_n also holds.

Theorem 5 *Let $1 \leq p \leq \infty$. Let w be a Freud-type weight function, and $Q := \log(1/w)$ satisfy the following Lipschitz condition:*

$$|Q'(a_y \cos\theta) - Q'(a_y \cos\phi)| \leq c\frac{y}{a_y}|\theta - \phi|^\lambda, \qquad y > 0,$$

where c and λ are positive constants independent of y. Let $L, \delta > 0$, $n \geq 1$ be an integer, and $(2+\delta)n \leq m \leq Ln$. Then for each $P \in \mathcal{P}_n$,

$$\|w^{(p-2)/p}P\|_{p,\mu} \leq c_1\|w^{(p-2)/p}P\|_{p,\nu_m} \leq c_2\|w^{(p-2)/p}P\|_{p,\mu}. \tag{8}$$

Further, for any $f : I\!R \to I\!R$,

$$\|w^{(p-2)/p}\tau_{n,m}(f)\|_{p,\mu} \leq c\|w^{(p-2)/p}f\|_{p,\nu_m}. \tag{9}$$

Proof. It is proved in [15] that (9) is satisfied if $p = \infty$. In the language of Section 2, this means that the condition $I_n(\infty)$ is satisfied. Clearly, the condition $C_n(\infty)$, which is known to hold [14], implies the condition $O_n(\infty)$. Theorem 1 then implies that the condition $I_n(1)$ holds as well, and hence, all the conditions $I_n(p)$ are satisfied; i.e., (9) holds. Since the reproduction property R_n also holds, Theorem 2 implies all the conditions $MZ_n(p)$; i.e., the estimates (8).

The conditions $MZ_n(p)$ in this context have been studied by Lubinsky and Matjila [9], [11] with a discrete measure depending upon less nodes than the degree of the polynomials involved. Again, they do not hold for $p = 1, \infty$. Lubinsky and Matjila [12] have also studied the mean convergence of Lagrange interpolation operators for the Freud weights. Typically, these results use very deep techniques. One sided approximation by polynomials is also studied in this context by Nevai (see [16], [14] for further references). In particular, we may obtain analogues of Theorem 3 in this case as well, but we do not find this worth pursuing at this time.

References

[1] V. M. Badkov, Convergence in the mean and almost everywhere of Fourier series in orthogonal polynomials, Mat. Sbornik 35 (1974), 229-262.

[2] J. Bergh and J. Löfström, “Interpolation spaces, an introduction”, Springer Verlag, Berlin, 1976.

[3] C. K. Chui, X. C. Shen and L. Zhong, On Lagrange interpolation at disturbed roots of unity, Trans. Amer. Math. Soc. 336 (1993), 817-830.

[4] C. K. Chui, and L. Zhong, Polynomial interpolation and Marcinkiewicz-Zygmund inequalities on the unit circle, CAT Report 365 (1995).

[5] G. Freud, Über einseitige Approximation durch Polynome, I, Acta Sci. Math. (Szeged) 6 (1955), 12-28.

[6] V. H. Hristov, Best onesided approximations and mean approximations by interpolating polynomials of periodic functions, Math.-Balkanica (N. S.) 3 no. 3-4 (1989), 418-429.

[7] S. O. Kallaev, De la Vallée Poussin means of Fourier-Gegenbauer series, Mat. Zametki 7 (1970), 19-30.

[8] G. G. Lorentz, "Approximation of Functions", Holt, Rinehart, and Winston, 1966.

[9] D. S. Lubinsky, Converse quadrature sum inequalities for polynomials with Freud weights, Acta. Sci. Math. Hung. 60 (1995), 527–557.

[10] D. S. Lubinsky, A. Máté and P. Nevai, Quadrature sums involving p^{th} powers of polynomials, SIAM J. Math. Anal. 18 (1987), 531-544.

[11] D. S. Lubinsky and D. Matjila, Full quadrature sums for pth powers of polynomials with Freud weights, J. Comput. Appl. Math. 60 (1995), 285–296.

[12] D. S. Lubinsky and D. Matjila, Necessary and sufficient conditions for mean convergence of Lagrange interpolation for Freud weights, SIAM J. Math. Anal. 26 (1995), 238–262.

[13] G. Mastroianni and P. Vértesi, Weighted L^p error of Lagrange interpolation, J. Approx. Theory 82 (1995), 321-339.

[14] H. N. Mhaskar, "An introduction to the theory of weighted polynomial approximation", World Scientific, Singapore, 1996.

[15] H. N. Mhaskar and J. Prestin, On bounded quasi-interpolatory polynomial operators, Submitted for publication.

[16] P. Nevai, Einseitige Approximation durch Polynome mit Anwendungen, Acta Math. Acad. Sci. Hungar. 23 (1972), 495-506.

[17] P. Nevai, "Orthogonal polynomials", Mem. Amer. Math. Soc. 18 no. 213 (1979).

[18] J. Prestin and E. Quak, On interpolation and best one-sided approximation by splines in L^p, in: "Approximation Theory," (Ed. G. Anastassiou), pp., 409-420, Lecture Notes in Pure and Applied Mathematics Vol. 138, 1992.

[19] B. Sendov and V. A. Popov, "The Averaged Moduli of Smoothness," Wiley, New York, 1988.

[20] M. Stojanova, The best onesided algebraic approximation in $L^p[-1,1]$ $(1 \leq p \leq \infty)$, Math. Balkanica (N.S.) 2 (1988), 101-113.

[21] A. F. Timan, "Theory of approximation of functions of a real variable", English translation Pergamon Press, 1963.

[22] A. K. Varma, P. Vértesi, Some Erdös-Feldheim type theorems on mean convergence of Lagrange interpolation, J. Math. Anal. Appl. 91 (1983), 68-79.

[23] A. K. Varma, J. Prasad, An analogue of a problem of P. Erdös and E. Feldheim on L_p convergence of interpolatory processes, J. Approx. Theory 56 (1989), 225-240.

[24] Y. Xu, Mean convergence of generalized Jacobi series and interpolating polynomials, I, J. Approx. Theory 72 (1993), 237-251.

[25] Y. Xu, Mean convergence of generalized Jacobi series and interpolating polynomials, II, J. Approx. Theory 76 (1994), 77-92.

[26] A. Zygmund, "Trigonometric Series", Cambridge University Press, Cambridge, 1977.