Polynomial frames for the detection of singularities

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Abstract

We propose a class of algebraic polynomial frames, which are computationally easier to implement than polynomial bases. We also discuss the weighted L^p- stability of our frames for $1 \le p \le \infty$. Our analysis is based on orthogonal polynomials with respect to the weight in question, but the frame bounds are independent of the system of orthogonal polynomials used. In spite of the fact that algebraic polynomials are inherently nonlocal, our frames provide good localization properties. In particular, they can be used to detect discontinuities in derivatives of all orders of a function. We describe asymptotic expressions for the frame coefficients in the vicinity of a discontinuity.

1 Introduction

Let I be an open real interval, $f : I \to I\!\!R$, and $r \ge 0$ be an integer. In this paper, a point $x_0 \in I$ will be called a singularity of f of order r if the $(r-1)$-st derivative, $f^{(r-1)}$, is absolutely continuous in a neighborhood of x_0, and $f^{(r)}$ is continuous in this neighborhood, except for a jump discontinuity at x_0. In many applications, for example, image and data compression, prediction of time series, and antenna technology, one needs to determine the location of the singularities of a function of various orders.

There has been a tremendous amount of research on the problem of detection of singularities using wavelets, see for example, the recent book of Meyer [14] and references therein. Of course, to apply the wavelet analysis, one needs either to know or to compute the wavelet coefficients of the target function. In some applications (e.g., [25]), the known information about the function consists of the Fourier coefficients of the function or the coefficients in some orthogonal polynomial expansion. Since the function is not smooth, it is not economical first to compute an approximation to the function, and then to use this approximation to compute the wavelet coefficients. We also observe that the ability of the wavelets to detect singularities stems from their vanishing moment property. Therefore, a single compactly supported wavelet cannot detect singularities of all orders. (cf. Figure 1).

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In [3], the authors constructed certain "wavelets" consisting of trigonometric polynomials. This idea has been developed in [21], [22]. For a more general approach to periodic multiresolution analysis and corresponding wavelets, compare [23], [12], [19], [9]. An important application of this domain of ideas is in the construction of optimal order bases for the class of 2π-periodic continuous functions [13], [10].

There have been efforts to construct a "multiresolution analysis" consisting of algebraic polynomials, for example [11]. However, it is not possible in general to construct wavelets consisting of translations of an algebraic polynomial. For example, it is proved in [16, Theorem 11.3.1] that it is not possible to construct a polynomial P of degree $2n$ and points $x_1, \cdots, x_n$ such that

$$\int_{I\!R} P(t - x_j) R(t) \exp(-t^2) dt = 0$$

for all polynomials R of degree at most n.

A different construction of polynomial wavelets was given in [5], where certain "generalized translates" were used instead of the usual translations. Motivated by the work [5], we propose in this paper a variety of polynomial "frames" which span the previously studied polynomial spaces. Compared to the wavelets studied in [5], it is computationally easier to obtain a decomposition of a function in terms of these frames. Further, unlike the wavelets in [5], the frame bounds (with respect to suitably weighted L^p norms) for our frames is independent of the system of orthogonal polynomials used. To examine the localization properties of our frames, we discuss how they can be used to detect singularities. In fact, we provide a precise quantitative description of how the frame coefficients may be used to detect the singularities of all orders.

Finally, we point out that our primary interest is to construct localized polynomials with properties similar to those of wavelets. Although we use the language of wavelet analysis, it should be kept in mind that the starting points in the construction of our frame operators and the classical wavelets, such as the Daubechies wavelets, are different, and hence, they are not really comparable.

We define our frames in a general setting in Section 2. In Section 3, we specialize to the case of Jacobi polynomials, and describe the detection of singularities. The stability properties of our frames are described in Section 4. In Section 5, we discuss a number of numerical experiments to demonstrate the advantages, disadvantages, and limitations of our frames, treating the Daubechies wavelets as a bench-mark.

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2 Polynomial frames

We denote the class of all algebraic polynomials of degree at most n by Π_n. Let μ be a positive Borel measure on $I\!R$ with infinitely many points of increase and having finite moments; i.e.,

$$\int |t|^r \, d\mu(t) < \infty, \qquad r = 0, 1, 2, \ldots \ .$$

Then there exists a unique system of polynomials

$$p_n(x) := p_n(d\mu; x) := \gamma_n(d\mu)x^n + \cdots, \qquad \gamma_n(d\mu) > 0,\ n = 0, 1, 2, \ldots,$$

such that

$$\int p_m(t)p_k(t)d\mu(t) = \begin{cases} 1, & \text{if } k = m, \\ 0, & \text{otherwise.} \end{cases}$$

In view of this definition, it is immediately clear that for any $P \in \Pi_n$, we have

$$P(x) = \int P(t)K_n(d\mu; x, t)d\mu(t), \tag{2.1}$$

where the *Christoffel-Darboux kernel* is given by

$$K_n(d\mu; x, t) = \sum_{k=0}^{n} p_k(x)p_k(t).$$

There are well known closed form formulas for $K_n(d\mu; x, t)$ ([6], [24]), which we do not need in this paper.

We will need another representation, which is a discretized version of the above representation. It is well known (see e.g. [6], [24]) that for each $m = 1, 2, \ldots,$ the polynomial $p_m(d\mu)$ has m distinct zeros, all in the smallest interval containing the support of $d\mu$. We denote these zeros by $x_{k,m} = x_{k,m}(d\mu)$, with the ordering

$$x_{m,m}(d\mu) < x_{m-1,m}(d\mu) < \cdots < x_{1,m}(d\mu)\,.$$

The *Cotes' numbers* are defined by

$$\lambda_{k,m} := \lambda_{k,m}(d\mu) := \Big\{K_m(d\mu; x_{k,m}, x_{k,m})\Big\}^{-1}.$$

In this paper, an important role is played by the (Gauss) quadrature formula

$$\int P(t)d\mu(t) = \sum_{k=1}^{m} \lambda_{k,m}P(x_{k,m}), \qquad P \in \Pi_{2m-1},\ m = 0, 1, 2, \cdots. \tag{2.2}$$

As a result of the quadrature formula, we have the following representation for all $P \in \Pi_n$, $m \geq n+1$, $n = 1, 2, \cdots,$

$$P(x) = \sum_{k=1}^{m} \lambda_{k,m}P(x_{k,m})K_n(d\mu; x, x_{k,m}), \tag{2.3}$$

which is actually the Lagrange interpolation formula (cf. [6, §I.4]).

Next, we define an analogue of the multiresolution analysis as in [5]. For integer $n \geq 0$, we let $N = 2^n$. We write $V_n := \Pi_N$, and

$$W_n := \{P \in V_{n+1}\ :\ \int P(t)R(t)d\mu(t) = 0, \quad R \in V_n\}.$$

We note that the dimensions of V_n and W_n are $N+1$ and N respectively. The formula (2.3) shows that the functions $K_N(d\mu; \cdot, x_{k,N+1})$ form a basis for V_n. In [5] (cf. [16]), it is shown that the functions $\sum_{j=N+1}^{2N} p_j(x_{k,N})p_j$ form a basis for W_n. Using the dual basis consisting of multiples of the Lagrange fundamental polynomials in W_n, one can obtain a representation similar to (2.3) with $m = N$. A simpler representation, with better stability properties, is obtained in the frame case, $m \geq 2N+1$, i.e., in the case when the "oversampling" m is bigger than two times the dimension of W_n. One of the major themes of this paper is to describe a class of such frames.

Let $G = \{g_{k,\ell}\}_{k=0,\cdots,\ell,\ell=1,2,\cdots}$ be a triangular matrix, and for integer $\ell \geq 0$, let

$$K_\ell(G;x,t) := K_\ell(d\mu, G; x,t) := \sum_{k=0}^{\ell} g_{k,\ell}p_k(x)p_k(t).$$

One may think of $K_\ell(G)$ as a summability kernel.

We now define the *frame operator* by

$$\tau_\ell(G;f,x) := \tau_\ell(d\mu, G; f, x) := \int f(t)K_\ell(G;t,x)d\mu(t),$$

whenever the integral is well defined, and write

$$\tau_{\ell,k,m}(G;f) := \tau_{\ell,k,m}(d\mu, G; f) := \int f(t)K_\ell(G;t,x_{k,m})d\mu(t). \tag{2.4}$$

We will say that G is a *scaling matrix* if $g_{k,\ell} \neq 0,\ k = 0,\ldots,\ell$, and that G is a *frame matrix* if $g_{k,2\ell} = 0,\ k = 0,\ldots,\ell;\ g_{k,2\ell} \neq 0,\ k = \ell+1,\ldots,2\ell$. More generally, for $0 < s < N$, the matrix G will be called a (s-) *partial frame matrix* if $g_{k,2\ell} = 0$, $0 \leq k < s$; $g_{k,2\ell} \neq 0$, $s \leq k \leq 2\ell$. For a triangular matrix G, the matrix $G^{[\sigma]}$ is defined for integer σ by

$$g_{k,\ell}^{[\sigma]} = \begin{cases} g_{k,\ell}^{\sigma}, & \text{if } g_{k,\ell} \neq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 2.1 *Let $\ell \geq 0$ be an integer, $G = \{g_{k,\ell}\}_{k=0,\cdots,\ell,\ell=1,2,\cdots}$, $H = \{h_{k,\ell}\}_{k=0,\cdots,\ell,\ell=1,2,\cdots}$ be triangular matrices, $GH := \{g_{k,\ell}h_{k,\ell}\}_{k=0,\cdots,\ell,\ell=1,2,\cdots}$. Then for $m \geq \ell+1$*

$$\sum_{k=1}^{m} \lambda_{k,m}K_\ell(G;t,x_{k,m})K_\ell(H;x,x_{k,m}) = K_\ell(GH;x,t). \tag{2.5}$$

If G is a scaling matrix, then for every $P \in V_n$, $N = 2^n$, and $m \geq N+1$,

$$P(x) = \sum_{k=1}^{m} \lambda_{k,m}\tau_{N,k,m}(G;P)K_N(G^{[-1]};x,x_{k,m})\ .$$

If G is a scaling or partial frame matrix, then for every $P \in W_n$ and $m \geq 2N+1$,

$$P(x) = \sum_{k=1}^{m} \lambda_{k,m}\tau_{2N,k,m}(G;P)K_{2N}(G^{[-1]};x,x_{k,m}). \tag{2.6}$$

Moreover, if G is a frame matrix, the functions $K_{2N}(G^{[-1]};\cdot,x_{k,m}) \in W_n$.

PROOF. Using the quadrature formula (2.2), we obtain

$$\sum_{k=1}^{m} \lambda_{k,m} K_N(G;t,x_{k,m}) K_N(H;x,x_{k,m}) = \int K_N(G;t,u) K_N(H;x,u) d\mu(u).$$

The formula (2.5) follows from the orthogonality relations. The remaining assertions of the theorem are now immediately clear from (2.1). □

If f is a Lebesgue measurable function, we may write

$$a_k(f) := a_k(d\mu; f) := \int f p_k d\mu \ , \quad k = 0, 1, \ldots,$$

whenever the integrals are well defined.

If $f \in L^2_{d\mu}$, and $n \geq 0$ is an integer, the orthogonal projection of f onto W_n is given by

$$\sum_{k=N+1}^{2N} a_k(f) p_k \ .$$

Further, if G is a frame matrix then we can use Theorem 2.1 to obtain for the frame decomposition (convergent in $L^2_{d\mu}$):

$$f = \sum_{r=0}^{2} a_r(f) p_k + \sum_{n=0}^{\infty} \sum_{k=1}^{m} \tau_{2N,k,m}(G; f) \Psi_{n,k,m},$$

where $m = m_n \geq 2N+1, n = 0, 1, \ldots$ and the frame elements $\Psi_{n,k,m}$ are defined for integers $k = 1, \ldots, m, \ \ n = 0, 1, \ldots$, by

$$\Psi_{n,k,m}(x) := \Psi_{n,k,m}(G; x) := \lambda_{k,m} K_{2N}(G^{[-1]}; x, x_{k,m}) \ .$$

3 Detection of singularities

In this section, we restrict our attention to the case of Jacobi polynomials. These are defined for $\alpha, \beta > -1$ by the *Rodrigues' formula:*

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) := \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n \left\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\right\}. \tag{3.1}$$

It is well known that

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) dx = \begin{cases} 0, & \text{if } k \neq n, \\ h_n^{(\alpha,\beta)}, & \text{if } k = n, \end{cases}$$

where the *Jacobi weight* $w_{\alpha,\beta}$ is defined by

$$w_{\alpha,\beta}(x) := \begin{cases} (1-x)^{\alpha}(1+x)^{\beta}, & \text{if } -1 < x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$h_n^{(\alpha,\beta)} := \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}.$$

Hence, $\{(h_n^{(\alpha,\beta)})^{-1/2}P_n^{(\alpha,\beta)}\}$ is the system of orthonormal polynomials with respect to $w_{\alpha,\beta}$.

Let $r \ge 0$ be an integer, and $f : [-1,1] \to I\!R$ have a singularity of order r at some point $y \in (-1,1)$, In this section, we examine the question of approximating y from the behavior of the frame coefficients $\{\tau_{2N,k,m}(G;f)\}$, or more generally, the functions $\{\tau_{2N}(G;f)\}$. A typical example is given by the *truncated power function* defined by

$$x_+^r := \begin{cases} x^r, & \text{if } x \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

In fact, in the general case, if f has a singularity at y with jump equal to d, then the function $f - \frac{d}{r!}(\cdot - y)_+^r$ is r times continuously differentiable at y. Repeating this process, most functions of practical interest can be written in the form

$$f(x) = \varphi(x) + \sum_{r=0}^{R}\frac{1}{r!}\sum_{k=1}^{k_r} d_{r,k}(x-y_{r,k})_+^r,$$

where φ is R times continuously differentiable on $[-1,1]$, $d_{r,k} \in I\!R$, and $y_{r,k} \in (-1,1)$. For functions of this form,

$$\tau_{2N}(G;f) = \tau_{2N}(G;\phi) + \sum_{r=0}^{R}\frac{1}{r!}\sum_{k=1}^{k_r} d_{r,k}\tau_{2N}(G;(\cdot - y_{r,k})_+^r). \tag{3.2}$$

In this section, we will examine the sequence of functions $\{\tau_{2N}(G;(\cdot - y)_+^r)\}$.

Let $g : [0,2\pi] \to I\!R$. The matrix G defined by

$$g_{k,\ell} := g\left(\frac{2\pi k}{\ell+1}\right), \qquad k = 0,\cdots,\ell,\ \ell = 0,1,\cdots,$$

will be called the matrix *generated by* g. If $g(0) = g(2\pi) = 0$, then G is a partial frame matrix. In this case, we may (and will) assume that g is extended to the whole real line as a 2π-periodic function, and $g_{k,\ell}$ are defined for all integer k using this extended function.

Let $q \ge 0$ be an integer. Let BV_0^q denote the class consisting of all 2π-periodic functions h, which can be expressed in the form

$$h(x) = \frac{1}{(q-1)!}\int_0^x (x-t)_+^{q-1}h^{(q)}(t)dt, \qquad \text{for all } x \in [-\pi,\pi],$$

where $h^{(q)}$ is a function having a bounded total variation on $[-\pi,\pi]$. We observe that if $h \in BV_0^q$, then $h^{(k)}(0) = h^{(k)}(2\pi) = 0$, $k = 0,\cdots,q-1$. The total variation of a function φ on $[-\pi,\pi]$ will be denoted by $V(\varphi)$.

Theorem 3.1 *Let $\alpha,\beta > -1$, r, q be nonnegative integers. Suppose that $g \in BV_0^q$, and for some $\kappa > 0$, $g(t) = 0$ for $t \in [0,\kappa]$. Let G be the matrix generated by g. Let $\delta > 0$, $y = \cos\phi$, $\phi \in [\delta,\pi-\delta]$.*

(a) *Uniformly for $\theta \in [\delta,\pi-\delta]$, $x = \cos\theta$, we have*

$$
\begin{aligned}
&N^r\tau_{2N}(w_{\alpha,\beta},G;(\cdot-y)_+^r,x)\\
&= \frac{\pi^r r!}{2^{\alpha+\beta+1}} w_{\alpha,\beta}(x)^{-1/2}w_{\alpha,\beta}(y)^{1/2}(1-x^2)^{-1/4}(1-y^2)^{r/2}\\
&\times \int_\kappa^{2\pi}\frac{g(t)}{t^{r+1}}\cos\Big(\big(\frac{2N+1}{2\pi}t+\frac{\alpha+\beta+1}{2}\big)(\phi-\theta)+(r+1)\phi-\frac{r+1}{2}\pi\Big)dt\\
&+ \mathcal{O}(1/N)\,. \qquad (3.3)
\end{aligned}
$$

(b) *There exists a sequence of complex valued continuous functions* $C_\nu := C_\nu(r,\alpha,\beta)$ *on* $[\delta,\pi-\delta]^2$ *such that uniformly for* x, y *as in part (a),*

$$
\begin{aligned}
&\Big(\frac{2N+1}{2\pi}\Big)^r\tau_{2N}(w_{\alpha,\beta},G;(\cdot-y)_+^r,x)\\
&= \sum_{\nu=0}^{q}\Big(\frac{2\pi}{2N+1}\Big)^\nu\Re\Big\{C_\nu(\theta,\phi)\int_\kappa^{2\pi}\frac{g(t)}{t^{r+1+\nu}}\exp\Big(i\frac{2N+1}{2\pi}(\phi-\theta)t\Big)dt\Big\}\\
&\quad+\mathcal{O}\Big(\frac{1}{N^{q+1}}\Big). \qquad (3.4)
\end{aligned}
$$

In the above theorem, the constants involved in the $\mathcal{O}(\cdot)$ terms depend on α, β, r, g, q, κ, and δ only. In the sequel, the symbols $c, c_1, \cdots$ will denote positive constants depending only on these and other explicitly indicated parameters.

We observe that (3.3) shows explicitly the leading term of the expansion in (3.4). In particular, it is not identically equal to zero.

Before proving Theorem 3.1, we point out a corollary to highlight some interesting aspects of the behavior of $\tau_{2N}(w_{\alpha,\beta},G;(\cdot-y)_+^r,x)$ near and away from y.

Corollary 3.1 *With the notation as in Theorem 3.1, there exist constants* $a, b \in I\!R$ *and a positive constant* c, *(all possibly dependent on* y*), such that*

$$
N^r|\tau_{2N}(w_{\alpha,\beta},G;(\cdot-y)_+^r,x)| \ge c, \qquad \text{if } \frac{2N+1}{2\pi}(\phi-\theta)\in(a,b). \qquad (3.5)
$$

If $\phi\neq\theta$, *then*

$$
N^r|\tau_{2N}(w_{\alpha,\beta},G;(\cdot-y)_+^r,x)| \le \frac{c_1}{(N|\phi-\theta|)^{q+1}}\,. \qquad (3.6)
$$

Proof of Corollary 3.1. In this proof, let

$$
\psi = \frac{\alpha+\beta+1}{2}(\phi-\theta)+(r+1)\phi-\frac{r+1}{2}\pi.
$$

The function

$$
z \to \int_\kappa^{2\pi}\frac{g(t)}{t^{r+1}}\cos(zt+\psi)dt
$$

is an entire function, and hence, nonzero on some interval $[a,b]$. For $|\phi-\theta|\le 1/N$, the numbers a and b may be chosen independent of θ. In light of this observation, (3.3) leads to (3.5). Since the functions $g(t)t^{-r-\nu-1}$, $\nu = 0, \cdots, q$ are all in BV_0^q, a repeated integration by parts shows that

$$\int_\kappa^{2\pi} \frac{g(t)}{t^{r+1+\nu}} \exp\Big(i\frac{2N+1}{2\pi}(\phi-\theta)t\Big)dt = \mathcal{O}\Big((N|\phi-\theta|)^{-q-1}\Big).$$

Therefore, (3.4) implies (3.6). □

The proof of Theorem 3.1 involves certain facts concerning the Jacobi polynomials, which we now recall.

Lemma 3.1 *Let $\alpha, \beta > -1$, r be a nonnegative integer, $-1 < y < 1$, $\delta > 0$ and $k \geq r+1$ be an integer.*
(a) *We have*

$$\int_y^1 (t-y)^r P_k^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t)dt = \frac{(k-r-1)!r!}{2^{r+1}k!} P_{k-r-1}^{(\alpha+r+1,\beta+r+1)}(y) w_{\alpha+r+1,\beta+r+1}(y). \quad (3.7)$$

(b) *Uniformly for $\theta \in [\delta, \pi-\delta]$, $x = \cos\theta$, we have*

$$P_k^{(\alpha,\beta)}(\cos\theta) = (\pi k/2)^{-1/2} w_{\alpha,\beta}(x)^{-1/2}(1-x^2)^{-1/4}\cos\Big((k+\lambda)\theta-\gamma\Big) + \mathcal{O}(k^{-3/2}), \quad (3.8)$$

where

$$\lambda := \frac{\alpha+\beta+1}{2}, \qquad \gamma := (\alpha+\frac{1}{2})\frac{\pi}{2}.$$

(c) *There exists a sequence of continuous complex valued functions ϕ_ν such that for any integer $p \geq 1$ and uniformly for $\theta \in [\delta, \pi-\delta]$, we have*

$$P_k^{(\alpha,\beta)}(\cos\theta) = 2\Re\Big\{e^{ik\theta}\sum_{\nu=0}^{p-1}\phi_\nu(e^{i\theta})k^{-\nu-1/2}\Big\} + \mathcal{O}\Big(\frac{1}{k^{p+1/2}}\Big). \quad (3.9)$$

PROOF. Part (a) can be proved easily from (3.1) by induction (cf. [15]). Parts (b) and (c) are Theorems 8.21.13 and 8.21.9 respectively in [24]. □

Next, we prove a lemma which plays a crucial role in our proof.

Lemma 3.2 *Let $q \geq 0$ be an integer, $h \in BV_0^q$, $0 < \delta < 1$, and $|\phi| \leq 2\pi(1-\delta)$, and $\psi \in I\!R$. Then for integer $m \geq 3/\delta$, we have*

$$\Big|\frac{1}{m}\sum_{k=0}^{m-1} h\Big(\frac{2\pi k}{m}\Big)\cos(k\phi+\psi) - \frac{1}{2\pi}\int_0^{2\pi} h(t)\cos\Big(\frac{m}{2\pi}\phi t+\psi\Big)dt\Big|$$
$$\leq c\frac{V(h^{(q)})}{m^{q+1}}. \quad (3.10)$$

PROOF. Let p be the integer part of $\delta m - 2$. In view of a well known theorem of Ganelius [8], there are trigonometric polynomials Q_1, Q_2 of order at most p, such that

$$Q_1(t) \le h(t) \le Q_2(t), \qquad t \in [0, 2\pi],$$

and

$$\int_0^{2\pi} (Q_2(t) - Q_1(t))dt \le c \frac{V(h^{(q)})}{m^{q+1}}. \tag{3.11}$$

Hence, if ℓ is an integer, $|\ell| \le m - p - 1$, then

$$\begin{aligned} &\frac{1}{m} \sum_{k=0}^{m-1} Q_1\Big(\frac{2\pi k}{m}\Big)\Big(2 + \cos\Big(\frac{2\pi k \ell}{m} + \psi\Big)\Big) \\ &\le \frac{1}{m} \sum_{k=0}^{m-1} h\Big(\frac{2\pi k}{m}\Big)\Big(2 + \cos\Big(\frac{2\pi k \ell}{m} + \psi\Big)\Big) \\ &\le \frac{1}{m} \sum_{k=0}^{m-1} Q_2\Big(\frac{2\pi k}{m}\Big)\Big(2 + \cos\Big(\frac{2\pi k \ell}{m} + \psi\Big)\Big), \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} Q_1(t)(2 + \cos(\ell t + \psi))dt \\ &\le \frac{1}{2\pi} \int_0^{2\pi} h(t)(2 + \cos(\ell t + \psi))dt \\ &\le \frac{1}{2\pi} \int_0^{2\pi} Q_2(t)(2 + \cos(\ell t + \psi))dt. \end{aligned}$$

Now, in view of a quadrature formula valid for trigonometric polynomials (cf. [26, Chapter X, Formula (2.5)]), we have for $j = 1, 2$,

$$\frac{1}{m} \sum_{k=0}^{m-1} Q_j\Big(\frac{2\pi k}{m}\Big)\Big(2 + \cos\Big(\frac{2\pi k \ell}{m} + \psi\Big)\Big) = \frac{1}{2\pi} \int_0^{2\pi} Q_j(t)(2 + \cos(\ell t + \psi))dt.$$

Hence, in view of (3.11), we obtain

$$\begin{aligned} &\left| \frac{1}{m} \sum_{k=0}^{m-1} h\left(\frac{2\pi k}{m}\right)\left(2 + \cos\left(\frac{2\pi k \ell}{m} + \psi\right)\right) - \frac{1}{2\pi} \int_0^{2\pi} h(t)(2 + \cos(\ell t + \psi))dt \right| \\ &\qquad \le c \int_0^{2\pi} (Q_2(t) - Q_1(t))dt \\ &\qquad \le c_1 \frac{V(h^{(q)})}{m^{q+1}}. \end{aligned} \tag{3.12}$$

Using (3.12) with $\ell = 0$, we conclude

$$\left|\frac{1}{m}\sum_{k=0}^{m-1} h\left(\frac{2\pi k}{m}\right) - \frac{1}{2\pi}\int_0^{2\pi} h(t)dt\right| \le c\frac{V(h^{(q)})}{m^{q+1}}.$$

Together with (3.12), this gives

$$\left|\frac{1}{m}\sum_{k=0}^{m-1} h\left(\frac{2\pi k}{m}\right)\cos\left(\frac{2\pi k\ell}{m}+\psi\right) - \frac{1}{2\pi}\int_0^{2\pi} h(t)\cos(\ell t+\psi)dt\right| \le c\frac{V(h^{(q)})}{m^{q+1}}.$$

Thus, we have proved (3.10) in the case when ϕ is of the form $2\pi\ell/m$ for some integer ℓ. (Since $|\phi| \le 2\pi(1-\delta)$, necessarily, $|\ell| \le m-p-1$.) In general,

$$\phi = \frac{2\pi(\ell+\epsilon)}{m}$$

for some $|\epsilon| < 1$ and integer ℓ, $|\ell| \le m-p-1$. The functions $h_c := h\cos(\epsilon\cdot)$ and $h_s := h\sin(\epsilon\cdot)$ are both in BV_0^q, and $V({h_c}^{(q)})$, $V({h_s}^{(q)})$ are both less than or equal to $cV(h^{(q)})$. Since

$$h\left(\frac{2\pi k}{m}\right)\cos(k\phi+\psi) = h_c\left(\frac{2\pi k}{m}\right)\cos\left(\frac{2\pi k\ell}{m}\right) - h_s\left(\frac{2\pi k}{m}\right)\sin\left(\frac{2\pi k\ell}{m}\right),$$

and

$$h(t)\cos\left(\frac{m\phi t}{2\pi}+\psi\right) = h_c(t)\cos(\ell t) - h_s(t)\sin(\ell t),$$

we may apply (3.10) once with h_c and once with h_s (and $\psi + \pi/2$ in place of ψ), and derive (3.10) in this general case by a simple computation. □

Proof of Theorem 3.1. Using the definition of τ_{2N} and (3.7), we obtain

$$\begin{aligned}
&\tau_{2N}(G;(\cdot-y)_+^r,x)\\
&= \sum_{k=0}^{2N} g_{k,2N}\{h_k^{(\alpha,\beta)}\}^{-1}P_k^{(\alpha,\beta)}(x)\int_y^1 (t-y)^r P_k^{(\alpha,\beta)}(t)w_{\alpha,\beta}(t)dt\\
&= \sum_{k=0}^{2N} g_{k,2N}\{h_k^{(\alpha,\beta)}\}^{-1}P_k^{(\alpha,\beta)}(x)\frac{(k-r-1)!r!}{2^{r+1}k!}\\
&\quad \times P_{k-r-1}^{(\alpha+r+1,\beta+r+1)}(y)w_{\alpha+r+1,\beta+r+1}(y).
\end{aligned} \tag{3.13}$$

We will prove part (b). The proof of part (a) is simpler, and we will indicate the differences. In the remainder of this proof, the symbols d_ν, will denote constants dependent on α, β, r, and q. They may or may not be positive, and their value may be different at different occurences, even within the same formula. In view of (3.9), there are continuous functions $\tilde\rho_\nu$ and functions $\tilde\gamma_\nu$ such that $\exp(i\tilde\gamma_\nu)$ are continuous, with the property that

$$\begin{aligned}
&P_{k-r-1}^{(\alpha+r+1,\beta+r+1)}(\cos\phi)\\
&= \sum_{\nu=0}^{q}(k-r-1)^{-\nu-1/2}\tilde\rho_\nu(\phi)\cos(k\phi-\tilde\gamma_\nu(\phi)) + \mathcal{O}(k^{-q-3/2})\\
&= \sum_{\nu=0}^{q}\sum_{\ell=0}^{q} d_\ell k^{-\nu-\ell-1/2}\tilde\rho_\nu(\phi)\cos(k\phi-\tilde\gamma_\nu(\phi)) + \mathcal{O}(k^{-q-1})\ .
\end{aligned}$$

Similarly, there are continuous functions ρ_j, and functions γ_j such that $\exp(i\tilde{\gamma}_j)$ are continuous, with the property that

$$P_k^{(\alpha,\beta)}(\cos\theta) = \sum_{j=0}^{q} k^{-j-1/2}\rho_j(\theta)\cos(k\theta - \gamma_j(\theta)) + \mathcal{O}(k^{-q-3/2}) \ .$$

Therefore,

$$\begin{aligned}
&P_k^{(\alpha,\beta)}(\cos\theta)P_{k-r-1}^{(\alpha+r+1,\beta+r+1)}(\cos\phi)\\
&= \sum_{j=0}^{q}\sum_{\nu=0}^{q}\sum_{\ell=0}^{q} d_\ell k^{-\nu-j-\ell-1}\tilde{\rho}_\nu(\phi)\rho_j(\theta)\cos(k\theta-\gamma_j(\theta))\cos(k\phi-\tilde{\gamma}_\nu(\phi)) + \mathcal{O}(k^{-q-1})\\
&= \frac{1}{2}\sum_{j=0}^{q}\sum_{\nu=0}^{q}\sum_{\ell=0}^{q} d_\ell k^{-\nu-j-\ell-1}\tilde{\rho}_\nu(\phi)\rho_j(\theta)\Big\{\cos\Big(k(\theta+\phi)-\gamma_j(\theta)-\tilde{\gamma}_\nu(\phi)\Big)\\
&\quad + \cos\Big(k(\phi-\theta)+\gamma_j(\theta)-\tilde{\gamma}_\nu(\phi)\Big)\Big\} + \mathcal{O}(k^{-q-1}) \ .
\end{aligned}$$

We expand each of the cosine terms using the addition formula, rearrange the triple sum, relabel the indices, and collect all $\mathcal{O}(k^{-q-1})$ terms together to obtain

$$\begin{aligned}
&P_k^{(\alpha,\beta)}(\cos\theta)P_{k-r-1}^{(\alpha+r+1,\beta+r+1)}(\cos\phi)\\
&= \sum_{\nu=0}^{q} k^{-\nu-1}\Re\Big\{\tilde{\omega}_\nu(\theta,\phi)\exp\Big(ik(\theta+\phi)\Big) + \omega_\nu(\theta,\phi)\exp\Big(ik(\phi-\theta)\Big)\Big\}\\
&\qquad + \mathcal{O}(k^{-q-1}) \ , \qquad (3.14)
\end{aligned}$$

where ω_ν and $\tilde{\omega}_\nu$ are continuous functions on $[\delta, \pi-\delta]^2$, dependent also on α, β, and r.

Next, by ([20], Eqn. (5.02)), we have

$$\begin{aligned}
&\{h_k^{(\alpha,\beta)}\}^{-1}\frac{(k-r-1)!}{k!}\\
&= \frac{2k+\alpha+\beta+1}{2^{\alpha+\beta+1}}\frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(k+\alpha+1)}\frac{\Gamma(k+1)}{\Gamma(k+\beta+1)}\frac{\Gamma(k-r)}{\Gamma(k+1)}\\
&= \frac{k^{-r}}{2^{\alpha+\beta}}\Big\{\sum_{\ell=0}^{q} d_\ell k^{-\ell} + \mathcal{O}(k^{-q-1})\Big\}. \qquad (3.15)
\end{aligned}$$

Substituting from (3.14) and (3.15) into (3.13) and simplifying as before, we get

$$\begin{aligned}
&\tau_{2N}(G; (\cdot-y)_+^r, x)\\
&= \frac{w_{\alpha+r+1,\beta+r+1}(y)r!}{2^{r+1+\alpha+\beta}}\sum_{k=0}^{2N}\sum_{\nu=0}^{q} k^{-r-\nu-1}g_{k,2N}\Big\{\sum_{\ell=0}^{q} d_\ell k^{-\ell} + \mathcal{O}(k^{-q-1})\Big\}\\
&\quad \times\Big(\Re\Big\{\tilde{\omega}_\nu(\phi,\theta)\exp\Big(ik(\theta+\phi)\Big) + \omega_\nu(\phi,\theta)\exp\Big(ik(\phi-\theta)\Big)\Big\} + \mathcal{O}(k^{-q-1})\Big)\\
&= \sum_{s=0}^{q}\sum_{k=0}^{2N} k^{-r-s-1}g_{k,2N}\Re\Big\{\tilde{C}_s(\phi,\theta)\exp\Big(ik(\theta+\phi)\Big) + C_s(\phi,\theta)\exp\Big(ik(\phi-\theta)\Big)\Big\}\\
&\quad + \mathcal{O}(k^{-r-q-1}) \ , \qquad (3.16)
\end{aligned}$$

where C_s and $\tilde{C}_s$ are continuous functions on $[\delta, \pi-\delta]^2$, dependent also on α, β, and r.

Using Lemma 3.2,

$$\begin{aligned}
&\sum_{k=0}^{2N} k^{-r-s-1} g_{k,2N} \exp(ik(\phi-\theta)) \\
&= \left(\frac{2\pi}{2N+1}\right)^{r+s+1} \sum_{k=0}^{2N} \left(\frac{2\pi k}{2N+1}\right)^{-r-s-1} g_{k,2N} \exp(ik(\phi-\theta)) \\
&= \left(\frac{2\pi}{2N+1}\right)^{r+s} \left\{ \int_{\kappa}^{2\pi} g(t) t^{-r-s-1} \exp\left(i \frac{(2N+1)(\phi-\theta)}{2\pi} t\right) dt \right. \\
&\quad \left. + \mathcal{O}(N^{-q-1}) \right\}. \qquad (3.17)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=0}^{2N} k^{-r-s-1} g_{k,2N} \exp(ik(\phi+\theta)) \\
&= \left(\frac{2\pi}{2N+1}\right)^{r+s} \left\{ \int_{\kappa}^{2\pi} g(t) t^{-r-s-1} \exp\left(i \frac{(2N+1)(\phi+\theta)}{2\pi} t\right) dt \right. \\
&\quad \left. + \mathcal{O}(N^{-q-1}) \right\}.
\end{aligned}$$

Since the functions $g(t)t^{-r-s-1}$ are all in BV_0^q, a repeated integration by parts gives

$$\begin{aligned}
&\sum_{k=0}^{2N} k^{-r-s-1} g_{k,2N} \exp(ik(\phi+\theta)) \\
&= \left(\frac{2\pi}{2N+1}\right)^{r+s} \left\{ \mathcal{O}\left(\frac{1}{N(\phi+\theta)}\right)^{q+1} + \mathcal{O}(N^{-q-1}) \right\} \\
&= \mathcal{O}(N^{-r-s-q-1}). \qquad (3.18)
\end{aligned}$$

Substituting from (3.17) and (3.18) into (3.16), we obtain

$$\begin{aligned}
&\tau_{2N}(G; (\cdot - y)_+^r, x) \\
&= \sum_{s=0}^{q} \left(\frac{2\pi}{2N+1}\right)^{r+s} \Re\left\{ C_s(\phi,\theta) \int_{\kappa}^{2\pi} g(t) t^{-r-s-1} \exp\left(i \frac{(2N+1)(\phi-\theta)}{2\pi} t\right) dt \right\} \\
&\quad + \mathcal{O}(N^{-r-q-1}).
\end{aligned}$$

This completes the proof of (3.4).

The proof of part (a) of the theorem is similar; we use the asymptotics (3.8) instead of (3.9) in (3.14), and keep only the $\mathcal{O}(k^{-1})$ terms in the rest of the proof. □

4 Stability

Let $n \geq 0$ be an integer, $P \in W_n$, and G be a triangular matrix. We are interested in comparing suitably weighted L^p norms of the sequence $\{\tau_{2N,k,m}(G;P)\}$ with suitably weighted L^p norms of P for $1 \leq p \leq \infty$. Obviously, the case $p = 2$ is the simplest, and the following proposition summarizes the results for an arbitrary mass distribution.

Theorem 4.1 *Let μ be a mass distribution, G be a frame matrix, $n \geq 0$ be an integer, and $P \in W_n$. Then for $m \geq 2N+1$*

$$B_{n,2}(G^{[-1]})^{-2} \int P(t)^2 d\mu(t) \leq \sum_{k=1}^{m} \lambda_{k,m} \tau_{2N,k,m}^2(G;P) \leq B_{n,2}(G)^2 \int P(t)^2 d\mu(t) \ , \qquad (4.1)$$

where

$$B_{n,2}(G) := \max_{0 \leq j \leq 2N} |g_{j,2N}| \ .$$

Moreover, there exists $P_1 \in W_n$ (respectively $P_2 \in W_n$) for which equality is attained in the lower (respectively upper) inequality of (4.1).

Proof. From formulas (2.4) and (2.5), we obtain

$$\begin{aligned} &\sum_{k=1}^{m} \lambda_{k,m} \tau_{N,k,m}^2(G;P) \\ &= \sum_{k=1}^{m} \lambda_{k,m} \int P(t) K_{2N}(G;t,x_{k,m}) d\mu(t) \int P(u) K_{2N}(G;u,x_{k,m}) d\mu(u) \\ &= \int \int P(t)P(u) \left(\sum_{k=1}^{m} \lambda_{k,m} K_{2N}(G;t,x_{k,m}) K_{2N}(G;u,x_{k,m}) \right) d\mu(t) d\mu(u) \\ &= \int \int P(t)P(u) K_{2N}(G^{[2]};t,u) d\mu(t) d\mu(u). \end{aligned}$$

If $P = \sum_{k=N+1}^{2N} a_k p_k$, we have

$$\begin{aligned} &\sum_{k=1}^{2N+1} \lambda_{k,2N+1} \tau_{2N,k,m}^2(G;P) \\ &= \int \int P(t)P(u) K_{2N}(G^{[2]};t,u) d\mu(t) d\mu(u) = \sum_{j=N+1}^{2N+1} g_{j,2N}^2 a_j^2. \end{aligned}$$

Formula (4.1) can now be deduced easily from the Parseval identity. It is also immediately clear that the bounds cannot be improved. ☐

We compare the bounds in (4.1) with Riesz bounds in the case of the basis when $m = N$ (cf. [5], Theorem 4.1). The frame bounds in (4.1) depend only on G; whereas in the basis case, the Riesz bounds depend heavily on the system of orthogonal polynomials. To explain this, we rewrite (4.1) in the form that for every non-vanishing

$$P(x) = \sum_{k=1}^{m} \sqrt{\lambda_{k,m}\tau_{2N,k,m}(G;P)} \left(\sqrt{\lambda_{k,m}} K_{2N}(G^{[-1]}; x, x_{k,m})\right),$$

we have

$$B_{n,2}(G^{[-1]})^{-2} \le \frac{\sum_{k=1}^{m} \lambda_{k,m}\tau_{2N,k,m}^2(G;P)}{\int P(t)^2 d\mu(t)} \le B_{n,2}(G)^2 . \tag{4.2}$$

On the other hand, it is proved in [5] that

$$\sqrt{\lambda_{s,N}} K_{2N}(G^{[-1]}; \cdot, x_{s,N}) , \qquad s = 1, \ldots, N ,$$

is a basis for W_n. (Actually, it is proved for a frame matrix G with $g_{k,2N} = 0$, $k = 0, \ldots, N$; $g_{k,2N} = 1$, $k = N+1, \ldots, 2N$, but the generalization is straightforward.) Hence, every $P \in W_n$ has the representation

$$P(x) = \sum_{s=1}^{N} \sqrt{\lambda_{s,N}} a_s \sqrt{\lambda_{s,N}} K_{2N}(G^{[-1]}; x, x_{s,N}) ,$$

where the coefficients $\sqrt{\lambda_{s,N}} a_s$ are the inner products with the dual functions. The basis is a Riesz-basis with the upper and lower estimates given by

$$B_{n,2}(G^{[-1]})^{-2}\Omega_{n,2}(A)^{-2} \le \frac{\sum_{s=1}^{N} \lambda_{s,N} a_s^2}{\int P(t)^2 d\mu(t)} \le B_{n,2}(G)^2 \Omega_{n,2}(A^{-1})^2, \tag{4.3}$$

where

$$A := \left(p_{N+k}(x_{s,N})\sqrt{\lambda_{s,N}}\right)_{k,s=1,\cdots,N},$$

A^{-1} denotes, as usual, the inverse matrix of A, and $\Omega_{n,2}(A)$ denotes the spectral norm of the matrix A. The proof of these estimates is again a simple extension of the proof given in [5] (Theorem 4.21).

The tightness of the stability estimates (4.2) and (4.3) is given by the quotients

$$\frac{B_{n,2}(G)}{B_{n,2}(G^{[-1]})^{-1}} = \frac{\max_{0 \le i \le 2N} |g_{i,2N}|}{\min_{0 \le i \le 2N} |g_{i,2N}|}, \qquad \frac{B_{n,2}(G)}{B_{n,2}(G^{[-1]})^{-1}} \frac{\Omega_{n,2}(A^{-1})}{\Omega_{n,2}(A)^{-1}}.$$

We observe that while the first ratio depends entirely on G, the second depends also on the particular system of orthogonal polynomials in question. If G is generated by a function $g \in BV_0^q$ such that $\|g\|_\infty = 1$ and $V(g^{(q)}) = V$, then it is easy to see that

$$\frac{B_{n,2}(G)}{B_{n,2}(G^{[-1]})^{-1}} \ge Vq! \left(\frac{2N+1}{2\pi}\right)^q .$$

Thus, there seems to be a trade off between the stability and the ability of detecting singularities.

At this time, we are able to analyze the case $p \ne 2$ only for special classical polynomials; in particular, the generalized Jacobi polynomials and Freud polynomials. We will discuss only the case of classical Jacobi polynomials; the generalized Jacobi and Freud polynomials can be dealt with using the same ideas.

In the remainder of this section, $\alpha, \beta > -1$ are fixed parameters, $d\mu$ denotes the measure $w_{\alpha,\beta}(x)dx$, and we write $w = w_{\alpha,\beta}$, $p_k = \{h_k^{(\alpha,\beta)}\}^{-1/2}P_k^{(\alpha,\beta)}$, etc. For integer $m \geq 1$, ν_m denotes the measure that associates the mass $\lambda_{k,m}$ with the point $x_{k,m}$, $k = 1, \cdots, m$. We observe that the estimates (4.1) can be written for $P \in W_n$, $m \geq 2N+1$ in the form

$$B_{n,2}(G^{[-1]})^{-1}\|P\|_{2,\mu} \leq \|\tau_{2N}(G;P)\|_{2,\nu_m} \leq B_{n,2}(G)\|P\|_{2,\mu}. \tag{4.4}$$

The upper estimates analogous to these in the L^p norms can be deduced using the multiplier criteria of Gasper and Trebels [7] in the case $1 < p < \infty$ and $\alpha, \beta \geq -1/2$. We will use easier ideas to prove estimates applicable for all values of $\alpha, \beta > -1$ and $1 \leq p \leq \infty$. In general, our estimates are not comparable to those obtained using the multiplier criteria.

The L^p analogues of the estimates (4.4) require certain additional weighting factors, which we now introduce. Let $-1 < \gamma < 1$. For integer $\ell \geq 1$ and $x \in [-1, 1]$, we write

$$\begin{aligned} u_{\gamma,\ell}(x) &:= \left(\sqrt{1-x^2} + \frac{1}{\sqrt{\ell}}\right)^{\gamma}\left(\sqrt{1-x} + \frac{1}{\ell}\right)^{\alpha+1/2}\left(\sqrt{1+x} + \frac{1}{\ell}\right)^{\beta+1/2}, \\ U_\gamma(x) &:= (1-x^2)^{\gamma/2}\sqrt{w(x)}. \end{aligned}$$

Theorem 4.2 *Let $\alpha, \beta > -1$, $1 \leq p \leq \infty$, $p' = p/(p-1)$ ($= \infty$ if $p = 1$), $-1 < \gamma < 1$, $L \geq 4$, $n \geq 0$ be an integer, $N = 2^n$, $4N + 1 \leq m \leq LN$ be an integer, $\kappa > 0$, and G be a κN-partial frame matrix. Let σ denote either μ or ν_m, and $f \in L^p_\mu$. We write*

$$B_{n,\infty}(G) := \sum_{j=0}^{2N}(j+1)\Big|g_{j,2N} - 2g_{j+1,2N} + g_{j+2,2N}\Big|,$$

and define

$$B_{n,p}(G) := \begin{cases} B_{n,2}(G)^{2/p}B_{n,\infty}(G)^{1-2/p}, & \text{if } 2 < p < \infty, \\ B_{n,p'}(G), & \text{if } 1 \leq p < 2. \end{cases}$$

(a) *If $2 \leq p \leq \infty$,*

$$\|u_{\gamma,N}^{1-2/p}\tau_{2N}(G;f)\|_{p,\sigma} \leq cB_{n,p}(G)\|U_\gamma^{1-2/p}f\|_{p,\mu}. \tag{4.5}$$

(b) *If $1 \leq p \leq 2$,*

$$\|U_\gamma^{2/p'-1}\tau_{2N}(G;f)\|_{p,\sigma} \leq cB_{n,p'}(G)\|u_{\gamma,N}^{2/p'-1}f\|_{p,\mu}. \tag{4.6}$$

(c) *If $P \in W_n$ and $2 \leq p \leq \infty$ then*

$$\|u_{\gamma,N}^{1-2/p}P\|_{p,\mu} \leq cB_{n,p}(G^{[-1]})\|U_\gamma^{1-2/p}\tau_{2N}(G;P)\|_{p,\nu_m}. \tag{4.7}$$

(d) *If $P \in W_n$ and $1 \leq p \leq 2$ then*

$$\|U_\gamma^{2/p'-1}P\|_{p,\mu} \leq cB_{n,p'}(G^{[-1]})\|u_{\gamma,N}^{2/p'-1}\tau_{2N}(G;P)\|_{p,\nu_m}.$$

REMARK. If the matrix G in Theorem 4.2 is generated by a function $g \in BV_0^1$ such that $g(t) = 0$ for all $t \in [0, \kappa]$, then it is not difficult to check that the quantities $B_{n,\infty}(G)$ and hence, $B_{n,p}(G)$ are bounded from above, independently of n. Further, if $[a, b] \subset (-1, 1)$, then (4.5) with $-\min(\alpha, \beta, 1) < \gamma < 1$ shows that

$$\|\tau_{2N}(G; f)\|_{\infty,[a,b]} \le c\|f\|_{\infty,[-1,1]}.$$

Since $\tau_{2N}(G; P) = 0$ for all $P \in V_n$, this leads to

$$\|\tau_{2N}(G; f)\|_{\infty,[a,b]} \le c \inf_{P \in V_n} \|f - P\|_{\infty,[-1,1]}.$$

In particular, it follows from the theorem of Jackson that for an R times continuously differentiable function $\varphi : [-1, 1] \to I\!R$, we have

$$\lim_{N\to\infty} N^R \|\tau_{2N}(G; \varphi)\|_{\infty,[a,b]} = 0. \tag{4.8}$$

It is not possible to choose γ so that the factor $u_{\gamma,N}$ is bounded from below as well. However, if $\max(\alpha, \beta) < 0$, and f is zero outside $[a, b]$, then instead of choosing γ as above, we may choose γ with $-1 < \gamma \le -2\max(\alpha, \beta) - 1$ and obtain

$$\|\tau_{2N}(G; f)\|_{\infty,[-1,1]} \le c\|f\|_{\infty,[a,b]} \le c\|f\|_{\infty,[-1,1]}.$$

Hence, in this case, one gets

$$\|\tau_{2N}(G; f)\|_{\infty,[-1,1]} \le c \inf_{P \in V_n} \|f - P\|_{\infty,[-1,1]}.$$

As before, if φ is R times continuously differentiable on $[-1, 1]$, and equal to zero outside $[a, b]$, then

$$\lim_{N\to\infty} N^R \|\tau_{2N}(G; \varphi)\|_{\infty,[-1,1]} = 0. \tag{4.9}$$

In view of (3.2), the formulas (4.8) (respectively (4.9)), (3.5), and (3.6) demonstrate why the operators $\tau_N(G; f)$ can be used to detect singularities of f. We illustrate this further in Section 5.

PROOF OF THEOREM 4.2. The proof relies upon a connection between the operators τ_ℓ and the operator of arithmetic averages of Jacobi expansions of a function. If σ denotes either μ or one of the measures ν_m, $\ell \ge 0$ is an integer, and $f \in L^1_\sigma$, we write

$$v_\ell(\sigma; f, x) := \int f(t) \left\{ \frac{1}{\ell + 1} \sum_{j=0}^{\ell} \sum_{k=0}^{j} p_k(x)p_k(t) \right\} d\sigma(t).$$

In [18], we have proved that

$$\|u_{\gamma,\ell} v_\ell(\mu, f)\|_{\infty,\mu} \le c\|U_\gamma f\|_{\infty,\mu}, \tag{4.10}$$

and

$$\|u_{\gamma,\ell} v_\ell(\nu_m, f)\|_{\infty,\mu} \le c\|U_\gamma f\|_{\infty,\nu_m}, \tag{4.11}$$

if $2\ell \le m \le L\ell$ for some $L \ge 2$. In this proof, we may assume that $\kappa N \ge 2$, so that $g_{2\ell+1,2\ell} = g_{2\ell+2,2\ell} = 0$ for all integer $\ell \ge 1$. Using summation by parts twice, we obtain the following connection between $\tau_{2\ell}$ and the operators v_j:

$$\tau_{2\ell}(G;f) = \sum_{j=0}^{2\ell} \Delta_{j,2\ell}(G) v_j(\mu;f), \tag{4.12}$$

where

$$\Delta_{j,\nu}(G) := (j+1)(g_{j,\nu} - 2g_{j+1,\nu} + g_{j+2,\nu}), \qquad \nu = 1, 2, \cdots.$$

Since G is a κN-partial frame matrix, the summation in the formula (4.12) for $\tau_{2N}(G;f)$ is actually for $\kappa N \le j \le 2N$. For these values, we have $u_{\gamma,j} \sim u_{\gamma,N}$. Hence, the estimate (4.10) yields

$$\begin{aligned} \|u_{\gamma,N}\tau_{2N}(G;f)\|_{\infty,\nu_m} &\le \|u_{\gamma,N}\tau_{2N}(G;f)\|_{\infty,\mu} \\ &\le \sum_{\kappa N \le j \le 2N} |\Delta_{j,2N}(G)| \|u_{\gamma,j} v_j(\mu;f)\|_{\infty,\mu} \le cB_{n,\infty}(G)\|U_\gamma f\|_{\infty,\mu}. \end{aligned} \tag{4.13}$$

Since $m \ge 2N+1$, we may use the quadrature formula and the Bessel inequality to deduce that

$$\begin{aligned} \|\tau_{2N}(G;f)\|_{2,\nu_m} &= \|\tau_{2N}(G;f)\|_{2,\mu} = \left\{ \sum_{\kappa N \le j \le 2N} g_{j,2N}^2 a_j^2(\mu;f) \right\}^{1/2} \\ &\le B_{n,2}(G)\|f\|_{2,\mu}\ . \end{aligned} \tag{4.14}$$

In view of the Stein-Weiss interpolation theorem (cf. [1], Theorem 5.5.2), (4.13) and (4.14) yield for $2 \le p \le \infty$,

$$\|u_{\gamma,N}^{1-2/p}\tau_{2N}(G;f)\|_{p,\sigma} \le cB_{n,2}(G)^{2/p} B_{n,\infty}(G)^{1-2/p} \|U_\gamma^{1-2/p} f\|_{p,\mu}.$$

This completes the proof of part (a).

Theorem 1 in [17] implies that the estimates (4.10) and (4.11) lead to

$$\|U_\gamma^{-1} v_\ell(\mu;f)\|_{1,\mu} \le c\|u_{\gamma,\ell}^{-1} f\|_{1,\mu},$$

and

$$\|U_\gamma^{-1} v_\ell(\mu;f)\|_{1,\nu_m} \le c\|u_{\gamma,\ell}^{-1} f\|_{1,\mu},$$

for $\ell = 1, 2, \cdots$, and $f \in L^1_\mu$, $2\ell \le m \le L\ell$. From (4.12), this yields as above

$$\|U_\gamma^{-1}\tau_{2N}(G;f)\|_{1,\sigma} \le cB_{n,\infty}(G)\|u_{\gamma,N}^{-1} f\|_{1,\mu}$$

for $\sigma = \mu, \nu_m$. Along with (4.14) and the Stein-Weiss interpolation theorem, this implies the estimate (4.6). The part (b) is now proved.

To prove parts (c) and (d), we define, in this proof only, $H := G^{[-1]}$, and

$$T(f;x) := \sum_{j=0}^{2N} h_{j,2N}\Big\{\int f(t)p_j(t)d\nu_m(t)\Big\}p_j(x).$$

Summing by parts twice, we obtain

$$T(f) = \sum_{j=0}^{2N} \Delta_{j,2N}(H)v_j(\nu_m, f). \tag{4.15}$$

Using (4.11), we get

$$\|u_{\gamma,N}T(f)\|_{\infty,\mu} \le cB_{n,\infty}(H)\|U_\gamma f\|_{\infty,\nu_m}. \tag{4.16}$$

Since $m \ge 4N+1$, the quadrature formula implies that

$$\int p_j p_k d\nu_m = \begin{cases} 1, & \text{if } j = k,\\ 0, & \text{otherwise.}\end{cases}$$

Hence, $\{p_j\}_{j=0}^{2N}$ is an orthonormal system of polynomials in $L^2_{\nu_m}$. Therefore, Parseval identity in L^2_μ, followed by Bessel inequality in $L^2_{\nu_m}$ lead to

$$\begin{aligned}\|T(f)\|^2_{2,\mu} &= \sum_{\kappa N\le j\le 2N} h^2_{j,2N}\Big\{\int f(t)p_j(t)d\nu_m(t)\Big\}^2\\ &\le B^2_{n,2}(H)\sum_{\kappa N\le j\le 2N}\Big\{\int f(t)p_j(t)d\nu_m(t)\Big\}^2\\ &\le B^2_{n,2}(H)\|f\|^2_{2,\nu_m};\end{aligned}$$

i.e.,

$$\|T(f)\|_{2,\mu} \le B_{n,2}(H)\|f\|_{2,\nu_m}. \tag{4.17}$$

Applying the Stein-Weiss interpolation theorem, we obtain from (4.16) and (4.17) that for $2 \le p \le \infty$,

$$\|u_{\gamma,N}^{1-2/p}T(f)\|_{p,\mu} \le cB_{n,p}(H)\|U_\gamma^{1-2/p}f\|_{p,\nu_m}\,. \tag{4.18}$$

If $P \in W_n$ and $f = \tau_{2N}(G;P)$, then (2.6) implies

$$\begin{aligned}P(x) &= \int \tau_{2N}(G;P,t)K_{2N}(H;x,t)d\nu_m(t)\\ &= \int f(t)K_{2N}(H;x,t)d\nu_m(t) = T(f,x)\,.\end{aligned}$$

Hence, (4.18) leads to (4.7). This completes the proof of part (c).

Theorem 1 in [17] and the estimates (4.10) and (4.11) lead to the estimate

$$\|U_\gamma^{-1}v_N(\nu_m;f)\|_{1,\mu} \le c\|u_{\gamma,N}^{-1}f\|_{1,\nu_m}.$$

Therefore, (4.15) implies

$$\|U_\gamma^{-1}T(f)\|_{1,\mu} \leq cB_{n,\infty}(H)\|u_{\gamma,N}^{-1}f\|_{1,\nu_m}. \tag{4.19}$$

The proof of part (d) is now completed using (4.19), (4.17), the Stein-Weiss interpolation theorem, and (4.19) as before. □

5 Numerical experiments

In this section, we present the results of some numerical computations. As a test case, we are interested in detecting the singularities of the function

$$f(x) := (x+1/2)_+^3 + (x-1/2)_+^4.$$

Figure 1 shows the wavelet coefficients with the standard Daubechies wavelets (cf. [4],[2]) ${}_3\psi$ and ${}_4\psi$. Here we have plotted

$${}_L\tau^D(x) = \int_{-1}^{1} f(t)\,{}_L\psi(64t-x)\,dt$$

for $x \in [-1,1]$ and $L = 3, 4$.

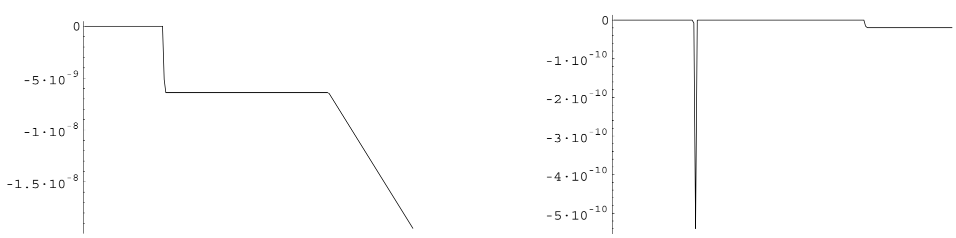

Figure 1: ${}_3\tau^D$ (left) and ${}_4\tau^D$ (right).

The singularity at $-1/2$ (but not the one at $1/2$) can easily be seen by a "spike" in the wavelet coefficients in the case $L = 4$. None of the two singularities produce a similar "spike" in the case $L = 3$.

In our experiments described below, we considered the case $\alpha = \beta = -1/2$. The matrix $G := G_{W,q}$ is generated by the B-spline of order $q+1$, supported on $[\pi, 2\pi]$. We recall that this function is in BV_0^q.

Figure 2 shows the behavior of $\tau_{512}(G_{W,0}; f)$ on $[-1,1]$ and (a zoomed in view on) $[0,1]$ respectively. Even though one detects a sharp rise in the value of the transform near $\pm 1/2$, the lack of localization is also clear, in spite of the high value of N. In contrast, Figure 3 shows the detection of the approximate location of both the singularities already with $N = 128$ and $q = 1, 3$ respectively.

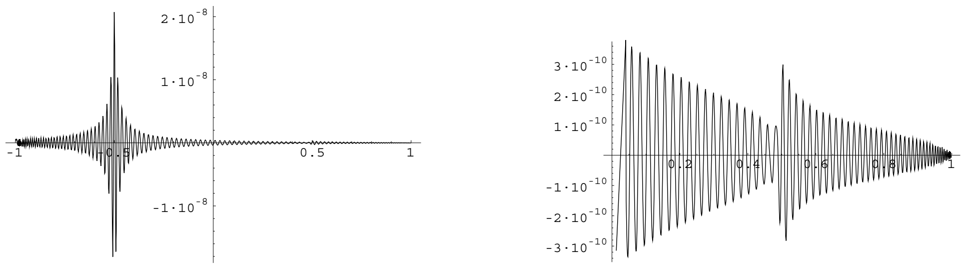

Figure 2: a) and b) $\tau_{512}(G_{W,0}; f, x)$.

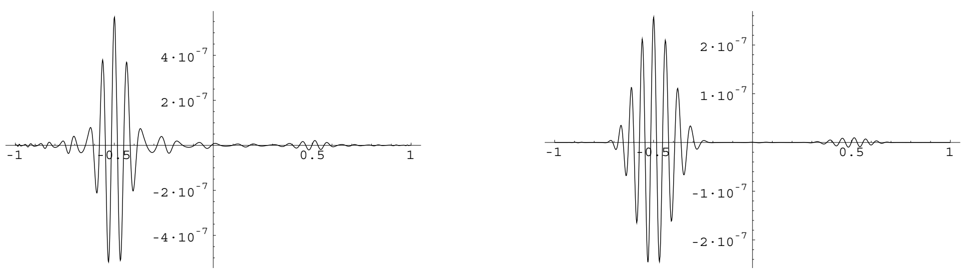

Figure 3: a) $\tau_{128}(G_{W,1}; f, x)$ and b) $\tau_{128}(G_{W,3}; f, x)$.

Figure 4 shows the graph of $\tau_{512}(G_{W,3}; f)$ on $[-1, 1]$ and (a zoomed in view on) $[0, 1]$ respectively, where a good localization is observed. Comparing these with Figure 5, we see that the localization is not significantly improved with $q = 5$, an indication of the importance of the unspecified constants in (3.5) and (3.6).

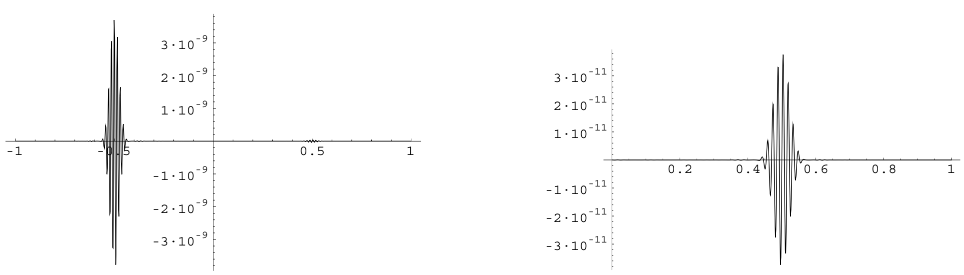

Figure 4: a) and b) $\tau_{512}(G_{W,3}; f, x)$.

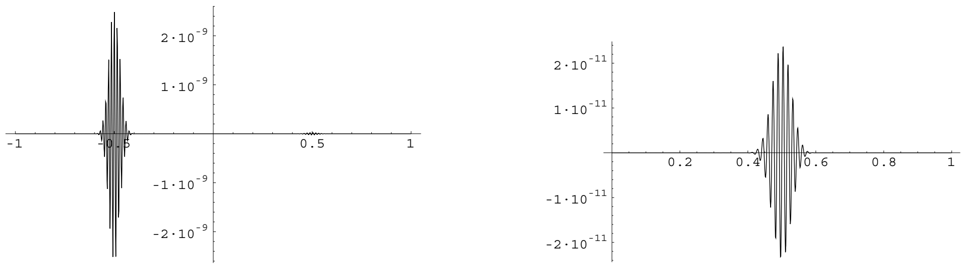

Figure 5: a) and b) $\tau_{512}(G_{W,5}; f, x)$.

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