On the representation of smooth functions on the sphere using finitely many bits

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Abstract

We discuss the construction of a parsimonious representation of smooth functions on the Euclidean sphere using finitely many bits, in the sense of metric entropy. The smoothness of the functions is measured by Besov spaces. The bit representation is obtained by uniform quantization on the values of a polynomial operator at scattered sites on the sphere. For each cap, one can identify a certain number of bits, commensurable with the local smoothness of the target function on that cap and the volume of that cap, and obtained using the values of the polynomial operator near that cap. The polynomial operator is calculated using either spherical harmonic coefficients or, in the case of uniform approximation, values of the function at scattered sites on the sphere. The localization properties of the polynomial operator are demonstrated by a characterization of local smoothness of the target function near a point in terms of the values of these operators near the point in question.

1 Introduction

A typical problem in classical approximation theory is to represent a function using finitely many real parameters, using which, the function can be reconstructed with a prescribed accuracy. In 1959, Kolmogorov and Tikhomirov [11] studied the question of representing functions using finitely many binary digits. Possible application areas for such a representation include signal and image processing (for example, [1, 2, 4, 7]), probability theory (for example, [8, 27]), learning theorey (for example, [3]), and quantum computation (for example, [9, 22]).

A fundamental notion in this theory is that of metric entropy, which we now define in an abstract setting following [13]. Let K be a compact subset of a metric space (X, d). Given $\epsilon > 0$, let $N_\epsilon(K)$ be the minimal number of balls of radius ϵ that cover K, and

*The research of this author was supported, in part, by grant DMS-0204704 from the National Science Foundation and grant DAAD19-01-1-0001 from the U.S. Army Research Office.

$g_1, \cdots, g_{N_\epsilon(K)}$ be a list of the centers of these balls. We note that a binary enumeration of these centers takes $N_\epsilon(K)$ integers, each having at most $\log_2 N_\epsilon(K) + 1$ bits. Given any $f \in K$, we find g_j such that $d(f, g_j) \le \epsilon$. We may then represent f using the binary representation of j, and use g_j as the reconstruction of f based on this representation. Thus, the metric entropy of K, defined by $H_\epsilon(K, X) := \log_2 N_\epsilon(K)$, gives (within one bit) the minimal number of bits necessary to represent any $f \in K$. Several examples where the metric entropy has been computed (in an asymptotic sense) are described in [13], where additional references can be found.

In recent years, there is a renewed interest in encoding compact subsets of function spaces in a practical manner, for example, using sigma-delta modulation [5] or uniform quantization of certain wavelet coefficients ([7, 10] and references therein). In [15, 19], we have studied this question in the context of band-limited functions on the real line and on the Euclidean sphere.

In this paper, we present a construction of bit representations of smooth functions defined on a Euclidean sphere. The smoothness of the functions is defined by Besov spaces. As in [7], the number of bits needed to represent and recover a function from the unit ball of a Besov space with a given accuracy are within a logarithmic factor of the metric entropy of the function class. In general, our construction depends upon global information about the target function in the form of its spherical harmonic coefficients. Nevertheless, our bit representation is local in the following sense. From the total number of bits in the representation, one can identify for each cap, a certain number of bits that represent the function on that cap. The number of bits in this subset is proportional to the volume of that cap, and the accuracy of reconstruction commensurate with the local smoothness of the function on that cap. In the case of uniform approximation, we may obtain our representation using the values of the function at scattered sites on the sphere instead of spherical harmonic coefficients. In this case, in order to obtain the bits needed to represent the function on a cap, one needs to use the values of the function on a slightly larger cap. In this sense, our representation is completely localized. The number of input parameters as well as the number of parameters in the reconstruction are within a logarithmic factor of those required by the widths of the Besov spaces. Our approximation to the function, based on its bit representation, is a spherical polynomial.

A principle tool in our paper is a variant of the spherical polynomial frame operator introduced in [20]. The operator introduced in this paper provides a multiscale analysis of each of the L^p spaces on the sphere ($1 \le p < \infty$, respectively the space of continuous functions in the case $p = \infty$). The localization properties of our frame operators are demonstrated by characterizing local Besov spaces on the sphere using the values of this operator at points near the point in question. In the analysis of continuous functions, we may use the values of the function at scattered sites to construct our operators. Our multiscale analysis is "built up" in the following sense. In applications of classical wavelet analysis, one usually projects the data on a large scaling space, and then uses the decomposition algorithms to obtain the wavelet coefficients. In our case, we may construct higher and higher scale operators using more and more data instead of starting with a large amount of data. As in [7], the bit representation is obtained by a uniform quantization of the values of our approximation operator at scattered sites on the sphere.

In Section 2, we recall some facts about spherical polynomials and Besov spaces on the sphere. The main results of this paper are discussed in Section 3, and the proofs of the new results in Sections 2 and 3 are given in Section 4.

Theorem 3.3 below is motivated by oral reports on an ongoing work of Narcowich, Petrushev, and Ward on a characterization of (global) Besov spaces on the sphere using polynomial frames, in particular, a lecture by Petrushev on the subject in December, 2003. We gratefully acknowledge subsequent discussions with Narcowich, Prestin, Reimer, Ward, and Xu, among others.

2 Background information

In this section, we review certain facts about polynomials on the sphere and quadrature formulas, and describe the notion of Besov spaces on the sphere.

2.1 Spherical Harmonics

Let $q \geq 1$ be an integer which will be fixed throughout the rest of this paper, and let $\mathbb{S}^q$ be the (surface of the) unit sphere in the Euclidean space $\mathbb{R}^{q+1}$. If ν is any (possibly signed) measure on $\mathbb{S}^q$, its total variation measure will be denoted by $|\nu|$. If $1 \leq p \leq \infty$, $A \subseteq \mathbb{S}^q$ is ν-measurable, and $f : A \to \mathbb{R}$ is ν-measurable, we define the $L^p(\nu; A)$ norms of f by

$$\|f\|_{\nu;A,p} := \begin{cases} \{\int_A |f(\xi)|^p d|\nu|(\xi)\}^{1/p}, & \text{if } 1 \leq p < \infty, \\ |\nu| - \text{ess sup}_{\xi\in A} |f(\xi)|, & \text{if } p = \infty. \end{cases} \tag{2.1}$$

The space of all ν-measurable functions on A such that $\|f\|_{\nu;A,p} < \infty$ will be denoted by $L^p(\nu; A)$, with the usual convention that two functions are considered equal as elements of this space if they are equal $|\nu|$-almost everywhere. The space of all uniformly continuous, bounded functions on A will be denoted by $C(A)$, and the symbol $X^p(\nu; A)$ will denote $L^p(\nu; A)$ if $1 \leq p < \infty$ and $C(A)$ if $p = \infty$ (equipped with the norm of $L^\infty(\nu; A)$).

The volume (surface area) measure on $\mathbb{S}^q$ will be denoted by μ_q^*. We note that the volume element is invariant under arbitrary coordinate changes. The volume of $\mathbb{S}^q$ is

$$\omega_q := \int_{\mathbb{S}^q} d\mu_q^* = \frac{2\pi^{(q+1)/2}}{\Gamma((q+1)/2)}. \tag{2.2}$$

We will omit the mention of the measure from the notations if the measure is μ_q^*. Thus, for example, $\|f\|_{A,p}$ will mean $\|f\|_{\mu_q^*;A,p}$, $X^p(A)$ will mean $X^p(\mu_q^*; A)$, etc.

For a fixed integer $\ell \geq 0$, the restriction to $\mathbb{S}^q$ of a homogeneous harmonic polynomial of degree ℓ is called a spherical harmonic of degree ℓ. Most of the following information is based on [21] and [25, §IV.2], although we use a different notation. The class of all spherical harmonics of degree ℓ will be denoted by $\mathbf{H}_\ell^q$, and for any $x \geq 0$, the class of all spherical harmonics of degree $\ell \leq x$ will be denoted by Π_x^q. The spaces $\mathbf{H}_\ell^q$'s are mutually orthogonal relative to the inner product of $L^2(\mathbb{S}^q)$. Of course, for any integer $n \geq 0$, $\Pi_n^q = \bigoplus_{\ell=0}^n \mathbf{H}_\ell^q$, and it comprises the restriction to $\mathbb{S}^q$ of all algebraic polynomials in $q+1$

variables of total degree not exceeding n. The dimension of $\mathbf{H}_\ell^q$ is given by

$$d_\ell^{\,q} := \dim \mathbf{H}_\ell^q = \begin{cases} \dfrac{2\ell+q-1}{\ell+q-1}\dbinom{\ell+q-1}{\ell}, & \text{if } \ell \geq 1, \\ 1, & \text{if } \ell = 0. \end{cases} \tag{2.3}$$

and that of Π_n^q is $\sum_{\ell=0}^n d_\ell^{\,q}$. Furthermore, $L^2(\mathbb{S}^q) = L^2\text{-closure}\{\bigoplus_\ell \mathbf{H}_\ell^q\}$. Hence, if we choose an orthonormal basis $\{Y_{\ell,k} : k = 1,\cdots,d_\ell^q\}$ for each $\mathbf{H}_\ell^q$, then the set $\{Y_{\ell,k} : \ell = 0,1,\dots \text{ and } k = 1,\cdots,d_\ell^{\,q}\}$ is an orthonormal basis for $L^2(\mathbb{S}^q)$. One has the well-known addition formula [21]:

$$\sum_{k=1}^{d_\ell^{\,q}} Y_{\ell,k}(\mathbf{x})\overline{Y_{\ell,k}(\mathbf{y})} = \frac{d_\ell^{\,q}}{\omega_q}\mathcal{P}_\ell(q+1;\mathbf{x}\cdot\mathbf{y}), \qquad \ell = 0,1,\cdots, \tag{2.4}$$

where $\mathcal{P}_\ell(q+1;x)$ is the degree-ℓ Legendre polynomial in $q+1$-dimensions.

The Legendre polynomials are normalized so that $\mathcal{P}_\ell(q+1;1) = 1$, and satisfy the orthogonality relations [21, Lemma 10]

$$\int_{-1}^{1} \mathcal{P}_\ell(q+1;x)\mathcal{P}_k(q+1;x)(1-x^2)^{\frac{q}{2}-1}dx = \frac{\omega_q}{\omega_{q-1}d_\ell^{\,q}}\delta_{\ell,k}. \tag{2.5}$$

They are related to the ultraspherical (Gegenbauer) polynomials $P_\ell^{(\frac{q-1}{2})}$ (cf. [26], [21, p. 33]), and the Jacobi polynomials, $P_\ell^{(\alpha,\beta)}$, with $\alpha = \beta = \frac{q}{2} - 1$, via

$$P_\ell^{(\frac{q-1}{2})}(x) = \binom{\ell+q-2}{\ell}\mathcal{P}_\ell(q+1;x) \quad (q \geq 2), \tag{2.6}$$

$$P_\ell^{(\frac{q}{2}-1,\frac{q}{2}-1)}(x) = \binom{\ell+\frac{q}{2}-1}{\ell}\mathcal{P}_\ell(q+1;x). \tag{2.7}$$

When $q = 1$, the Legendre polynomials $\mathcal{P}_\ell(2;x)$ coincide with the Chebyshev polynomials $T_\ell(x)$; the ultraspherical polynomials $P_\ell^{(0)}(x) = (2/\ell)T_\ell(x)$, if $\ell \geq 1$. For $\ell = 0$, $P_0^{(0)}(x) = 1$.

2.2 Quadrature

In this subsection, we review certain facts about polynomials which will be utilized often in the sequel. In the remainder of this paper, we adopt the following convention regarding constants. The letters $c, c_1, \cdots$ will denote positive constants depending only on the dimension q, the different norms involved in the formula, and other explicitly mentioned quantities, if any. Their value will be different at different occurrences, even within the same formula. The expression $A \sim B$ will mean $cA \leq B \leq c_1 A$.

A (possibly signed) measure ν will be called an *M–Z (Marcinkiewicz–Zygmund) quadrature measure of order n* if each of the following conditions are satisfied.

$$\|P\|_{\nu;\mathbb{S}^q,p} \leq c\|P\|_{\mathbb{S}^q,p}, \qquad P \in \Pi_n^q,\ 1 \leq p \leq \infty, \tag{2.8}$$

and

$$\int_{\mathbb{S}^q} P(\mathbf{x})d\mu_q^*(\mathbf{x}) = \int_{\mathbb{S}^q} P(\mathbf{x})d\nu(\mathbf{x}), \qquad P\in\Pi_n^q. \tag{2.9}$$

Clearly, μ_q^* itself is an M–Z quadrature measure of order n for every integer n. The following theorem, proved in several parts in [18], shows the existence of positive M–Z quadrature measures supported at a finite set of scattered sites.

Theorem 2.1 *There exist constants α_q and N_q with the following property. Let $\mathcal{C}$ be a finite set of distinct points on $\mathbb{S}^q$,*

$$\delta_{\mathcal{C}} := \sup_{\mathbf{x}\in\mathbb{S}^q} \mathrm{dist}(\mathbf{x},\mathcal{C}), \tag{2.10}$$

and n be an integer with $N_q \le n \le \alpha_q\delta_{\mathcal{C}}^{-1}$. Then there exists a nonnegative M–Z quadrature measure $\nu_{\mathcal{C}}$ of order n, supported on a subset $\mathcal{C}_1$ of $\mathcal{C}$ with

$$|\mathcal{C}_1| \sim n^q \sim \dim(\Pi_n^q). \tag{2.11}$$

Further, $\delta_{\mathcal{C}} \le \delta_{\mathcal{C}_1} \le c\delta_{\mathcal{C}}$ and $\min_{\xi,\zeta\in\mathcal{C}_1,\xi\neq\zeta}\mathrm{dist}(\xi,\zeta) \ge c_1\delta_{\mathcal{C}}$.

2.3 Besov spaces

There are many ways to define Besov spaces on the sphere [24, 12]. We find it convenient to adopt the following definition, motivated by the equivalence theorem in [12, Theorem 3.2]. For any $x \ge 0$, $1\le p\le\infty$ and $f\in L^p(\mathbb{S}^q)$, we write

$$E_{\mathbb{S}^q,x,p}(f) := \min_{P\in\Pi_x^q} \|f-P\|_{\mathbb{S}^q,p}. \tag{2.12}$$

We will define the Besov spaces in terms of the sequence $\{E_{\mathbb{S}^q,2^n,p}(f)\}$. Let $0<\rho\le\infty$, $\gamma>0$, and $\mathbf{a}=\{a_n\}_{n=0}^\infty$ be a sequence of real numbers. We define

$$\|\mathbf{a}\|_{\rho,\gamma} := \begin{cases} \{\sum_{n=0}^\infty 2^{n\gamma\rho}|a_n|^\rho\}^{1/\rho}, & \text{if } 0<\rho<\infty, \\ \sup_{n\ge 0} 2^{n\gamma}|a_n|, & \text{if } \rho=\infty. \end{cases} \tag{2.13}$$

The space of sequences $\mathbf{a}$ for which $\|\mathbf{a}\|_{\rho,\gamma}<\infty$ will be denoted by $\mathsf{b}_{\rho,\gamma}$. If $1\le p\le\infty$, the Besov space $B_{\mathbb{S}^q,p,\rho,\gamma}$ consists of all functions $f\in L^p(\mathbb{S}^q)$ for which $\{E_{\mathbb{S}^q,2^n,p}\}\in\mathsf{b}_{\rho,\gamma}$. A spherical cap centered at a point $\mathbf{x}_0\in\mathbb{S}^q$, and radius $\alpha\in[0,\pi]$ is defined by

$$\mathbb{S}^q_\alpha(\mathbf{x}_0) := \{\mathbf{x}\in\mathbb{S}^q \ : \ \mathbf{x}\cdot\mathbf{x}_0 \ge \cos\alpha\} = \{\mathbf{x}\in\mathbb{S}^q \ : \ \|\mathbf{x}-\mathbf{x}_0\| \le 2\sin(\alpha/2)\}, \tag{2.14}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{q+1}$. For a cap C, the space $C_0^\infty(C)$ consists of infinitely differentiable functions ϕ on $\mathbb{S}^q$ such that $\phi(\mathbf{x})=0$ if $\mathbf{x}\notin C$. If $\mathbf{x}_0\in\mathbb{S}^q$, the local Besov space $B_{\mathbb{S}^q,p,\rho,\gamma}(\mathbf{x}_0)$ consists of functions $f\in L^p(\mathbb{S}^q)$ for which there exists a cap C centered at $\mathbf{x}_0$ such that for every $\phi\in C_0^\infty(C)$, $f\phi\in B_{\mathbb{S}^q,p,\rho,\gamma}$.

We end this section with a result on the metric entropy for the unit balls of the Besov spaces. We find it easier to prove the following Theorem 2.2 than finding a reference for it. For $1 \le p \le \infty$, $0 < \gamma < \infty$, $0 < \rho \le \infty$, let

$$\overline{B}_{\mathbb{S}^q,p,\rho,\gamma} := \left\{ f \in L^p(\mathbb{S}^q) \ : \ \|f\|_{\mathbb{S}^q,p} + \|\{E_{\mathbb{S}^q,2^n,p}(f)\}\|_{\rho,\gamma} \le 1 \right\} \tag{2.15}$$

be the unit ball of the Besov space $B_{\mathbb{S}^q,p,\rho,\gamma}$.

Theorem 2.2 *Let $0 < \epsilon \le 1$, $1 \le p \le \infty$, $0 < \gamma < \infty$, $0 < \rho \le \infty$. Then*

$$c_1(\log(1/\epsilon))^{-(3q)/(2\gamma\rho)}(1/\epsilon)^{q/\gamma} \le H_\epsilon(\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}, L^p(\mathbb{S}^q)) \le c_2(1/\epsilon)^{q/\gamma}. \tag{2.16}$$

3 Main Results

In this section, we discuss a variant of the polynomial frames introduced in [20], and demonstrate their localization by the fact that their behavior characterizes local Besov spaces on the sphere. We will further demonstrate that a uniform quantization process leads to a representation of functions in local Besov spaces using a finite number of bits, which is within a logarithmic factor of the metric entropy estimate corresponding to the space.

We start by defining certain kernel functions. Let $h : [0,\infty) \to \mathbb{R}$, $h(x) = 0$ if $x > c$. We define the kernels

$$\Phi_n(h,t) := \sum_{\ell=0}^{\infty} h(\ell/2^n)\frac{d_\ell^q}{\omega_q}\mathcal{P}_\ell(t), \qquad t \in \mathbb{R}, \ n = 0, 1, \cdots, \tag{3.1}$$

and for $n \ge 2$,

$$\tilde{\Phi}_n(h,t) = \Phi_{n+1}(h,t) - \Phi_{n-2}(h,t), \qquad t \in \mathbb{R}. \tag{3.2}$$

We define $\tilde{\Phi}_n(h,t) = \Phi_n(h,t)$, $n = 0, 1$, and $\tilde{\Phi}_n(h,t) = \Phi_n(h,t) = 0$ if $n < 0$.

Let $\{\mu_n\}$ be a sequence of (possibly signed) finite, Borel measures on $\mathbb{S}^q$. We define the following operators. The *summability operator* is defined by

$$\sigma_n(\mu_n, h, f, \mathbf{x}) := \int_{\mathbb{S}^q} \Phi_n(h, \mathbf{x}\cdot\xi) f(\xi) d\mu_n(\xi), \tag{3.3}$$

and with $\sigma_t(\mu_n, h, f) = 0$ if $t < 0$, the *polynomial frame operator* is defined by

$$\tau_n(\mu_n, h, f) := \sigma_n(\mu_n, h, f) - \sigma_{n-1}(\mu_{n-1}, h, f). \tag{3.4}$$

We note that the operator τ_n uses two measures: μ_n and μ_{n-1}. Although we need to mention the measures in order to emphasize, for example, whether the operator is constructed using spherical harmonic coefficients or values of the function, we prefer to keep the notation simple by mentioning only μ_n with the understanding that two measures are actually involved in the definition.

In our discussions below, the interesting cases are when $\mu_n = \mu_q^*$ for all n, and when μ_n is an M–Z quadrature measure of order $6(2^n)$, supported on a set of scattered sites as in Theorem 2.1. Our discussion will be more concise by considering two sequences of M–Z quadrature measures, which need not be distinct. On the occasions when our statements require that the operators be constructed using μ_q^*, we will write $\sigma_n^*(h,f) := \sigma(\mu_q^*, h, f)$ and $\tau_n^*(h,f) := \sigma_n^*(h,f) - \sigma_{n-1}^*(h,f)$.

We will write $\mu_n \preceq_p \mu_q^*$ if every μ_q^*-measurable function is also μ_n-measurable, and $\|f\|_{\mu_n;A,p} \le c(A,p,\{\mu_n\})\|f\|_{A,p}$ for every μ_q^*-measurable subset $A \subseteq \mathbb{S}^q$ and $f \in X^p(A)$. In addition to the case when $\mu_n = \mu_q^*$, it is clear that any sequence of measures $\{\mu_n\}$ supported on finite subsets of $\mathbb{S}^q$, and with $|\mu_n|(\mathbb{S}^q) \le c$, satisfies $\mu_n \preceq_\infty \mu_q^*$.

The operators will be shown to characterize local Besov spaces if h is sufficiently smooth. Let $Q \ge 1$ be an integer. We will write $h \in \mathcal{A}_Q$ if each of the following conditions is satisfied: (i) $h : [0,\infty) \to \mathbb{R}$, (ii) for some integer $K \ge Q+q$, h is a K times iterated integral of a function of bounded variation, (iii) $h'(x) = 0$ if $x \le c_1$, and (iv) $h(x) = 0$ if $x > c$. Further, we will write $h \in \mathcal{A}_Q^*$ if $h \in \mathcal{A}_Q$, $h(x) = 1$ for $x \in [0,1/2]$ and $h(x) = 0$ for $x > 1$. In the sequel, all unspecified constants may depend upon h and sequences of measures as well as the norms and q.

Our first theorem below is a simple fact regarding the expansions of functions in $X^p(\mathbb{S}^q)$ in terms of the various operators just introduced.

Theorem 3.1 *Let $1 \le p \le \infty$, $f \in X^p(\mathbb{S}^q)$, $\gamma > 0$, $0 < \rho \le \infty$, $Q > \max(1,\gamma)$, $h \in \mathcal{A}_Q^*$, and $\mathbf{x}_0 \in \mathbb{S}^q$. For $n \ge 0$, let μ_n, ν_n be M–Z quadrature measures of order $6(2^n)$, and in addition, $\mu_n \preceq_p \mu_q^*$. Then*

$$f = \sum_{n=0}^{\infty} \tau_n(\mu_n, h, f), \tag{3.5}$$

with the series converging in the sense of $X^p(\mathbb{S}^q)$. In the special case when $\mu_n = \mu_q^*$ for all n, we have

$$f = \sum_{n=0}^{\infty} \int_{\mathbb{S}^q} \tau_n^*(h,f,\xi)\tilde{\Phi}_n(h, (\cdot)\cdot\xi)d\nu_n(\xi), \tag{3.6}$$

with the series converging in the sense of $X^p(\mathbb{S}^q)$.

We note that in the case when each ν_n is supported on a finite set of points, the values $\{\tau_n^*(h,f,\xi)\}$ in (3.6) appear as the coefficients in a multiscale expansion of f. A similar expansion was given in [20] using a different set of functions than $\tilde{\Phi}_n$. While that expansion was more in keeping with the multiresolution paradigm, in the sense that the terms in the inner series were orthogonal to $\Pi_{2^{n-1}}^q$, rather than to $\Pi_{2^{n-3}}^q$ as in the case of (3.6), the expansion in (3.6) has an important stability property, summarized in the following theorem.

Theorem 3.2 *Let $Q \ge 1$ be an integer, $h \in \mathcal{A}_Q^*$, and V denote the total variation of h. For $n \ge 0$, let ν_n be an M–Z quadrature measure of order $6(2^n)$. Then for $f \in L^2(\mathbb{S}^q)$,*

$$\|f\|_{\mathbb{S}^q,2}^2 \le 9V^2 \sum_{n=0}^{\infty} \|\tau_n^*(h,f)\|_{\mathbb{S}^q,2}^2 = 9V^2 \sum_{n=0}^{\infty} \|\tau_n^*(h,f)\|_{\nu_n;\mathbb{S}^q,2}^2 \le 9V^4\|f\|_{\mathbb{S}^q,2}^2. \tag{3.7}$$

Thus, we have defined a frame expansion for functions in $L^2(\mathbb{S}^q)$, where both the frame elements $\tilde\Phi_n$ and the coefficient functionals $\tau_n^*(h,f)$ are polynomials, and the number of vanishing moments at level n tends to infinity as $n\to\infty$.

Next, we illustrate the localization properties of our frame operators by describing a characterization of the local Besov spaces in terms of the values of this operator.

Theorem 3.3 *Let $1\le p\le\infty$, $f\in X^p(\mathbb{S}^q)$, $\gamma>0$, $0<\rho\le\infty$, $Q>\max(1,\gamma)$, $h\in\mathcal{A}_Q^*$, and $\mathbf{x}_0\in\mathbb{S}^q$. For $n\ge 0$, let μ_n,ν_n be M–Z quadrature measures of order $6(2^n)$, and in addition, $\mu_n\preceq_p\mu_q^*$. Then the following are equivalent.*
(a) $f\in B_{\mathbb{S}^q,p,\rho,\gamma}(\mathbf{x}_0)$.
(b) *There exists a cap C, centered at $\mathbf{x}_0$, such that for every $\phi\in C_0^\infty(C)$, $\{\|\tau_n(\mu_n,h,f\phi)\|_{\mathbb{S}^q,p}\}\in\mathsf{b}_{\rho,\gamma}$.*
(c) *There exists a cap C, centered at $\mathbf{x}_0$, such that for every $\phi\in C_0^\infty(C)$, $\{\|\tau_n(\mu_n,h,f\phi)\|_{\nu_n;\mathbb{S}^q,p}\}\in\mathsf{b}_{\rho,\gamma}$.*
(d) *There exists a cap C, centered at $\mathbf{x}_0$, such that $\{\|\tau_n(\mu_n,h,f)\|_{C,p}\}\in\mathsf{b}_{\rho,\gamma}$.*
(e) *There exists a cap C, centered at $\mathbf{x}_0$, such that $\{\|\tau_n(\mu_n,h,f)\|_{\nu_n;C,p}\}\in\mathsf{b}_{\rho,\gamma}$.*

We remark that the condition (e) characterizes the local Besov space at a point in terms of samples of the polynomials $\tau_n(\mu_n,h,f)$ near that point. In the case when $p=\infty$, we may take μ_n above either as μ_q^* or an M–Z quadrature measure supported on a set of scattered sites, perhaps different from those in the support of ν_n, as guaranteed in Theorem 2.1. In this case, the conditions (b) and (c) characterize the local Besov space at a point, based on samples of the function near the point. In particular, condition (c) involves only discrete measures.

To complete the discussion about the localization properties of our frame operators, our next theorem gives a characterization of local Besov spaces in terms of the coefficients in an "arbitrary" series expansion of the target function.

Theorem 3.4 *Let $1\le p\le\infty$, $f\in X^p(\mathbb{S}^q)$, $\gamma>0$, $0<\rho\le\infty$, $Q>\max(1,\gamma)$, $h\in\mathcal{A}_Q^*$, and $\mathbf{x}_0\in\mathbb{S}^q$. For each $n\ge 0$, let ν_n be an M–Z quadrature measure of order $6(2^n)$. Let d_n be a ν_n measurable function, $\|d_n\|_{\nu_n;\mathbb{S}^q,p}\le c$, and*

$$f=\sum_{n=0}^\infty\int_{\mathbb{S}^q}d_n(\xi)\tilde\Phi_n(h,(\cdot)\cdot\xi)d\nu_n(\xi), \tag{3.8}$$

with convergence in the sense of $L^p(\mathbb{S}^q)$. If there exists a cap C, centered at $\mathbf{x}_0$, such that $\{\|d_n\|_{\nu_n;C,p}\}\in\mathsf{b}_{\rho,\gamma}$, then $f\in B_{\mathbb{S}^q,p,\rho,\gamma}(\mathbf{x}_0)$.

Finally, the following theorem shows that a uniform quantization of the values of the summability operators leads to a bit representation of functions in local Besov spaces, with the number of bits within a logarithmic factor of the metric entropy. In view of Theorem 3.3, we find it convenient to define the unit ball of a local Besov space as follows. Let C be a spherical cap. For integer $n\ge 0$, let μ_n be an M–Z quadrature measure of order $6(2^n)$. We define

$$\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}(\{\mu_m\},C):=\left\{f\in L^p(\mathbb{S}^q)\ :\ \|f\|_{\mathbb{S}^q,p}+\|\{\|\tau_n(\mu_n,h,f)\|_{C,p}\}\|_{\rho,\gamma}\le 1\right\}. \tag{3.9}$$

Theorem 3.5 *Let $1 \le p \le \infty$, $\gamma > 0$, $0 < \rho \le \infty$, $Q > \max(1, \gamma)$, $h \in \mathcal{A}_Q^*$. For $n \ge 0$, let μ_n be an M–Z quadrature measure of order $6(2^n)$, with $\mu_n \preceq_p \mu_q^*$. Suppose that for each integer $n \ge 0$, $\mathcal{C}_n$ is a finite set of points on $\mathbb{S}^q$ such that there exists an M–Z quadrature measure ν_n of order $6(2^n)$, supported on a subset of $\mathcal{C}_n$ as in Theorem 2.1. Let C be a spherical cap. If $n \ge 0$, and $f \in L^p(\mu_n; \mathbb{S}^q)$, we define*

$$I_n(\mu_n, h, f, \xi) := \lfloor 2^{nQ} \sigma_n(\mu_n, h, f, \xi) \rfloor, \qquad \xi \in \mathbb{S}^q, \tag{3.10}$$

and

$$\begin{aligned} \sigma_n^\circ(C, h, f, \mathbf{x}) &:= \sigma_n^\circ(\mu_n, \nu_n; C, h, f, \mathbf{x}) \\ &:= 2^{-nQ} \int_C I_n(\mu_n, h, f, \xi) \Phi_{n+1}(h, \mathbf{x} \cdot \xi) d\nu_n(\xi), \quad \mathbf{x} \in \mathbb{S}^q. \end{aligned} \tag{3.11}$$

If $f \in \overline{B}_{p,\rho,\gamma}(\{\mu_m\}, C)$, then for a cap C', concentric with C and having radius strictly less than that of C, $\{\|f - \sigma_n^\circ(C, h, f)\|_{C',p}\} \in \mathsf{b}_{\rho,\gamma}$, and in particular,

$$\|f - \sigma_n^\circ(C, h, f)\|_{C',p} \le c(C, C') 2^{-n\gamma}. \tag{3.12}$$

If n is chosen so that the right hand side above is at most ϵ, then the number of bits needed to represent all the integers $\{I_n(\mu_n, h, f, \xi),\ \xi \in C \cap supp(\nu_n)\}$ does not exceed $c_1(\log(1/\epsilon))^c (1/\epsilon)^{q/\gamma} \mu_q^*(C)$.

If $n \ge 0$ and $g \in L^p(\mu_n; \mathbb{S}^q)$, then

$$\|f - \sigma_n^\circ(C, h, g)\|_{C',p} \le c(C, C')\{2^{-n\gamma} + \|f - g\|_{\mu_n;\mathbb{S}^q,p}\}. \tag{3.13}$$

We remark that an obvious analogue of the above theorem is clearly valid also for the global Besov space, obtained by taking the caps to be the whole sphere. We observe that in the case when $\mu_n^* = \mu_q^*$ for each n, the polynomials σ_n are constructed using the global information contained in the spherical harmonic coefficients of the function. Nevertheless, in the approximate representation of the function on a cap, we need to retain only those numbers $I_n(\mu_q^*, h, f, \xi)$ for which ξ is in a slightly larger cap. In the case when $p = \infty$, we may apply the above theorem with μ_n to be an M–Z quadrature measure supported on a finite set of scattered sites, as in Theorem 2.1. We may also apply the theorem with $f\phi$ in place of f for a suitable infinitely differentiable function ϕ supported on the larger cap C, and equal to 1 on C'. Thus, only the values of f on C are needed in the calculation of $I_n(\mu_n, h, f\phi)$, and in turn, only the values of these operators on C are needed to obtain the bit representation suitable for approximation on the smaller cap C'. Finally, we remark that neither the quantization nor the reconstruction requires an a priori knowledge of the specific Besov spaces to which the target function belongs on different caps, in the sense that for a fixed n, the number of bits on each cap is proportional to its volume, and the error in reconstruction is commensurate with the smoothness of the target function on that cap.

4 Proofs

Theorem 2.2 is a consequence of a very general theorem due to Lorentz [13, Theorem 3.3(i), Chapter 15]. To describe this theorem, let X be a Banach space, $\{\phi_k\}$ be a sequence of linearly independent functions in X whose span is dense in X, $X_n :=$ span $(\{\phi_1, \cdots, \phi_n\})$, $X_0 = \{0\}$,

$$E(f, X_n) := \inf_{g \in X_n} \|f - g\|_X, \qquad n = 0, 1, 2, \cdots. \tag{4.1}$$

If $\{\delta_k\}$ is a nonincreasing sequence of positive numbers with $\delta_k \to 0$ as $k \to \infty$, we define

$$A(X; \{\delta_k\}, \{\phi_k\}) := \{f \in X \ : \ E(f, X_n) \le \delta_n, \ n = 0, 1, 2, \cdots\}. \tag{4.2}$$

The theorem of Lorentz [13, Theorem 3.3(i), Chapter 15] that we wish to apply can be stated as follows.

Theorem 4.1 *Let $\{\delta_k\}$ be a nonincreasing sequence of positive numbers with $\delta_k \to 0$ as $k \to \infty$, and $\delta_{2n} \le c\delta_n$, $n = 1, 2, \cdots$ for some $c \in (0, 1)$. For $x \ge 0$, let*

$$N_x := \min\{k \ : \ \delta_k \le e^{-x}\}. \tag{4.3}$$

Let X be a Banach space, $\{\phi_k\}$ be a sequence of linearly independent functions in X whose span is dense in X, $\epsilon > 0$, and $j := 2 + \lfloor \log(1/\epsilon) \rfloor$. Then

$$H_\epsilon(A(X; \{\delta_k\}, \{\phi_k\}), X) \sim \sum_{k=1}^{j} N_k. \tag{4.4}$$

We will apply this theorem with $\delta_k = Ck^{-\gamma/q}(\log k)^{-\beta/\rho}$, once with $\beta = 3/2$ and once with $\beta = 0$. The computations are summarized in the following lemma.

Lemma 4.1 *Let $\delta_k = ck^{-a}(\log k)^{-b}$ for some $a > 0$, $b \ge 0$, and N_x be defined as in (4.3). Then*

$$\sum_{k=1}^{j} N_k \sim e^{j/a} j^{-b/a}. \tag{4.5}$$

Proof. Let y be the solution of the equation

$$cy^{-a}(\log y)^{-b} = e^{-x}. \tag{4.6}$$

With $u = \log y$, and $v = x/a + (\log c)/a$, we get $u + (b/a)\log u = v$. Clearly, $u/v \to 1$ as $v \to \infty$. So, $u = v - (b/a)\log(v(1 + o(1))) = v - (b/a)\log v + o(1)$. This yields $y \sim x^{-b/a}e^{x/a}$. Thus, $N_x \sim x^{-b/a}e^{x/a}$. Therefore, denoting (in this proof only), B to be the smallest integer $> 2b$, we have

$$\sum_{k=1}^{j} N_k \sim \sum_{k=1}^{B} N_k + \int_B^j x^{-b/a} e^{x/a} dx. \tag{4.7}$$

Integrating by parts, we see that

$$\int_B^j x^{-b/a}e^{x/a}dx = aj^{-b/a}e^{j/a} - c_1 + b\int_B^j x^{-b/a-1}e^{x/a}dx.$$

Therefore,

$$\left|\int_B^j x^{-b/a}e^{x/a}dx - aj^{-b/a}e^{j/a} + c_1\right| \le (b/B)\int_B^j x^{-b/a}e^{x/a}dx \le (1/2)\int_B^j x^{-b/a}e^{x/a}dx,$$

and we conclude that

$$(1/2)\int_B^j x^{-b/a}e^{x/a}dx \le aj^{-b/a}e^{j/a} - c_1 \le (3/2)\int_B^j x^{-b/a}e^{x/a}dx.$$

Hence, (4.7) leads to (4.5). $\Box$

PROOF OF THEOREM 2.2. In Theorem 4.1, we take $X = L^p(\mathbb{S}^q)$ and $\{\phi_k\}$ be an enumeration of $\{Y_{\ell,\nu}\}$ in which degree $(\phi_j) <$ degree (ϕ_k) implies $j < k$. Let $X_k := \text{span}\{\phi_1, \cdots, \phi_k\}$. In this proof only, let $D_n = \sum_{\ell=0}^{2^n} d_\ell^q$ denote the dimension of $\Pi_{2^n}^q$. Then (cf. (2.3))

$$D_n \sim D_{n+1} - D_n \sim 2^{nq}.$$

So, for $D_n \le k \le D_{n+1}$, $n = 0, 1, \cdots$, and $f \in X$,

$$2^{(n+1)\gamma}E_{\mathbb{S}^q,2^{n+1},p}(f) \le ck^{\gamma/q}E(f, X_k) \le c_1 2^{n\gamma}E_{\mathbb{S}^q,2^n,p}(f).$$

Therefore, there exist constants C_1 and C_2, such that with $\delta_{1,k} := C_1 k^{-\gamma/q}(\log k)^{-3/(2\rho)}$, $\delta_{2,k} := C_2 k^{-\gamma/q}$, we have

$$A(X; \{\delta_{1,k}\}, \{\phi_k\}) \subseteq \overline{B}_{\mathbb{S}^q,p,\rho,\gamma} \subseteq A(X; \{\delta_{2,k}\}, \{\phi_k\}),$$

and hence,

$$H_\epsilon(A(X; \{\delta_{1,k}\}, \{\phi_k\}), X) \le H_\epsilon(\overline{B}_{\mathbb{S}^q,p,\rho,\gamma}, X) \le H_\epsilon(A(X; \{\delta_{2,k}\}, \{\phi_k\}), X).$$

Both $\{\delta_{1,k}\}$ and $\{\delta_{2,k}\}$ are nonincreasing, and $\delta_{j,2k} \le c_j\delta_{j,k}$ for some $c_j \in (0,1)$, $j = 1, 2$ as required in Theorem 4.1. Therefore, the estimates (4.4) and (4.5) lead to (2.16). $\Box$

In order to prove the theorems in Section 3, we first establish certain basic facts regarding the kernel functions and the associated operators. For the convenience of the reader we reproduce below two of the results from [16].

Lemma 4.2 *Let m_1, m_2 be signed measures (having bounded variation) on a measure space S, supported on S_1 and S_2 respectively, $\Psi : S \times S \to \mathbb{R}$ be a bounded, $|m_1| \times |m_2|$ measurable function, $\Psi(x,t) = \Psi(t,x)$ for $x, t \in S$, $1 \le p \le \infty$, $f \in L^p(|m_1|)$, and let*

$$T_f(x) := \int f(t)\Psi(x,t)dm_1(t).$$

Then with

$$A = \max(\sup_{x\in S_1} \|\Psi(x,\cdot)\|_{|m_2|;1}, \sup_{x\in S_2} \|\Psi(x,\cdot)\|_{|m_1|;1}),$$

we have

$$\|T_f\|_{|m_2|;p} \le A\|f\|_{|m_1|;p}. \tag{4.8}$$

PROOF. This is Lemma 4.1 in [16]. □

Lemma 4.3 *Let $Q \ge 1$ and $h \in \mathcal{A}_Q$. Then*

$$\int_{-1}^{1} |\Phi_n(h,t)|(1-t^2)^{q/2-1}dt \le c, \qquad n = 0, 1, \cdots, \tag{4.9}$$

and for any $\eta > 0$,

$$\sup_{-1\le t\le 1-\eta} 2^{nQ}|\Phi_n(h,t)| \le c(\eta), \qquad n = 0, 1, \cdots. \tag{4.10}$$

PROOF. This lemma is a consequence of Theorem 3.1 in [16], where the notations are different. In this proof only, let p_ℓ denote the orthonormalized Jacobi polynomial, orthogonal on $[-1,1]$ with respect to the weight $(1-t^2)^{q/2-1}$, $t \in [-1,1]$. Then (2.5) implies that

$$\Phi_n(h,t) = \omega_{q-1}^{-1} \sum_{\ell=0}^{\infty} h(\ell/2^n)p_\ell(t)p_\ell(1). \tag{4.11}$$

Under the assumptions of the lemma, the conditions of Theorem 3.1 of [16] are satisfied with $\alpha = \beta = (q/2) - 1$. Therefore, the estimates (2.10) and (2.11) of [16] are satisfied. The estimate (4.9) follows by using (2.10) of [16] with $x = 1$. The estimate (4.10) is obtained by using the values $x_0 = x = 1$ in (2.11) of [16]. □

The following proposition summarizes certain properties of the kernel functions Φ_n and the operators σ_n.

Proposition 4.1 *Let $\mathbf{x}_0 \in \mathbb{S}^q$, $0 < \alpha' < \alpha$, $Q \ge 1$ be an integer, $h \in \mathcal{A}_Q$, and $h(x) = 0$ if $x > A$. For each integer $n \ge 0$, let μ_n be an M–Z quadrature measure of order $2A(2^n)$. Then*

$$\int_{\mathbb{S}^q} |\Phi_n(h, \mathbf{x}\cdot\mathbf{x}_0)|d|\mu_n|(\mathbf{x}) \le c, \tag{4.12}$$

and for each $1 \le p \le \infty$, $f \in L^p(\mu_n;\mathbb{S}^q)$, we have

$$\|\sigma_n(\mu_n, h, f)\|_{\mathbb{S}^q,p} \le c\|f\|_{\mu_n;\mathbb{S}^q,p}. \tag{4.13}$$

For $\mathbf{x} \in \mathbb{S}^q_{\alpha'}(\mathbf{x}_0)$ and $\mathbf{y} \in \mathbb{S}^q \setminus \mathbb{S}^q_\alpha(\mathbf{x}_0)$,

$$|\Phi_n(h, \mathbf{x}\cdot\mathbf{y})| \le c(\alpha,\alpha')2^{-nQ}. \tag{4.14}$$

Further, if $\mu_n \preceq_p \mu_q^*$ then for $f \in L^p(\mathbb{S}^q)$,

$$\|\sigma_n(\mu_n, h, f)\|_{\mathbb{S}^q,p} \le c\|f\|_{\mathbb{S}^q,p}, \qquad n = 0, 1, \cdots. \tag{4.15}$$

In addition, if $h \in \mathcal{A}_Q^*$, then

$$E_{\mathbb{S}^q,2^n,p}(f) \le \|f - \sigma_n(\mu_n, h, f)\|_{\mathbb{S}^q,p} \le cE_{\mathbb{S}^q,2^{n-1},p}(f). \tag{4.16}$$

PROOF. Using the fact that μ_n satisfies (2.8) with $p=1$, and the rotation invariance of μ_q^*, we obtain

$$\begin{aligned}\int_{\mathbb{S}^q}|\Phi_n(h,\mathbf{x}\cdot\mathbf{x}_0)|d|\mu_n|(\mathbf{x}) &\le c\int_{\mathbb{S}^q}|\Phi_n(h,\mathbf{x}\cdot\mathbf{x}_0)|d\mu_q^*(\mathbf{x})\\ &= c\int_{-1}^1|\Phi_n(h,t)|(1-t^2)^{q/2-1}dt.\end{aligned}$$

Therefore, (4.9) implies (4.12). We note that (4.12) is now proved also in the case when μ_n is replaced by μ_q^*. The estimate (4.13) is obtained by using Lemma 4.2 with $\Phi_n(h,\mathbf{x}\cdot\mathbf{y})$ in place of $\Psi(\mathbf{x},\mathbf{y})$, μ_n in place of m_1 and μ_q^* in place of m_2.

Next, let $\mathbf{x}\in\mathbb{S}^q_{\alpha'}(\mathbf{x}_0)$ and $\mathbf{y}\in\mathbb{S}^q\setminus\mathbb{S}^q_\alpha(\mathbf{x}_0)$. Then

$$\|\mathbf{x}-\mathbf{y}\|\ge\|\mathbf{y}-\mathbf{x}_0\|-\|\mathbf{x}-\mathbf{x}_0\|\ge 2(\sin(\alpha/2)-\sin(\alpha'/2)).$$

Therefore,

$$\mathbf{x}\cdot\mathbf{y}=1-(\|\mathbf{x}-\mathbf{y}\|^2/2)\le 1-2(\sin(\alpha/2)-\sin(\alpha'/2))^2,$$

and (4.10) implies (4.14).

If $\mu_n\preceq_p\mu_q^*$, then (4.15) follows from (4.13). Finally, let $h\in\mathcal{A}_Q^*$. Since $h(x)=1$ for $x\in[0,1/2]$, and μ_n satisfies (2.9), it is easy to deduce that $\sigma_n(\mu_n,h,P)=P$ for all $P\in\Pi^q_{2^{n-1}}$. Therefore, for any $P\in\Pi^q_{2^{n-1}}$, we have from (4.12) that

$$E_{\mathbb{S}^q,2^n,p}(f)\le\|f-\sigma_n(\mu_n,h,f)\|_{\mathbb{S}^q,p}=\|f-P-\sigma_n(\mu_n,h,f-P)\|_{\mathbb{S}^q,p}\le c\|f-P\|_{\mathbb{S}^q,p}.$$

This proves (4.16). □

We are now in a position to prove Theorem 3.1 and Theorem 3.2.

PROOF OF THEOREM 3.1. The equation (3.5) follows from (4.16) and the definitions. Since $h(k/2^{n+1})-h(k/2^{n-2})=1$ for $2^{n-2}\le k\le 2^n$, and $h(k/2^n)-h(k/2^{n-1})=0$ otherwise, the quadrature formula leads to the fact that for $n\ge 0$ for $\mathbf{x}\in\mathbb{S}^q$,

$$\tau_n^*(h,f,\mathbf{x})=\int_{\mathbb{S}^q}\tau_n^*(h,f,\xi)\tilde\Phi_n(h,\mathbf{x}\cdot\xi)d\nu_n(\xi). \tag{4.17}$$

Therefore, using (3.5) with the choice $\mu_n=\mu_q^*$ for all n, we get

$$f=\sum_{n=0}^\infty\tau_n^*(h,f)=\sum_{n=0}^\infty\int_{\mathbb{S}^q}\tau_n^*(h,f,\xi)\tilde\Phi_n(h,(\cdot)\cdot\xi)d\nu_n(\xi).$$

This completes the proof of (3.6). □

PROOF OF THEOREM 3.2. It is enough to prove (3.7) if f is an arbitrary polynomial. Therefore, in this proof, we will assume that f is a polynomial, and hence, all the sums will be finite sums. In this proof only, let

$$y_{\ell,n}=\begin{cases}h(\ell), & \text{if } n=0,\\ h(\ell/2), & \text{if } n=1,\\ h(\ell/2^{n+1})-h(\ell/2^{n-2}), & \text{if } n=2,3,\cdots,\end{cases}$$

and

$$g_{\ell,n} = \begin{cases} h(\ell), & \text{if } n = 0, \\ h(\ell/2^n) - h(\ell/2^{n-1}), & \text{if } n = 1, 2 \cdots. \end{cases}$$

Then $\sum_{n=0}^{\infty} y_{\ell,n} g_{\ell,n} = \sum_{n=0}^{\infty} g_{\ell,n} = 1$. Moreover, for $n \geq 2$, $y_{\ell,n} = g_{\ell,n+1} + g_{\ell,n} + g_{\ell,n-1}$, and hence, for $\ell \geq 0$,

$$\sum_{n=2}^{\infty} g_{\ell,n}^2 \leq \left(\sum_{n=2}^{\infty} |g_{\ell,n}|\right)^2 \leq V^2, \ \sum_{n=2}^{\infty} y_{\ell,n}^2 \leq \left(\sum_{n=2}^{\infty} |y_{\ell,n}|\right)^2 \leq 9V^2.$$

Therefore,

$$\begin{aligned}
\|f\|_{\mathbb{S}^q,2}^2 &= \sum_{\ell=0}^{1} \sum_k |\hat{f}(\ell,k)|^2 + \sum_{\ell=2}^{\infty} \sum_k \left| \sum_{n=2}^{\infty} y_{\ell,n} g_{\ell,n} \hat{f}(\ell,k) \right|^2 \\
&\leq \sum_{\ell=0}^{1} \sum_k |\hat{f}(\ell,k)|^2 + 9V^2 \sum_{\ell=2}^{\infty} \sum_k \sum_{n=2}^{\infty} |g_{\ell,n} \hat{f}(\ell,k)|^2 \\
&= \sum_{\ell=0}^{1} \sum_k |\hat{f}(\ell,k)|^2 + 9V^2 \sum_{n=2}^{\infty} \|\tau_n^*(h,f)\|_{\mathbb{S}^q,2}^2 \\
&= \sum_{n=0}^{1} \|\tau_n^*(h,f)\|_{\mathbb{S}^q,2}^2 + 9V^2 \sum_{n=2}^{\infty} \|\tau_n^*(h,f)\|_{\mathbb{S}^q,2}^2 \\
&\leq \sum_{\ell=0}^{1} \sum_k |\hat{f}(\ell,k)|^2 + 9V^4 \sum_{\ell=2}^{\infty} \sum_k |\hat{f}(\ell,k)|^2 \leq 9V^4 \|f\|_{\mathbb{S}^q,2}^2.
\end{aligned}$$

□

In order to prove Theorem 3.3, we need a more detailed analogue of the Marcinkiewicz–Zygmund inequalities (2.8). We note that the following lemma does not require that $\mu_n \preceq_p \mu_q^*$, and hence, the roles of μ_n, ν_n are interchangeable in the estimates (4.18) and (4.19) below.

Lemma 4.4 *For each integer $n \geq 0$, let μ_n, ν_n be M–Z quadrature measures of order $6(2^n)$. For $P \in \Pi_{2^{n+1}}^q$,*

$$\|P\|_{\mu_n;\mathbb{S}^q,p} \leq c\|P\|_{\nu_n;\mathbb{S}^q,p}. \tag{4.18}$$

If $C_1 \subset C_2 \subseteq \mathbb{S}^q$ are concentric spherical caps, and $Q \geq 1$, then for $P \in \Pi_{2^n}^q$,

$$\|P\|_{\mu_n;C_1,p} \leq c(Q, C_1, C_2)\{\|P\|_{\nu_n;C_2,p} + 2^{-nQ}\|P\|_{\nu_n;\mathbb{S}^q,p}\}. \tag{4.19}$$

Proof. Let $Q \geq 1$. We choose, and fix in this proof, $h \in \mathcal{A}_Q^*$. Using the quadrature formula (2.9), we see that for $\mathbf{x} \in \mathbb{S}^q$,

$$P(\mathbf{x}) = \int_{\mathbb{S}^q} P(\xi)\Phi_{n+2}(h, \mathbf{x} \cdot \xi) d\nu_n(\xi).$$

We now use Lemma 4.2 with $\Phi_{n+2}(h, \mathbf{x} \cdot \mathbf{y})$ in place of $\Psi(\mathbf{x}, \mathbf{y})$, ν_n in place of m_1 and μ_n in place of m_2 to arrive at (4.18).

Let $P \in \Pi_{2^n}^q$, and $\phi \in C_0^\infty(C_2)$ be chosen so that $\phi(\mathbf{x}) = 1$ if $x \in C_1$. By the direct theorem of approximation theory [23], there exists $R \in \Pi_{2^n}^q$ such that

$$\|\phi - R\|_{\mathbb{S}^q,\infty} \le c(Q, C_1, C_2)2^{-nQ}.$$

Therefore, using (4.18) for the polynomial $PR \in \Pi_{2^{n+1}}^q$,

$$\begin{aligned} \|P\|_{\mu_n;C_1,p} &= \|P\phi\|_{\mu_n;\mathbb{S}^q,p} \le \|PR\|_{\mu_n;\mathbb{S}^q,p} + \|P(\phi - R)\|_{\mu_n;\mathbb{S}^q,p} \\ &\le c(Q, C_1, C_2)\{\|PR\|_{\nu_n;\mathbb{S}^q,p} + 2^{-nQ}\|P\|_{\mu_n;\mathbb{S}^q,p}\} \\ &\le c(Q, C_1, C_2)\{\|P\phi\|_{\nu_n;\mathbb{S}^q,p} + 2^{-nQ}\|P\|_{\nu_n;\mathbb{S}^q,p}\} \\ &\le c(Q, C_1, C_2)\{\|P\|_{\nu_n;C_2,p} + 2^{-nQ}\|P\|_{\nu_n;\mathbb{S}^q,p}\} \end{aligned}$$

This proves (4.19). □

Proof of Theorem 3.3. We will prove first that (a) ⇔ (b) ⇔ (c).

Let part (a) hold, C be as required in the definition of the local Besov space, and $\phi \in C_0^\infty(C)$. In view of (4.16),

$$\|\tau_n(\mu_n, h, f\phi)\|_{\mathbb{S}^q,p} \le cE_{\mathbb{S}^q,2^{n-2},p}(f\phi).$$

Therefore, part (a) implies part (b). Conversely, let part (b) hold, C be as in part (b), and $\phi \in C_0^\infty(C)$. Then (3.5) implies that

$$E_{\mathbb{S}^q,2^{n+1},p}(f\phi) \le \|f\phi - \sigma_n(\mu_n, h, f\phi)\|_{\mathbb{S}^q,p} \le \sum_{m=n+1}^{\infty} \|\tau_m(\mu_m, h, f\phi)\|_{\mathbb{S}^q,p}.$$

In view of the discrete Hardy inequality [6, Lemma 3.24, p. 27], this implies part (a).

The implication (c) ⇒ (b) follows from (4.18), applied with μ_q^* in place of μ_n. The converse implication follows by applying (4.18) with ν_n in place of μ_n and μ_q^* in place of ν_n there.

Next, we show that (b) implies (d) and (d) implies (a). Let (b) hold, $C = \mathbb{S}_\alpha^q(\mathbf{x}_0)$ be as in part (b), $\alpha < \pi/4$, and $C_{1/2}$ (respectively $C_{1/4}$) be the cap centered at $\mathbf{x}_0$ and radius $\alpha/2$ (respectively $\alpha/4$). Let $\psi \in C_0^\infty(C)$ be chosen so that $\psi(\mathbf{x}) = 1$ for $\mathbf{x} \in C_{1/2}$ and $\|\psi\|_\infty = 1$. For $\mathbf{x} \in C_{1/4}$, we have from (4.14) that for any integer $m \ge 1$,

$$\begin{aligned} &\left|\int_{\mathbb{S}^q} f(\xi)(1 - \psi(\xi))\Phi_m(h, \xi\cdot\mathbf{x})d\mu_m(\xi)\right| \\ &\quad= \left|\int_{\mathbb{S}^q\setminus C_{1/2}} f(\xi)(1 - \psi(\xi))\Phi_m(h, \xi\cdot\mathbf{x})d\mu_m(\xi)\right| \\ &\quad\le c(C)2^{-mQ}\int_{\mathbb{S}^q} |f(\xi)|d|\mu_m|(\xi) \\ &\quad\le c(C)2^{-mQ}\|f\|_{\mu_m;\mathbb{S}^q,p} \le c(C)2^{-mQ}\|f\|_{\mathbb{S}^q,p}. \end{aligned}$$

Applying this inequality once with $m = n$ and once with $m = n - 1$, we deduce that

$$\|\tau_n(\mu_n, h, (1 - \psi)f)\|_{C_{1/4},\infty} \le c(C)2^{-nQ}\|f\|_{\mathbb{S}^q,p}.$$

Therefore,

$$\begin{aligned}\|\tau_n(\mu_n,h,f)\|_{C_{1/4},p} &\le \|\tau_n(\mu_n,h,\psi f)\|_{\mathbb{S}^q,p}+c\|\tau_n(\mu_n,h,(1-\psi)f)\|_{C_{1/4},\infty}\\ &\le \|\tau_n(\mu_n,h,\psi f)\|_{\mathbb{S}^q,p}+c(C)2^{-nQ}\|f\|_{\mathbb{S}^q,p}.\end{aligned}$$

Since both the sequences $\{\|\tau_n(\mu_n,h,\psi f)\|_{\mathbb{S}^q,p}\}$ and $\{2^{-nQ}\}$ are in $\mathsf{b}_{\rho,\gamma}$, part (d) is proved.

Next, let part (d) hold, C be the cap as in that part, and $\phi\in C_0^\infty(C)$. By the direct theorem of approximation theory [23], there exists $R\in\Pi_{2^n}^q$ such that

$$\|\phi-R\|_{\mathbb{S}^q,\infty}\le c(Q,\phi)2^{-nQ}.$$

Therefore, using (4.15) and (3.5), we obtain

$$\begin{aligned}E_{2^{n+1},p}(f\phi) &\le \|f\phi-R\sigma_n(\mu_n,h,f)\|_{\mathbb{S}^q,p}\\ &\le \|(f-\sigma_n(\mu_n,h,f))\phi\|_{\mathbb{S}^q,p}+\|(\phi-R)\sigma_n(\mu_n,h,f)\|_{\mathbb{S}^q,p}\\ &\le c(Q,\phi)\{\|f-\sigma_n(\mu_n,h,f)\|_{C,p}+2^{-nQ}\|f\|_{\mathbb{S}^q,p}\}\\ &\le c(Q,\phi)\left\{\sum_{m=n+1}^\infty\|\tau_m(\mu_m,h,f)\|_{C,p}+2^{-nQ}\|f\|_{\mathbb{S}^q,p}\right\}.\end{aligned}$$

The discrete Hardy inequality [6, Lemma 3.4, p. 27] now shows that $f\phi\in B_{\mathbb{S}^q,p,\rho,\gamma}$. Thus, part (d) implies part (a).

Thus, parts (a), (b), (c), (d) are equivalent. The equivalence between parts (d) and (e) is a simple consequence of (4.19) as in the proof of the equivalence between parts (b) and (c).

□

Proof of Theorem 3.4. In this proof only, we will write $p_n:=(d_n^q/\omega)\mathcal{P}_n(q+1;\cdot)$. Using (2.4), it is easy to deduce that for $\mathbf{x},\mathbf{y}\in\mathbb{S}^q$ and integer $\ell,k\ge 0$,

$$\int_{\mathbb{S}^q}p_\ell(\mathbf{y}\cdot\xi)p_k(\mathbf{x}\cdot\mathbf{y})d\mu_q^*(\mathbf{y})=\begin{cases}0, & \text{if } \ell\ne k,\\ p_k(\mathbf{x}\cdot\xi), & \text{if } \ell=k.\end{cases}\tag{4.20}$$

In this proof only, let $g_{k,m}=h(k/2^m)-h(k/2^{m-1})$, $y_{k,n}=h(k/2^{n+1})-h(k/2^{n-2})$, and ψ_j be defined for $j\in\mathbb{Z}$ by

$$\psi_j(x)=\left(h(2^jx)-h(2^{j+1}x)\right)\left(h(x/2)-h(4x)\right).$$

Then $g_{k,m}=0$ if $k\le 2^{m-2}$ or $k>2^m$, and $y_{k,n}=0$ if $k\le 2^{n-3}$ or $k>2^{n+1}$. If $m\ge n+3$, then $n+1\le m-2$. If $k\le 2^{m-2}$, then $g_{k,m}=0$, otherwise, $k>2^{n+1}$, and $y_{k,n}=0$. Thus, $m\ge n+3$ implies that $g_{k,m}y_{k,n}=0$ for all k. Similarly, $g_{k,m}y_{k,n}=0$ for all k if $m\le n-3$. Therefore, for $\mathbf{x}\in\mathbb{S}^q$, (3.8) and (4.20) imply that for $m\ge 2$,

$$\begin{aligned}\tau_m(\mu_q^*,h,f,\mathbf{x}) &= \sum_{k=0}^\infty g_{k,m}\int_{\mathbb{S}^q}f(\mathbf{y})p_k(\mathbf{x}\cdot\mathbf{y})d\mu_q^*(\mathbf{y})\\ &= \sum_{n=0}^\infty\int_{\mathbb{S}^q}d_n(\xi)\sum_{k=0}^\infty y_{k,n}g_{k,m}p_k(\mathbf{x}\cdot\xi)d\nu_n(\xi)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=m-2}^{m+2} \int_{\mathbb{S}^q} d_n(\xi) \sum_{k=0}^{\infty} y_{k,n} g_{k,m} p_k(\mathbf{x}\cdot\xi) d\nu_n(\xi) \\
&= \sum_{j=-2}^{2} \int_{\mathbb{S}^q} d_{m+j}(\xi) \sum_{k=0}^{\infty} y_{k,m+j} g_{k,m} p_k(\mathbf{x}\cdot\xi) d\nu_{m+j}(\xi) \\
&= \sum_{j=-2}^{2} \int_{\mathbb{S}^q} d_{m+j}(\xi) \sum_{k=0}^{\infty} \psi_j(k/2^{m+j}) p_k(\mathbf{x}\cdot\xi) d\nu_{m+j}(\xi) \\
&= \sum_{j=-2}^{2} \int_{\mathbb{S}^q} d_{m+j}(\xi) \Phi_{m+j}(\psi_j, \mathbf{x}\cdot\xi) d\nu_{m+j}(\xi). \qquad (4.21)
\end{aligned}
$$

Now, we observe that each of the functions $\psi_j \in \mathcal{A}_Q$, and $\psi_j(x) = 0$ if $x \ge 2$. Therefore, in view of Proposition 4.1, (4.12) and (4.14) hold for each of the functions ψ_j in place of h, and $\{\nu_n\}$ in place of $\{\mu_n\}$, $|j| \le 2$. Let C_1 be the cap, concentric with C and having radius equal to half that of C. Then for $|j| \le 2$, $\mathbf{x} \in C_1$, and $\xi \in \mathbb{S}^q \setminus C$,

$$
|\Phi_{m+j}(\psi_j, \mathbf{x}\cdot\xi)| \le c(C) 2^{-mQ}.
$$

Hence, for $j = 0, \pm 1, \pm 2$, and $\mathbf{x} \in C_1$,

$$
\left| \int_{\xi\in\mathbb{S}^q\setminus C} d_{m+j}(\xi) \Phi_{m+j}(\psi_j, \mathbf{x}\cdot\xi) d\nu_{m+j}(\xi) \right| \le c(C) 2^{-mQ} \|d_{m+j}\|_{\nu_{m+j};\mathbb{S}^q,p} \le c(C) 2^{-mQ}. \qquad (4.22)
$$

Therefore, denoting by χ the characteristic function of C, we obtain that for $\mathbf{x} \in C_1$ and $j = 0, \pm 1, \pm 2$,

$$
\begin{aligned}
&\left| \int_{\mathbb{S}^q} d_{m+j}(\xi) \Phi_{m+j}(\psi_j, \mathbf{x}\cdot\xi) d\nu_{m+j}(\xi) \right| \\
&\le \left| \int_{\mathbb{S}^q} d_{m+j}(\xi) \chi(\xi) \Phi_{m+j}(\psi_j, \mathbf{x}\cdot\xi) d\nu_{m+j}(\xi) \right| + \frac{c(C)}{2^{mQ}} \qquad (4.23)
\end{aligned}
$$

Using (4.13), we obtain that

$$
\left\| \int_{\mathbb{S}^q} d_{m+j}(\xi) \chi(\xi) \Phi_{m+j}(\psi_j, \mathbf{x}\cdot\xi) d\nu_{m+j}(\xi) \right\|_{\mathbb{S}^q,p} \le c \|d_{m+j}\chi\|_{\nu_{m+j};p} = c \|d_{m+j}\|_{\nu_{m+j};C,p}. \qquad (4.24)
$$

Along with (4.21), this implies that

$$
\|\tau_m(\mu_q^*, h, f)\|_{C_1,p} \le c(C) \left\{ \sum_{j=-2}^{2} \|d_{m+j}\|_{\nu_{m+j};C,p} + 2^{-mQ} \right\}.
$$

Therefore, $\{\|\tau_m(\mu_q^*, h, f)\|_{C_1,p}\} \in \mathsf{b}_{\rho,\gamma}$, and Theorem 3.3 implies that $f \in B_{p,\rho,\gamma}(\mathbf{x}_0)$.

□

Proof of Theorem 3.5. Let $f \in L^p(\mu_n; \mathbb{S}^q)$. In view of the quadrature formula,

$$
\sigma_n(\mu_n, h, f, \mathbf{x}) = \int_{\mathbb{S}^q} \sigma_n(\mu_n, h, f, \xi) \Phi_{n+1}(h, \mathbf{x}\cdot\xi) d\nu_n(\xi), \qquad \mathbf{x} \in \mathbb{S}^q. \qquad (4.25)
$$

In view of (4.14), (4.18) (applied with the roles of ν_n and μ_n interchanged) and (4.13), we get for $\mathbf{x} \in C'$,

$$\begin{aligned}\left|\int_{\mathbb{S}^q\setminus C} \sigma_n(\mu_n, h, f, \xi)\Phi_{n+1}(h, \mathbf{x}\cdot\xi)d\nu_n(\xi)\right| &\le c2^{-nQ}\|\sigma_n(\mu_n, h, f)\|_{\nu_n;\mathbb{S}^q,p} \\ &\le c2^{-nQ}\|\sigma_n(\mu_n, h, f)\|_{\mu_n;\mathbb{S}^q,p} \le c2^{-nQ}\|f\|_{\mu_n;\mathbb{S}^q,p}. \qquad (4.26)\end{aligned}$$

Further,

$$|\sigma_n(\mu_n, h, f, \xi) - 2^{-nQ}I_n(\mu_n, h, f, \xi)| \le 2^{-nQ}, \qquad \xi \in \mathbb{S}^q. \qquad (4.27)$$

Therefore, using (4.12),

$$\begin{aligned}&\|\sigma_n(\mu_n, h, f) - \sigma_n^\circ(C, h, f)\|_{C',p} \\ &= \left\|\sigma_n(\mu_n, h, f) - \int_C 2^{-nQ}I_n(\mu_n, h, f, \xi)\Phi_{n+1}(h, (\cdot)\cdot\xi)d\nu_n(\xi)\right\|_{C',p} \\ &\le \left\|\sigma_n(\mu_n, h, f) - \int_C \sigma_n(\mu_n, h, f, \xi)\Phi_{n+1}(h, (\cdot)\cdot\xi)d\nu_n(\xi)\right\|_{C',p} + c2^{-nQ} \\ &\le \left\|\sigma_n(\mu_n, h, f) - \int_{\mathbb{S}^q} \sigma_n(\mu_n, h, f, \xi)\Phi_{n+1}(h, (\cdot)\cdot\xi)d\nu_n(\xi)\right\|_{C',p} + c2^{-nQ}(1 + \|f\|_{\mu_n;\mathbb{S}^q,p}) \\ &= c2^{-nQ}(1 + \|f\|_{\mu_n;\mathbb{S}^q,p}). \qquad (4.28)\end{aligned}$$

Now, let $f \in \overline{B}_{p,\rho,\gamma}(\{\mu_n\}, C)$. Then $\|f\|_{\mu_n;\mathbb{S}^q,p} \le c\|f\|_{\mathbb{S}^q,p} \le c$. Therefore, in view of (3.5),

$$\begin{aligned}\|f - \sigma_n^\circ(C, h, f)\|_{C',p} &\le \|f - \sigma_n(\mu_n, h, f)\|_{C',p} + \|\sigma_n(\mu_n, h, f) - \sigma_n^\circ(C, h, f)\|_{C',p} \\ &\le \sum_{m=n+1}^{\infty} \|\tau_m(\mu_m, h, f)\|_{C',p} + c2^{-nQ}.\end{aligned}$$

Since both the terms in the last expression above are in $\mathsf{b}_{\rho,\gamma}$, the sequence $\{\|f - \sigma_n^\circ(C, h, f)\|_{C',p}\}$ is also in $\mathsf{b}_{\rho,\gamma}$. In view of the Nikolskii inequalities [17, Proposition 2.1], we see that

$$|I_n(\mu_n, h, f, \xi)| \le 2^{nQ}\|\sigma_n(\mu_n, h, f)\|_{\mathbb{S}^q,\infty} \le c2^{n(Q+1)}\|\sigma_n(\mu_n, h, f)\|_{\mathbb{S}^q,p} \le c_1 2^{cn}.$$

Thus, the number of bits needed to represent each $I_n(\mu_n, h, f, \xi)$ is at most cn. In view of Theorem 2.1, the set $C \cap \mathrm{supp}(\nu_n)$, and hence the set $\{I_n(\mu_n, h, f, \xi),\ C \cap \mathrm{supp}(\nu_n)\}$ contains at most $c2^{nq}\mu_q^*(C)$ elements. Hence, the total number of bits needed to represent all of these integers is at most $cn2^{nq}\mu_q^*(C)$. The desired estimate on this number is now clear.

If $g \in L^p(\mu_n; \mathbb{S}^q)$, then (4.13) and (4.28) imply that

$$\begin{aligned}\|f - \sigma_n^\circ(C, h, g)\|_{C',p} &\le \|f - \sigma_n^\circ(C, h, f)\|_{C',p} + \|\sigma_n^\circ(C, h, f) - \sigma_n^\circ(C, h, g)\|_{C',p} \\ &\le \|f - \sigma_n^\circ(C, h, f)\|_{C',p} + c\|f - g\|_{\mu_n;\mathbb{S}^q,p} + c_1 2^{-nQ}.\end{aligned}$$

□

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