Matrix–free interpolation on the sphere

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October 5, 2005

Abstract

We study a subspace of bivariate trigonometric polynomials for interpolating functions on the sphere. We give an explicit construction for a system of interpolation nodes, and the corresponding basis for this space, that allows a (discrete) fast Fourier transform–type formula for the interpolant. We prove that the uniform norm of our interpolation operator is of the order $(\log M)^2$, where M is the number of interpolation points. We also construct a minimal quadrature rule for our space (with number of points equal to the dimension of the space), and describe an associated interpolation operator.

Key Words: Sphere; interpolation; discrete orthogonal projection, minimal quadrature.

1 Introduction

Approximation of differentiable functions on the sphere, and analysis of the error in the uniform norm are important ingredients for solving partial differential equations on spherical geometries [4, 5, 6]. Advances in approximation theory on the sphere are required,

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*The research of this author was supported, in part, by the Australian Research Council.

†The research of this author was supported, in part, by grant DMS-0204704 from the National Science Foundation and grant W911NF-04-1-0339 from the U.S. Army Research Office.

for example, to measure earth's atmospheric flow and gravitational potential, to simulate sound waves scattered by spherical geometries, to identify the shape of the scattering objects, and for image reconstructions in cosmology.

One popular strategy for approximation of functions is interpolation. Construction of an interpolatory approximation to an unknown function f defined on a set Ω consists of (i) designing a set of nodes $x_j \in \Omega$ and a class of functions ϕ_j defined on Ω, $j = 1, \cdots, M$ that forms a basis for a space V; and (ii) computing an approximant $I_M f = \sum_{j=1}^{M} a_j(f)\phi_j \in V$ such that $I_M f(x_j) = f(x_j), j = 1, \cdots, M$. The matrix $\mathbf{A} := [\phi_k(x_j)]_{j,k=1,\cdots,M}$ is called the *interpolation matrix*. We say that $\{x_1, \cdots, x_M\}$ is a *set of uniqueness* for V if $\mathbf{A}$ is invertible, or equivalently, for $v \in V$, the fact that $v(x_j) = 0$ for $j = 0, \cdots, M$ implies that $v \equiv 0$. It is clear that the vector $\mathbf{a}$ of coefficients in $I_M f$ satisfies the matrix equation

$$\mathbf{A}\mathbf{a} = [f(x_1), \cdots, f(x_M)]^T . \tag{1.1}$$

Thus, some of the important problems in this theory are: an efficient evaluation of the coefficient vector $\mathbf{a}$ and an estimation of the *Lebesgue constant* (uniform norm) of the operator I_M, defined to be the infimum of all Λ_M for which

$$\sup_{x\in\Omega} |I_M f(x)| \le \Lambda_M \sup_{x\in\Omega} |f(x)|.$$

A classical example is the interpolation on the unit circle, $\{e^{ix} \ : \ x \in [0, 2\pi)\}$. We define the nodes x_j, and the basis ϕ_j by

$$x_j := \frac{2(j+N)\pi}{2N+1}, \qquad \phi_j(x) := \frac{\exp(ijx)}{\sqrt{2N+1}}, \qquad -N \le j \le N.$$

In this case (using a well–known identity for complex exponentials, see (4.1)), the inverse of the resulting interpolation matrix A_{trig} is $\overline{A_{trig}^T}$. Hence, for a function f defined on $[0, 2\pi]$, the solution of the interpolation problem can be written down explicitly as follows.

$$I_{2N+1} f(x) = \sum_{j=-N}^{N} a_j(f)\phi_j(x) = \sum_{\ell=-N}^{N} f(x_\ell) K_N(x, x_\ell), \tag{1.2}$$

where

$$a_j(f) := \sum_{\ell=-N}^{N} f(x_\ell)\overline{\phi_j(x_\ell)}, \quad -N \le j \le N. \qquad K_N(x,y) = \sum_{j=-N}^{N} \phi_j(x)\overline{\phi_j(y)}.$$

We note that each $a_j(f)$ is the discrete Fourier transform (DFT), and hence, for a fixed x, the $\mathcal{O}(N^2)$ summations in $I_{2N+1}f(x)$ can be computed with $\mathcal{O}(N \log N)$ operations, using the Fast Fourier Transform (FFT). The Lebesgue constant in this case is $\mathcal{O}(\log N)$.

Formulas similar to (1.2) hold also in the case of interpolation at the zeros of an arbitrary orthogonal polynomial system on a real interval [3, Section I.4]. This fact is a simple consequence of the Gauss–Jacobi quadrature formula, which, although based on the zeros of a polynomial of degree n, is exact for integrating polynoomials of degree $2n - 1$. In Proposition 2.1 below, we will use the same arguments as in [3, Section I.4] and [10, Lemma 3] to observe that the existence of suitable quadrature formulas imply the existence of interpolation formulas similar to (1.2). One common feature in all these examples is that one does not need to solve a matrix equation to solve the interpolation problem. We say that the interpolation is *matrix-free* when a system of interpolation nodes and basis functions is given explicitly, so that a formula analogous to the FFT-type representation (1.2) defines an interpolation operator.

If Ω is the unit sphere $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$, one may think of a simple strategy for interpolation by considering the coordinate transformation

$$\hat{\mathbf{x}} = \boldsymbol{p}(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}, \qquad \hat{\mathbf{x}} \in \mathbb{S}^2. \tag{1.3}$$

We may then think of a function on $\mathbb{S}^2$ as a periodic function of θ and ϕ, and use bivariate trigonometric interpolation to fit the data. One problem with this straightforward approach is that not all the bivariate trigonometric polynomials are continuous functions of points on $\mathbb{S}^2$. Several authors have considered approximation by special classes of bivariate trigonometric polynomials and special bases for the same. We refer to [2, Sections 18.26, 18.27] and references therein for an interesting account.

In [5], a subspace of bivariate trigonometric polynomials was introduced for facilitating interpolation of functions on the sphere. However, a matrix-free interpolant construction was not discussed in [5]. In this paper, we introduce a different subspace spanned by a basis of the form $Q_n^m(\cos\theta)\exp(im\phi)$, where Q_n^m is the associated Chebyshev or Legendre functions of degree n and order at most 2. These two bases lead to FFT-type matrix-free interpolation, corresponding to two systems of points on the sphere. This fact will be proved using a suitable quadrature formula as a consequence of Proposition 2.1.

We prove that the uniform norm of the interpolation operator, based on one of the systems of points is $\mathcal{O}(\log^2 N)$. The other system of points allows us to construct quadrature formulas that are exact for integration of elements of a higher order space with respect to the area measure on $\mathbb{S}^2$. We demonstrate numerically that the two interpolation operators provide the same degree of approximation for the ten benchmark functions studied in [11] and references therein.

It might be interesting to compare interpolation from our spaces with that from the classical spherical polynomials. First, we observe that our space $\mathcal{X}_N$ defined after Theorem 2.1 below consists of all bivariate trigonometric polynomials of coordinatewise degree at most N that are functions on the sphere, except for only a few of these polynomials. In particular, if $\mathbb{P}_n$ denotes the class of all spherical polynomials of degree at most n (i.e., the class of the restrictions to $\mathbb{S}^2$ of all trivariate algebraic polynomials of total degree at most n), then $\mathbb{P}_{N-1} \subset \mathcal{X}_N$. Moreover, the dimension of $\mathcal{X}_N$ is approximately twice that of $\mathbb{P}_{N-1}$.

The case of spherical polynomial interpolation is very different from our constructions. Sloan [10] has proved that it is not possible to construct a matrix–free interpolation operator onto $\mathbb{P}_N$, for $N \geq 3$, using a quadrature formula that is exact on $\mathbb{P}_{2N}$ ([10, Lemma 3]). Several authors including Südermann [13], Golitschek and Light [7], Xu [17, 18] and Laín Fernández [8] have constructed point systems with various symmetry properties for which a spherical polynomial interpolation matrix is guaranteed to be invertible. To the best of our knowledge, the norm of the resulting spherical polynomial interpolation operator, as well as the computationally important condition number of the interpolation matrix have not been investigated.

The problem of finding points on the sphere for which a polynomial interpolation matrix is invertible and well conditioned has been studied computationally by Sloan and Womersley [11, 12], by imposing such additional constraints as to minimize the Lebesgue constant of the resulting operator or to maximize the determinant of the interpolatory matrix. It is conjectured that the Lebesgue constant of the best quality interpolation operator constructed in [11, 12] is $\mathcal{O}(N)$. In view of the smoothing properties of the surface integral operators arising in applications in elasticity theory, potential theory, and scattering of sound waves from three dimensional smooth spherical geometries, it is important for the projection operators to have Lebesgue constants that are $\mathcal{O}(N^\alpha)$, for some $\alpha < 1$ [4, 5, 6].

In the next section, we discuss the construction of the space, and discuss the degree of approximation, interpolation, and quadrature in this space. Numerical experiments are presented in Section 3, and the proofs of the results in Section 2 are given in Section 4. We would like to thank the referees for their careful reading and many suggestions for an improvement of the first draft of this paper.

2 Main results

In this paper, let C^* denote the class of all continuous functions on $\mathbb{R}^2$ that are 2π-periodic in each of their variables, equipped with the uniform norm $\| \circ \|_\infty^*$. The space of all continuous functions on $\mathbb{S}^2$, equipped with the uniform norm $\| \circ \|_\infty$, will be denoted by $C(\mathbb{S}^2)$. For $f \in C(\mathbb{S}^2)$, let $f^*(\theta, \phi) := f(\boldsymbol{p}(\theta, \phi))$. It is clear that $f \in C(\mathbb{S}^2)$ if and only if $f^* \in C^*$, and f^* satisfies the following symmetry conditions:

$$f^*(-\theta, \phi + \pi) = f^*(\theta, \phi), \qquad \theta, \phi \in \mathbb{R}, \tag{2.1}$$

and

$$f^*(0, \phi),\ f^*(\pi, \phi) \text{ are independent of } \phi. \tag{2.2}$$

We will denote by C° the subspace of C^* comprising of functions satisfying the above two conditions. If $g \in C^\circ$, there exists a unique $f \in C(\mathbb{S}^2)$ such that $g = f^*$. We will write $f = g^\circ$. For integer $N \geq 2$, the space $\mathbb{H}_N$ denotes the class of all bivariate trigonometric polynomials of order at most N; i.e., the span of $\{e^{i\ell\theta} e^{im\phi}\}_{|\ell|, |m| \leq N}$. The space of all univariate algebraic polynomials of degree at most N will be denoted by Π_N.

Our first objective is to obtain a detailed description of $C^\circ \cap \mathbb{H}_N$.

Theorem 2.1 *Let $N \geq 0$ be an integer. We have $T \in C^\circ \cap \mathbb{H}_N$ if and only if*

$$\begin{aligned} T(\theta, \phi) &= S_0(\cos\theta) + \sin^2\theta \sum_{\substack{|\ell| \leq N,\ \ell \neq 0 \\ \ell \text{ even}}} Q_\ell(\cos\theta) \exp(i\ell\phi) + \sin\theta \sum_{\substack{|\ell| \leq N \\ \ell \text{ odd}}} R_\ell(\cos\theta) \exp(i\ell\phi) \qquad (2.3) \\ &= L(\cos\theta) + \sin^2\theta \sum_{\substack{|\ell| \leq N \\ \ell \text{ even}}} Q_\ell(\cos\theta) \exp(i\ell\phi) + \sin\theta \sum_{\substack{|\ell| \leq N \\ \ell \text{ odd}}} R_\ell(\cos\theta) \exp(i\ell\phi), \qquad (2.4) \end{aligned}$$

where $S_0 \in \Pi_N$, $L \in \Pi_1$, *and for* $|\ell| \le N$, $Q_\ell \in \Pi_{N-2}$, $R_\ell \in \Pi_{N-1}$.

In this paper, we are interested in the spaces

$$\mathcal{X}_N^* = \{T \in C^\circ \cap \mathbb{H}_N : T \ \text{ satisfies } \ (2.3) \ \text{ with } \ R_\ell \in \Pi_{N-2}\}, \quad \mathcal{X}_N = \{T^\circ : T \in \mathcal{X}_N^*\}. \tag{2.5}$$

Thus, $\mathcal{X}_N$ comprises those T° for which the terms corresponding to $\sin N\theta$ are absent from the expansion of T. It is easy to see from (2.5) and (2.4) that the dimension of $\mathcal{X}_N$ is given by

$$d_N := 2 + (N-1)(2N+1) = 2N^2 - N + 1. \tag{2.6}$$

If $P \in \mathbb{P}_{N-1}$, then $P^* \in \mathbb{H}_{N-1} \cap C^\circ$. Therefore, $\mathbb{P}_{N-1} \subset \mathcal{X}_N$. The dimension of $\mathbb{P}_{N-1}$ is N^2, which is slightly less than half the dimension of $\mathcal{X}_N$. This is partly because $\mathcal{X}_N$ consists of polynomials with coordinatewise degree at most N, rather than total degree at most N, but clearly, $\mathcal{X}_N$ contains many elements which are not spherical polynomials of any degree.

Next, we describe the construction of a matrix free interpolation scheme for the space $\mathcal{X}_N$. Before launching into the details of these constructions, we formulate in an abstract setting a proposition that shows a close connection between minimal quadrature formulas and interpolation. If Ω is a nonempty set, and V is a vector space of functions on Ω, we recall that a subset $\mathcal{C} \subseteq \Omega$ is a *set of uniqueness* for V if $P \in V$ and $P(x) = 0$ for all $x \in \mathcal{C}$ imply that $P \equiv 0$. The following proposition summarizes standard constructions in the theory of interpolation at the zeros of orthogonal polynomials on a real interval [3, Section I.4], and appears in essence also as [10, Lemma 3].

Proposition 2.1 *Let* $d \ge 1$ *be an integer,* Ω *be a set containing at least* d *elements,* V *be a* d*–dimensional vector space of functions on* Ω, $\mathcal{C} = \{x_1, \cdots, x_d\} \subseteq \Omega$ *be a set of uniqueness for* V, *and* $w_1, \cdots, w_d > 0$. *Let* $\{\Phi_1, \cdots, \Phi_d\}$ *be an orthonormal basis for* V *with respect to the inner product*

$$\langle P, Q\rangle = \sum_{k=1}^{d} w_k P(x_k)\overline{Q(x_k)},$$

and

$$K(x,y) := \sum_{k=1}^{d} \Phi_k(x)\overline{\Phi_k(y)}.$$

Then

$$w_k^{-1} = \sum_{j=1}^{d} |\Phi_j(x_k)|^2, \qquad k = 1, \cdots, d. \tag{2.7}$$

If $\mathcal{Y} := \{y_1, \cdots, y_d\} \subset \mathbb{C}$ *and*

$$g(\mathcal{Y}, x) := \sum_{k=1}^{d} w_k y_k K(x, x_k), \tag{2.8}$$

then $g(\mathcal{Y})$ *is the unique element of* V *satisfying* $g(\mathcal{Y}, x_j) = y_j$, $j = 1, \cdots, d$.

We now describe our matrix free constructions for interpolation from the space $\mathcal{X}_N$. In the sequel, we will assume that $N \geq 2$, and write

$$N_0 = N, \qquad N_m = N-2, \qquad \text{for } m \neq 0. \tag{2.9}$$

First, we describe an orthonormal basis for $\mathcal{X}_N$. In the remainder of this section, let P_n denote the Legendre polynomial of degree n, normalized so that $P_n(1) = 1$. The associated Legendre function of degree n and order m is defined by

$$P_n^m(x) := (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x). \tag{2.10}$$

We recall that the classical spherical harmonics are defined by

$$Y_n^m(\boldsymbol{p}(\theta,\phi)) := \alpha_n^m P_n^{|m|}(\cos\theta)\exp(im\phi), \qquad |m| \leq n, \tag{2.11}$$

where the *Condon-Shortley phase* α_n^m is given by

$$\alpha_n^m := \begin{cases} (-1)^m \sqrt{\dfrac{2n+1}{4\pi}\dfrac{(n-m)!}{(n+m)!}}, & \text{if } m \geq 0, \\ (-1)^m \alpha_n^{|m|}, & \text{if } m < 0. \end{cases} \tag{2.12}$$

The polynomials $\{Y_n^m \ : \ n = 0, \cdots, N, \ |m| \leq n\}$ form an orthonormal basis for the space of all spherical polynomials of degree N, are eigenfunctions of the Laplace–Beltrami operator, and play a very important role in the theory of functions on the sphere. Following the representation (2.3), we define

$$G_n^m(\boldsymbol{p}(\theta,\phi)) := \begin{cases} Y_n^0(\boldsymbol{p}(\theta,0)), & \text{if } m = 0, \\ Y_{n+1}^1(\boldsymbol{p}(\theta,0))\exp(im\phi), & \text{if } m \text{ is odd}, \\ Y_{n+2}^2(\boldsymbol{p}(\theta,0))\exp(im\phi), & \text{if } m \text{ is even}, m \neq 0. \end{cases} \tag{2.13}$$

Using (2.5) and (2.3), it is not difficult to check that $\{G_n^m \ : \ n = 0, \cdots, N_m, |m| \leq N\}$ is an orthonormal basis for $\mathcal{X}_N$:

$$\int_{\mathbb{S}^2} G_n^m(\boldsymbol{p}(\theta,\phi)) G_{n'}^{m'}(\boldsymbol{p}(\theta,\phi)) d(\boldsymbol{p}(\theta,\phi)) = \delta_{(n,m),(n',m')}, \tag{2.14}$$

where $d(\boldsymbol{p}(\theta,\phi))$ is the area element of $\mathbb{S}^2$, given by $\sin\theta d\theta d\phi$. They satisfy the following differential equations on $\mathbb{S}^2$ and in the unit ball in $\mathbb{R}^3$, where Δ_* denotes the Laplace–Beltrami operator, and Δ denotes the Laplacian in three variables.

$$(\Delta_* + n(n+1))\, G_n^m(\boldsymbol{p}(\theta,\phi)) = \begin{cases} 0, & \text{if } m = 0, \\ \dfrac{1-m^2}{\sin^2(\theta)} G_n^m(\boldsymbol{p}(\theta,\phi)), & \text{if } m \text{ is odd}, \\ \dfrac{4-m^2}{\sin^2(\theta)} G_n^m(\boldsymbol{p}(\theta,\phi)), & \text{if } m \text{ is even}, m \neq 0, \end{cases} \tag{2.15}$$

$$\Delta\left(r^n G_n^m(\boldsymbol{p}(\theta,\phi))\right) = \begin{cases} 0, & \text{if } m = 0, \\ \dfrac{1-m^2}{r^2\sin^2(\theta)} G_n^m(\boldsymbol{p}(\theta,\phi)), & \text{if } m \text{ is odd}, \\ \dfrac{4-m^2}{r^2\sin^2(\theta)} G_n^m(\boldsymbol{p}(\theta,\phi)), & \text{if } m \text{ is even}, m \neq 0. \end{cases} \tag{2.16}$$

In accordance with Proposition 2.1, we obtain a quadrature formula based on d_N points that is exact for integration of elements of $\mathcal{X}_{2N-1}$, and describe an interpolation operator for these nodes. This construction is based on the "Gauss–Lobatto quadrature rule", given by the zeros of $P_N^1(\cos\theta)$. Thus, let $\widehat{\theta}_0 < \cdots < \widehat{\theta}_{N-2}$ be points on $[0,\pi]$, such that $P_N^1(\cos\widehat{\theta}_n) = 0$, $n = 0, \cdots, N-2$, $\widehat{\theta}_{N-1} = 0$, $\widehat{\theta}_N = \pi$, and

$$\widetilde{\phi}_m := \frac{2(m+N)\pi}{2N+1}, \qquad -N \le m \le N. \tag{2.17}$$

Let

$$\mathcal{C}_N^q = \{\boldsymbol{p}(\widehat{\theta}_n, \widetilde{\phi}_m) \ : \ n = 0, \cdots, N_m, \ |m| \le N\} \cup \{\hat{\mathbf{n}}, \hat{\mathbf{s}}\}. \tag{2.18}$$

We note that $\mathcal{C}_N^q$ contains exactly d_N elements. We define the corresponding discrete inner product by

$$\begin{aligned} \langle f, g\rangle_N^q \;\; &= \;\; \frac{4\pi}{N(N+1)(2N+1)} \sum_{|m|\le N} \sum_{n=0}^{N-2} \frac{f(\boldsymbol{p}(\widehat{\theta}_n, \widetilde{\phi}_m))\overline{g(\boldsymbol{p}(\widehat{\theta}_n, \widetilde{\phi}_m))}}{[P_N(\cos\widehat{\theta}_n)]^2} \\ &\quad + \frac{4\pi}{N(N+1)} \left\{ f(\hat{\mathbf{n}})\overline{g(\hat{\mathbf{n}})} + f(\hat{\mathbf{s}})\overline{g(\hat{\mathbf{s}})} \right\}. \end{aligned} \tag{2.19}$$

With

$$(g_n^m)^{-1} := \begin{cases} 2 + 1/N, & \text{if } m = 0 \text{ and } n = N, \\ 2 - 3/(N+2), & \text{if } 0 \ne m \text{ even and } n = N-2, \\ 1, & \text{otherwise,} \end{cases} \tag{2.20}$$

we will show in the proof of Theorem 2.2 below that the functions $\{\sqrt{g_n^m} G_n^m\}$ form an orthonormal basis for $\mathcal{X}_N$, orthonormalized with respect to the inner product $\langle \circ, \circ \rangle_N^q$. In view of Proposition 2.1, a matrix–free interpolation operator can now be defined easily with the kernel

$$\mathcal{K}_N^q(\hat{\mathbf{x}}, \hat{\mathbf{y}}) := \sum_{|m|\le N} \sum_{n=0}^{N_m} g_n^m G_n^m(\hat{\mathbf{x}}) G_n^m(\hat{\mathbf{y}}). \tag{2.21}$$

The following theorem summarizes some facts regarding our constructions so far.

Theorem 2.2 *Let $N \ge 2$ be an integer. For $T \in \mathcal{X}_{2N-1}$, we have*

$$\begin{aligned} &\int_{\mathbb{S}^2} T\left(\boldsymbol{p}(\theta,\phi)\right) d(\boldsymbol{p}(\theta,\phi)) \\ &\quad = \frac{4\pi}{N(N+1)(2N+1)} \sum_{|m|\le N} \sum_{n=0}^{N-2} \frac{T\left(\boldsymbol{p}(\widehat{\theta}_n, \widetilde{\phi}_m)\right)}{\left[P_N(\cos\widehat{\theta}_n)\right]^2} + \frac{4\pi}{N(N+1)} \left\{T(\hat{\mathbf{n}}) + T(\hat{\mathbf{s}})\right\}. \end{aligned} \tag{2.22}$$

For $f \in C(\mathbb{S}^2)$, let

$$\begin{aligned} \mathcal{G}_N f(\hat{\mathbf{x}}) :=& \frac{4\pi}{N(N+1)(2N+1)} \sum_{|m|\le N} \sum_{n=0}^{N-2} \left[P_N(\cos\widehat{\theta}_n)\right]^{-2} f\left(\boldsymbol{p}(\widehat{\theta}_n, \widetilde{\phi}_m)\right) \mathcal{K}_N^q\left(\hat{\mathbf{x}}, \boldsymbol{p}(\widehat{\theta}_n, \widetilde{\phi}_m)\right) \\ &+ \frac{4\pi}{N(N+1)} \left\{ f(\hat{\mathbf{n}}) \mathcal{K}_N^q(\hat{\mathbf{x}}, \hat{\mathbf{n}}) + f(\hat{\mathbf{s}}) \mathcal{K}_N^q(\hat{\mathbf{x}}, \hat{\mathbf{s}}) \right\}. \end{aligned} \tag{2.23}$$

Then $\mathcal{G}_N f$ is the unique element of $\mathcal{X}_N$ that satisfies $\mathcal{G}_N f(\xi) = f(\xi)$ for $\xi \in \mathcal{C}_N^q$.

We believe that our proof of (2.31) below can be adapted to show that the Lebesgue constant of $\mathcal{G}_N$ is $\mathcal{O}(\sqrt{N})$, but hope to report on this in near future, along with similar constructions for spheres embedded in Euclidean spaces of dimensions higher than 3. On the other hand, we remark that a direct application of the representation (2.23) leads only an estimate $\mathcal{O}(N)$, using standard techniques as in [14].

Next, we describe another construction for the nodes that leads to a matrix free interpolation operator with uniform norm $\mathcal{O}((\log N)^2)$. Let

$$\widetilde{\theta}_n := \frac{(n+1)\pi}{N}, \ n = 0, \cdots, N-2, \qquad \widetilde{\theta}_{N-1} := 0, \ \widetilde{\theta}_N := \pi, \tag{2.24}$$

and $\widetilde{\phi}_m$ be defined by (2.17). For the points of interpolation, we choose the set

$$\mathcal{C}_N^i :== \left\{ \boldsymbol{p}(\widetilde{\theta}_n, \widetilde{\phi}_m) \ : n = 0, \cdots, N-2, \ m = -N, \cdots, N \right\} \cup \{\hat{\mathbf{n}}, \hat{\mathbf{s}}\},$$

where $\hat{\mathbf{n}}, \hat{\mathbf{s}}$ are the north and south poles respectively.

To describe a basis for $\mathcal{X}_N$, we recall that the formula $T_n(\cos\theta) = \cos n\theta$ defines a unique polynomial T_n of degree n, called the Chebyshev polynomial (of first kind). Analogous to the associated Legendre functions, we define the associated Chebyshev functions by

$$C_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} T_n(x).$$

Our basis functions are defined by

$$Z_n^m(\boldsymbol{p}(\theta, \phi)) := \begin{cases} C_n^0(\cos\theta), & \text{if } m = 0, \\ C_{n+1}^1(\cos\theta) e^{im\phi}, & \text{if } m \text{ is odd}, \\ C_{n+2}^2(\cos\theta) e^{im\phi}, & \text{if } m \neq 0, \ m \text{ is even}. \end{cases} \tag{2.25}$$

These functions are not orthogonal with respect to the standard L^2 inner product on $\mathbb{S}^2$. However, we observe that an application of Proposition 2.1 requires only the orthogonality of the functions with respect to a discrete inner product based at the points in question. Accordingly, we define

$$\langle f, g \rangle_N := \frac{2\pi^2}{N(2N+1)} \sum_{|m| \le N} \sum_{n=0}^{N-2} f(\boldsymbol{p}(\widetilde{\theta}_n, \widetilde{\phi}_m)) \overline{g(\boldsymbol{p}(\widetilde{\theta}_n, \widetilde{\phi}_m))} + \frac{2\pi^2}{2N} \left\{ f(\hat{\mathbf{n}}) \overline{g(\hat{\mathbf{n}})} + f(\hat{\mathbf{s}}) \overline{g(\hat{\mathbf{n}})} \right\}. \tag{2.26}$$

With the normalization factors

$$(z_{n,N}^m)^{-1} := \begin{cases} 2\pi^2, & \text{if } m = 0, \ n = 0, N, \\ \pi^2, & \text{if } m = 0, \ n = 1, \cdots, N, \\ (n+1)^2 \pi^2, & \text{if } m \text{ odd}, \ n = 0, \cdots, N-2, \\ (n+2)^4 \pi^2 (1 - (n+2)^{-2}), & \text{if } m \text{ even}, \ m \neq 0, \ n = 0, \cdots, N-3, \\ 2(n+2)^4 \pi^2 (1 - (n+2)^{-1}), & \text{if } m \text{ even}, \ m \neq 0, \ n = N-2, \end{cases} \tag{2.27}$$

we will show in the proof of Theorem 2.3 below that the functions $\{\sqrt{z_n^m} Z_n^m\}$ is an orthonormal basis for $\mathcal{X}_N$ with respect to this inner product. The interpolation operator can now be described using the kernel

$$\mathcal{K}_N(\hat{\mathbf{x}}, \hat{\mathbf{y}}) := \sum_{|m| \le N} \sum_{n=0}^{N_m} z_{n,N}^m Z_n^m(\hat{\mathbf{x}}) Z_n^m(\hat{\mathbf{y}}). \tag{2.28}$$

Theorem 2.3 *Let $N \ge 2$ be an integer, $f \in C(\mathbb{S}^2)$, and*

$$\mathcal{I}_N f(\hat{\mathbf{x}}) := \frac{2\pi^2}{N(2N+1)} \sum_{|m| \le N} \sum_{n=0}^{N-2} f\left(\boldsymbol{p}(\widetilde{\theta}_n, \widetilde{\phi}_m)\right) \mathcal{K}_N\left(\hat{\mathbf{x}}, \boldsymbol{p}(\widetilde{\theta}_n, \widetilde{\phi}_m)\right)$$
$$+ \frac{\pi^2}{N}\left\{f(\hat{\mathbf{n}})\mathcal{K}_N(\hat{\mathbf{x}}, \hat{\mathbf{n}}) + f(\hat{\mathbf{s}})\mathcal{K}_N(\hat{\mathbf{x}}, \hat{\mathbf{s}})\right\}. \tag{2.29}$$

Then $\mathcal{I}_N f$ is the unique element of $\mathcal{X}_N$ that satisfies $\mathcal{I}_N f(\xi) = f(\xi)$ for each $\xi \in \mathcal{C}_N^i$.

In the next theorem, we discuss the approximation properties of the operator $\mathcal{I}_N$. If $V \subset C^*$, we define

$$\text{dist } (f, V) := \inf_{P \in V} \|f - P\|_\infty^*, \qquad f \in C^*, \tag{2.30}$$

with a similar definition for dist (f, V) when $f \in C(\mathbb{S}^2)$ and $V \subset C(\mathbb{S}^2)$. Throughout this paper, c denotes a generic constant, independent of N. Its value may be different at different occurrences, even within a single formula.

Theorem 2.4 *For integer $N \ge 2$ and $f \in C(\mathbb{S}^2)$, we have*

$$\|\mathcal{I}_N f\|_\infty \le c(\log N)^2 \|f\|_\infty, \tag{2.31}$$

and

$$\|f - \mathcal{I}_N f\|_\infty \le c(\log N)^2 \text{ dist } (f^*, \mathbb{H}_{N-1}) \le c(\log N)^2 \text{ dist } (f, \mathbb{P}_{N-1}). \tag{2.32}$$

The connection between the smoothness of f^* and the superalgebraic convergence of dist $(f^*, \mathbb{H}_{N-1})$ to zero has been investigated in detail in classical approximation theory [16]. We note that since the space $\mathbb{P}_{N-1}^*$ corresponding to spherical polynomials of degree at most $N-1$ is contained in $\mathbb{H}_{N-1}$, dist $(f^*, \mathbb{H}_{N-1}) \le$ dist $(f^*, \mathbb{P}_{N-1}^*) =$ dist $(f, \mathbb{P}_{N-1})$.

3 Numerical experiments

We demonstrate the quality of our interpolatory operators by computing estimates of the uniform norm errors $\|\mathcal{I}_N f_i - f_i\|_\infty$ and $\|\mathcal{G}_N f_i - f_i\|_\infty$, $i = 1, \cdots 10$, where $f_1, \cdots, f_{10}$ are the benchmark functions on the sphere (see [11] and references therein), defined for $\hat{\mathbf{x}} = (x_1, x_2, x_3) \in \mathbb{S}^2$, by

$$f_1(\hat{\mathbf{x}}) = x_1 x_2 x_3, \qquad f_2(\hat{\mathbf{x}}) = \exp(x_1), \qquad f_3(\hat{\mathbf{x}}) = \exp(x_1 + x_2 + x_3)/10,$$

$$
\begin{aligned}
f_4(\hat{\mathbf{x}}) &= -5\sin(1+10x_3), \;\; f_5(\hat{\mathbf{x}}) = \sum_{i=1}^{5} \alpha_i \exp(-\beta_i \mathrm{dist}(\hat{\mathbf{x}}, \hat{\mathbf{y}}_i)^{2\gamma_i}), \;\; f_6(\hat{\mathbf{x}}) = \frac{1}{101-100x_3}, \\
f_7(\hat{\mathbf{x}}) &= |x_1| + |x_2| + |x_3|, \quad f_8(\hat{\mathbf{x}}) = \frac{1}{f_7(\hat{\mathbf{x}})}, \qquad f_9(\hat{\mathbf{x}}) = \frac{\sin^2(1+f_7(\hat{\mathbf{x}}))}{10},
\end{aligned}
$$

and

$$
f_{10}(\hat{\mathbf{x}}) = \begin{cases} \cos^2\left(\frac{3\pi}{2}\mathrm{dist}(\hat{\mathbf{x}}, \boldsymbol{p}(\pi/4, 5\pi/4)\right), & \text{if } \mathrm{dist}(\hat{\mathbf{x}}, \boldsymbol{p}(\pi/4, 5\pi/4) < 1/3, \\ 0, & \text{if } \mathrm{dist}(\hat{\mathbf{x}}, \boldsymbol{p}(\pi/4, 5\pi/4) \geq 1/3, \end{cases}
$$

where the $\mathrm{dist}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = \cos^{-1}(\hat{\mathbf{x}} \cdot \hat{\mathbf{y}})$ is the geodesic distance between two points $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in \mathbb{S}^2$, and the parameters $\hat{\mathbf{y}}_i = (y_{i,1}, y_{i,2}, y_{i,3})$ and $\alpha_i, \beta_i, \gamma_i,\ i = 1, \cdots, 5$ in the test function f_5 are in Table 1. The functions $f_i,\ i = 1 \cdots, 6$ are analytic; f_7, f_8, f_9 are continuous, but not continuously differentiable, and the locally supported cosine cap function f_{10} is once (but not twice) continuously differentiable function on $\mathbb{S}^2$.

i	$y_{i,1}$	$y_{i,2}$	$y_{i,3}$	α_i	β_i	γ_i
1	0	0	1	2	5	1
2	0.932039	0	0.362358	0.5	7	1
3	-0.362154	0.619228	0.696707	-2	6	2
4	0.904035	0.279651	-0.323290	-2	5	1
5	-0.0479317	-0.424684	-0.904072	0.2	2.1	1

Table 1: Parameters for f_5

We computed approximations of these functions in $\mathcal{X}_N$, using the interpolation operators $\mathcal{I}_N$ and $\mathcal{G}_N$. For each $i = 1, \cdots, 10$, the uniform norm errors $\|f_i - \mathcal{I}_N f_i\|_\infty$ and $\|f_i - \mathcal{G}_N f_i\|_\infty$ were estimated by taking the maximum of errors over 12,000 points on the sphere.

The results in Tables (2)–(5) show that both $\mathcal{I}_N f_i$ and $\mathcal{G}_N f_i$ provide a similar quality of approximation of f_i, $i = 1, \cdots 10$. The tables clearly demonstrate also that our operators yield a better reconstruction of these functions with their various smoothness properties than the (matrix–dependent) interpolatory and non–interpolatory polynomial approximations of the same functions discussed in [11, p. 222-223]. (The functions denoteed here by f_5 and f_6 are denoted in [11, p. 222-223] by f_6 and f_5 respectively.) We note again that the construction of the interpolatory operators $\mathcal{I}_N f_i,\ \mathcal{G}_N f_i,\ i = 1, \cdots 10$, does not require a numerical solution of any linear system of equations. Moreover, Theorem 2.4 shows that the Lebesgue constant of $\mathcal{I}_N$ is $\mathcal{O}\left((\log N)^2\right)$.

N	$\|f_1-\mathcal{I}_Nf_1\|_\infty$	$\|f_2-\mathcal{I}_Nf_2\|_\infty$	$\|f_3-\mathcal{I}_Nf_3\|_\infty$	$\|f_4-\mathcal{I}_Nf_4\|_\infty$	$\|f_5-\mathcal{I}_Nf_5\|_\infty$
4	1.5193e-15	1.0193e-03	5.4374e-02	1.1526e+01	7.0812e-01
8	1.5193e-15	2.1948e-08	1.9515e-05	6.7137e+00	1.3019e-01
16	1.5193e-15	8.2158e-15	2.2205e-14	7.1530e-03	2.6437e-03
32	1.5193e-15	1.5543e-14	2.2205e-14	1.0658e-13	5.9918e-07
64	1.5193e-15	4.4631e-14	2.0872e-14	2.3714e-13	2.1585e-11

Table 2: **Error in approximation of f_i by $\mathcal{I}_Nf_i,\ i=1,\cdots,5$.**

N	$\|f_1-\mathcal{G}_Nf_1\|_\infty$	$\|f_2-\mathcal{G}_Nf_2\|_\infty$	$\|f_3-\mathcal{G}_Nf_3\|_\infty$	$\|f_4-\mathcal{G}_Nf_4\|_\infty$	$\|f_5-\mathcal{G}_Nf_5\|_\infty$
4	1.6653e-16	1.2257e-03	6.5224e-02	1.0562e+01	7.5962e-01
8	1.6653e-16	3.4587e-08	3.0874e-05	5.6223e+00	1.0930e-01
16	1.9429e-16	5.2180e-14	1.4522e-13	5.6956e-03	2.2566e-03
32	3.6082e-16	5.4622e-14	1.2346e-13	2.2027e-13	6.4830e-07
64	3.0531e-16	1.7064e-13	5.3824e-13	6.6702e-13	1.4792e-11

Table 3: **Error in approximation of f_i by $\mathcal{G}_Nf_i,\ i=1,\cdots,5$.**

4 Proofs

PROOF OF THEOREM 2.1. It is clear that any expression of the form on the right hand side of (2.4) is in $C^\circ\cap\mathbb{H}_N$. Let $T\in C^\circ\cap\mathbb{H}_N$, and

$$T(\theta,\phi)=:\sum_{|\ell|,|k|\le N}a_{\ell,k}e^{ik\theta}e^{i\ell\phi}.$$

Then, recalling that for integers $k\ge 0$, $\cos k\theta$ and $\sin k\theta/\sin\theta$ are polynomials in $\cos\theta$ of degree k and $k-1$ respectively, we obtain

$$\begin{aligned} T(\theta,\phi) &= (1/2)(T(\theta,\phi)+T(-\theta,\phi+\pi)) \\ &= \sum_{|\ell|,|k|\le N}a_{\ell,k}\frac{e^{ik\theta}+(-1)^\ell e^{-ik\theta}}{2}e^{i\ell\phi} \\ &= \sum_{\substack{|\ell|\le N\\ \ell \text{ even}}}\sum_{k=0}^{N}a_{\ell,k}\cos k\theta e^{i\ell\phi}+i\sum_{\substack{|\ell|\le N\\ \ell \text{ odd}}}\sum_{k=1}^{N}a_{\ell,k}\sin k\theta e^{i\ell\phi} \\ &= \sum_{\substack{|\ell|\le N\\ \ell \text{ even}}}S_\ell(\cos\theta)e^{i\ell\phi}+\sin\theta\sum_{\substack{|\ell|\le N\\ \ell \text{ odd}}}R_\ell(\cos\theta)e^{i\ell\phi}, \end{aligned}$$

where $S_\ell\in\Pi_N$ and $R_\ell\in\Pi_{N-1}$, $|\ell|\le N$. There exist $Q_\ell\in\Pi_{N-2}$, $L_\ell\in\Pi_1$ such that $S_\ell(\cos\theta)=(1-\cos^2\theta)Q_\ell(\cos\theta)+L_\ell(\cos\theta)$. Since $T(0,\phi)$ and $T(\pi,\phi)$ are independent of ϕ, we have $S_\ell(\pm 1)=0$ if $\ell\ne 0$. Therefore, $L_\ell=0$ for $|\ell|\le N$, ℓ even, and $\ell\ne 0$. □

PROOF OF PROPOSITION 2.1. Let $\mathbf{A}=[a_{j,k}]$ be the collocation matrix defined by $a_{j,k}=\Phi_k(x_j)$. Since $\mathcal{C}$ is a set of uniqueness, $\mathbf{A}$ is invertible. Also, the orthonormality

N	$\|f_6 - \mathcal{I}_N f_6\|_\infty$	$\|f_7 - \mathcal{I}_N f_7\|_\infty$	$\|f_8 - \mathcal{I}_N f_8\|_\infty$	$\|f_9 - \mathcal{I}_N f_9\|_\infty$	$\|f_{10} - \mathcal{I}_N f_{10}\|_\infty$
8	3.4509e-01	1.0125e-02	9.6995e-02	8.0456e-03	1.0205e-01
16	1.0003e-01	5.2213e-03	5.1501e-02	4.8926e-03	1.6087e-01
32	1.0757e-02	2.9341e-03	2.6055e-02	2.5565e-03	2.3648e-03
64	1.1523e-04	1.3091e-03	1.3079e-02	9.9133e-04	2.5106e-04
128	1.3881e-08	6.5481e-04	6.5467e-03	4.9564e-04	3.6030e-05

Table 4: **Error in approximation of f_i by $\mathcal{I}_N f_i$, $i = 6, \cdots, 10$.**

N	$\|f_6 - \mathcal{G}_N f_6\|_\infty$	$\|f_7 - \mathcal{G}_N f_7\|_\infty$	$\|f_8 - \mathcal{G}_N f_8\|_\infty$	$\|f_9 - \mathcal{G}_N f_9\|_\infty$	$\|f_{10} - \mathcal{G}_N f_{10}\|_\infty$
8	3.8245e-01	1.0690e-02	1.0331e-01	8.4990e-03	1.5496e-01
16	1.3193e-01	5.7501e-03	5.6980e-01	5.0567e-03	2.7559e-02
32	1.2847e-02	2.9679e-03	2.9636e-02	2.5629e-03	6.4724e-03
64	1.2910e-04	1.4640e-03	1.4645e-02	1.1080e-03	8.3897e-04
128	1.4840e-08	7.2705e-04	7.2712e-03	5.5022e-04	1.8309e-04

Table 5: **Error in approximation of f_i by $\mathcal{G}_N f_i$, $i = 6, \cdots, 10$.**

of $\{\Phi_k\}$ is equivalent to the statement that $\overline{\mathbf{A}^T} D \mathbf{A} = I$, where $D = \mathrm{diag}\,[w_1, \cdots, w_d]$. This leads to $\overline{\mathbf{A}^T} D = \mathbf{A}^{-1}$, and $D^{-1} = \mathbf{A}\overline{\mathbf{A}^T}$. This is (2.7). With $\mathbf{y} = [y_1, \cdots, y_d]^T$, and $\mathbf{b}(x) = [\Phi_1(x), \cdots, \Phi_d(x)]^T$, we have

$$g(\mathcal{Y}, x) = \mathbf{b}(x)^T \overline{\mathbf{A}^T} D \mathbf{y} = \mathbf{b}(x)^T \mathbf{A}^{-1} \mathbf{y}.$$

This completes the proof. □

The next lemma describes some sets of uniqueness for $\mathcal{X}_N$.

Lemma 4.1 *Let $N \geq 2$ be an integer, $\theta_0, \cdots, \theta_{N-2}$ be distinct points in $(0, \pi)$, $\theta_{N-1} = 0$, $\theta_N = \pi$, and ϕ_m, $|m| \leq N$ be distinct points on $[0, 2\pi)$. Then the set*

$$\mathcal{C} = \{\boldsymbol{p}(\theta_n, \phi_m) \ : \ n = 0, \ldots, N-2, \ |m| \leq N\} \cup \{\hat{\mathbf{n}}, \hat{\mathbf{s}}\}$$

consists of d_N distinct elements, and is a set of uniqueness for $\mathcal{X}_N$.

Proof. Let $T \in \mathcal{X}_N^*$, and for $|\ell| \leq N$, $Q_\ell, R_\ell \in \Pi_{N-2}$, $Q_0 \in \Pi_N$ be found so that

$$T(\theta, \phi) = Q_0(\cos\theta) + \sin^2\theta \sum_{\substack{|\ell| \leq N, \ \ell \neq 0 \\ \ell \text{ even}}} Q_\ell(\cos\theta) \exp(i\ell\phi) + \sin\theta \sum_{\substack{|\ell| \leq N \\ \ell \text{ odd}}} R_\ell(\cos\theta) \exp(i\ell\phi),$$

and $T^\circ(\boldsymbol{p}(\theta_n, \phi_m)) = 0$, $n = 0, \cdots, N$, $|m| \leq N$. For any n, $T(\theta_n, \circ)$ is a trigonometric polynomial of degree at most N. Since this polynomial has $2N + 1$ distinct zeros, $\{\phi_m\}_{|m| \leq N}$, it must be identically zero. This yields $Q_0(\cos\theta_n) = 0$ for $n = 0, \cdots, N$, and $Q_\ell(\cos\theta_n) = R_\ell(\cos\theta_n) = 0$, $n = 0, \cdots, N-2$, $\ell \neq 0$. Since $Q_0 \in \Pi_N$ and $Q_\ell, R_\ell \in \Pi_{N-2}$ for $\ell \neq 0$, this implies that each of these polynomials is identically equal to zero. Thus, $T \equiv 0$, and hence, $T^\circ \equiv 0$. □

We are now in a position to prove Theorem 2.2 and Theorem 2.3. We observe that for any integer $M \geq 1$, and integer k,

$$\frac{1}{M}\sum_{m=0}^{M-1}\exp(2\pi ikm/M) = \begin{cases} 1, & \text{if } k = 0 \bmod M, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

In particular,

$$\int_0^{2\pi} e^{ik\phi}d\phi = \frac{2\pi}{M}\sum_{m=0}^{M-1}\exp(2\pi ikm/M), \qquad |k| \leq M-1. \tag{4.2}$$

Proof of Theorem 2.2. It is well known [1, (25.4.32), p. 888] that for $P \in \Pi_{2N-1}$, we have

$$\begin{aligned} &\int_0^{\pi} P(\cos\theta)\sin\theta d\theta \\ &= \frac{2}{N(N+1)}\sum_{n=0}^{N-2}\frac{P(\cos\widehat{\theta}_n)}{\left[P_N(\cos\widehat{\theta}_n)\right]^2} + \frac{2}{N(N+1)}\left\{P(\cos\widehat{\theta}_0) + P(\cos\widehat{\theta}_N)\right\}. \end{aligned} \tag{4.3}$$

(The notation in [1] is somewhat different.) The equation (2.22) follows from the definition of the space $\mathcal{X}_N$, and the quadrature formulas (4.2) and (4.3).

In this proof only, we adopt the notation P_n^0 for the Legendre polynomial P_n, and write

$$F_n^m(x) := \begin{cases} \alpha_n^0\ P_n^0(x), & \text{if } m = 0, \\ \alpha_{n+1}^1\ P_{n+1}^1(x), & \text{if } m \text{ is odd}, \\ \alpha_{n+2}^2\ P_{n+2}^2(x), & \text{if } m \text{ is even}, m \neq 0, \end{cases} \tag{4.4}$$

and Let $-N \leq m, j \leq N$, and $n = 0, \cdots, N_m$, $l = 0, \cdots, N_j$. Using (2.19), (2.13), (4.4) and (4.1) we get

$$\begin{aligned} \langle G_n^m, G_l^j\rangle_N^q &= \delta_{m,j}\left[\frac{4\pi}{N(N+1)}\sum_{q=0}^{N-2}\frac{F_n^m(\cos\widehat{\theta}_q)F_l^j(\cos\widehat{\theta}_q)}{\left[P_N^0(\cos\widehat{\theta}_q)\right]^2}\right. \\ &\quad \left. + \frac{4\pi}{N(N+1)}\left[F_n^m(-1)F_l^j(-1) + F_n^m(1)F_l^j(1)\right]\right. . \end{aligned} \tag{4.5}$$

Let $j = m$. In view of (2.9) and (4.4), $F_n^m(x)F_l^m(x)$ is a polynomial of degree at most $2N-1$ on $[-1,1]$, for all $l = 0, \cdots, N_m$, $0 \leq n < N_m$, and also for $n = N_m$, if m is odd. Since the associated Legendre functions are orthonormal, (4.3) and (4.5) show that

$$\begin{aligned} \langle G_n^m, G_l^m\rangle_N^q &= 0, \quad \text{if } n \neq l, \\ \langle G_n^m, G_n^m\rangle_N^q &= 1, \quad \text{if } n \neq N_m, \qquad \langle G_n^m, G_n^m\rangle_N^q = 1, \quad \text{if } n = N_m \text{ and } m \text{ odd.} \end{aligned}$$

Thus, we have shown that

$$\langle G_n^m, G_l^j\rangle_N^q = (g_n^m)^{-1}\delta_{n,l}\delta_{m,j}, \tag{4.6}$$

for all n, m, l, j in question, except for the case when m is even, and $n = l = N_m$.

Next, let $m = 0$, and $n = l = N$. Using (4.5), (2.9) and (4.4), we have

$$\begin{aligned}\langle G_N^0, G_N^0\rangle_N^q &= \frac{2N+1}{N(N+1)}\left(\sum_{q=0}^{N-2}\frac{\left[P_N^0(\cos\widehat{\theta}_q)\right]^2}{\left[P_N^0(\cos\widehat{\theta}_q)\right]^2} + \left[P_N^0(-1)\right]^2 + \left[P_N^0(1)\right]^2\right)\\ &= \frac{2N+1}{N(N+1)}(N-1+1+1) = 2 + 1/N. \qquad (4.7)\end{aligned}$$

Finally, let m be even, $m \neq 0$, and $l = n = N_m = N-2$. In view of (4.4) and (2.12), we see that

$$[F_n^m(x)]^2 = \frac{2N+1}{4\pi}\frac{(N-2)!}{(N+2)!}\left[P_N^2(x)\right]^2.$$

Since $P_N^2(x) = (1-x^2)\dfrac{d^2}{dx^2}P_N^0(x)$, and P_N^0 is a solution of the Legendre differential equation [15, (4.2.1) with $\alpha = \beta = 0$], we have $P_N^2(x) = 2x\dfrac{d}{dx}P_N^0(x) - N(N+1)P_N^0(x)$. Since the Gauss-Lobatto quadrature points $x_q := \cos\widehat{\theta}_q \in (-1,1)$, $q = 0, \cdots, N-2$ are zeros of the derivative of P_N^0, we have $P_N^2(x_q) = -N(N+1)P_N^0(x_q)$, for $q = 0, \cdots, N-2$. We substitute this expression for $P_N^2(x_q)$ in (4.5), and recall that $P_N^2(\pm 1) = 0$ to obtain for even m

$$\langle G_{N-2}^m, G_{N-2}^m\rangle_N^q = \frac{2N+1}{(N-1)(N+2)}\left(\sum_{q=0}^{N-2}\frac{\left[P_N^0(x_q)\right]^2}{\left[P_N^0(x_q)\right]^2}\right) = \frac{2N+1}{(N+2)} = 2 - 3/(N+2). \qquad (4.8)$$

The equations (4.6), (4.7), (4.8), and (2.20) show that $\{\sqrt{g_n^m}G_n^m\}$ is a basis for $\mathcal{X}_N$, orthonormalized with respect to the inner product $\langle\circ,\circ\rangle_N^q$. Lemma 4.1 shows that $\mathcal{C}_N^q$ is a set of uniqueness for $\mathcal{X}_N$. Therefore, the proof of Theorem 2.2 is complete in view of Proposition 2.1. □

The proof of Theorem 2.3 is very similar to that of Theorem 2.2, although the details are somewhat different. First, we obtain an analogue of (4.3).

Lemma 4.2 *Let $N \geq 1$ be an integer. For $P \in \Pi_{2N-1}$, we have*

$$\int_0^\pi P(\cos\theta)d\theta = \frac{\pi}{N}\sum_{j=0}^{N-2}P(\cos\widetilde{\theta}_j) + \frac{\pi}{2N}\left\{P(\cos\widetilde{\theta}_{N-1}) + P(\cos\widetilde{\theta}_N)\right\}. \qquad (4.9)$$

Proof. We observe that for $k = 0, \cdots, 2N-1$

$$\begin{aligned}&\frac{1}{N}\sum_{j=0}^{N-2}\cos k\widetilde{\theta}_j + \frac{1}{2N}\left\{\cos k\widetilde{\theta}_{N-1} + \cos k\widetilde{\theta}_N\right\}\\ &= \frac{1}{2N}\left\{\sum_{\ell=1}^{N-1}\exp(\pi ik\ell/N) + \exp(\pi ik(2N-\ell)/N)\right\} + \frac{1}{2N}\{\cos k(0) + \cos k\pi\}\\ &= \frac{1}{2N}\sum_{\ell=0}^{2N-1}\exp(2\pi ik\ell/(2N)) = \delta_{k,0}.\end{aligned}$$

This implies (4.9) when $P(\cos\theta) = \cos k\theta$, $k = 0, \cdots, 2N-1$. $\square$

In the remainder of this section, we will write

$$\langle f, g\rangle_N^i := \frac{\pi}{N}\sum_{j=0}^{N-2} f(\cos\widetilde{\theta}_j)\overline{g(\cos\widetilde{\theta}_j)} + \frac{\pi}{2N}\left\{f(1)\overline{g(1)} + f(-1)\overline{g(-1)}\right\}. \tag{4.10}$$

The following lemma summarizes certain properties of the associated Chebyshev functions that we will need.

Lemma 4.3 *Let $N \ge 2$ be an integer. Then $\langle C_n^m, C_{n'}^m\rangle_N^i = 0$ if $m = 0, 1, 2$, $n, n' = m, \cdots, N$, $n \ne n'$. Further,*

$$\langle C_n^m, C_n^m\rangle_N^i = \begin{cases} \pi, & \text{if } m = 0,\ n = 0, N, \\ \pi/2, & \text{if } m = 0,\ n = 1, \cdots, N-1, \\ n^2\pi/2, & \text{if } m = 1,\ n = 1, \cdots, N-1, \\ (n^4\pi/2)(1 - n^{-2}), & \text{if } m = 2,\ n = 2, \cdots, N-1, \\ n^4\pi(1 - n^{-1}), & \text{if } m = 2,\ n = N. \end{cases} \tag{4.11}$$

Proof. In this proof only, we introduce the polynomials

$$U_n(\cos\theta) := \frac{\sin(n+1)\theta}{\sin\theta}, \qquad V_n(\cos\theta) := (1/2)U'_{n+1}(\cos\theta). \tag{4.12}$$

We note that

$$C_n^0 = T_n,\ C_n^1(\cos\theta) = n\sin n\theta = n\sin\theta U_{n-1}(\cos\theta),\ C_n^2(\cos\theta) = 2n\sin^2\theta V_{n-2}(\cos\theta).$$

Further, the polynomials U_n and V_n are the ultraspherical polynomials denoted in [15, p. 80] by $P_n^{(1)}$ and $P_n^{(2)}$ respectively. Therefore, using a straightforward computation in the case of C_n^0 and C_n^1, and the orthogonality of $\{P_n^{(2)}\}$ in the case of C_n^2, we obtain

$$\int_0^\pi C_n^m(\cos\theta)C_{n'}^m(\cos\theta)d\theta = 0, \qquad m = 0, 1, 2,\ n \ne n',\ n = m, m+1, \cdots. \tag{4.13}$$

Similarly, using [15, (4.7.15), p. 81] in the case of C_n^2, we get

$$\int_0^\pi C_n^m(\cos\theta)^2 d\theta = \begin{cases} \pi, & \text{if } n = m = 0, \\ \pi/2, & \text{if } m = 0,\ n = 1, 2, \cdots, \\ n^2\pi/2, & \text{if } m = 1,\ n = 1, 2, \cdots, \\ (n^4\pi/2)(1 - n^{-2}), & \text{if } m = 2,\ n = 2, 3, \cdots. \end{cases} \tag{4.14}$$

The quadrature formula (4.9) now shows that $\langle C_n^m, C_{n'}^m\rangle_N^i = 0$ if $m = 0, 1, 2$, $n, n' = m, \cdots, N$, $n \ne n'$, as well as all the equations in (4.11), except for the cases $m = 0$, $n = N$, and $m = 2$, $n = N$. The case $m = 0$, $n = N$ is clear from the definitions. Let $m = 2$, $n = N$. From the differential equation for Chebyshev polynomials, we see that with $x = \cos\theta$,

$$C_N^2(x) = (1 - x^2)T''_N(x) = xT'_N(x) - N^2T_N(x) = N\cot\theta\sin(N\theta) - N^2\cos(N\theta).$$

Thus, $C_N^2(\pm 1) = 0$, and $C_N^2(\cos\widetilde{\theta}_j) = (-1)^j N^2$. The last equation in (4.11) is now easy to obtain from the definitions. $\square$

PROOF OF THEOREM 2.3. In this proof only, let

$$F_n^m(x) := \begin{cases} C_n^0(x), & \text{if } m = 0,\ n = 0, \cdots, N, \\ C_{n+1}^1(x), & \text{if } m \text{ is odd, } |m| \le N,\ n = 0, \cdots, N-2, \\ C_{n+2}^2(x), & \text{if } m \text{ is even, } |m| \le N,\ m \ne 0,\ n = 0, \cdots, N-2. \end{cases}$$

Using (4.1) with $M = 2N+1$, and the fact that $C_{n+m}^m(\pm 1) = 0$ if $m = 1, 2$, we obtain as in the proof of Theorem 2.2 that for integers $|m|, |m'| \le N$, $n = 0, \cdots, N_m$, $n' = 0, \cdots, N_{m'}$,

$$\langle Z_{n'}^{m'}, Z_n^m \rangle_N = \begin{cases} 0, & m \ne m' \text{ or } n \ne n' \\ 2\pi \langle F_n^m, F_n^m \rangle_N^i, & \text{if } n = n',\ m = m'. \end{cases} \tag{4.15}$$

Together with (2.27) and (4.11), this shows that $\{\sqrt{z_n^m} Z_n^m\}$ is an orthonormal basis for $\mathcal{X}_N$. Lemma 4.1 shows that $\mathcal{C}_N^i$ is a set of uniqueness for $\mathcal{X}_N$. Therefore, the proof of Theorem 2.2 is complete in view of Proposition 2.1. $\square$

In order to prove Theorem 2.4, we need a representation for $\mathcal{I}_N f$ in (2.29) using the Dirichlet kernels. For integer $m \ge 1$, let

$$\begin{aligned} D_m^*(\theta) &= \frac{1}{2} + \sum_{k=1}^{m-1} \cos k\theta + \frac{1}{2}\cos m\theta = \frac{\sin m\theta}{2\tan(\theta/2)}, \\ D_m(\phi) &= \sum_{|k| \le m} \exp(ik\phi) = \frac{\sin(m+1/2)\phi}{\sin(\phi/2)} = D_{m,e}(\phi) + D_{m,o}(\phi), \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} D_{m,e}(\phi) &= \sum_{|2k| \le m} \exp(2ki\phi) = \frac{1}{2}\{D_m(\phi) + D_m(\phi + \pi)\}, \\ D_{m,o}(\phi) &= \sum_{|2k+1| \le m} \exp((2k+1)i\phi) = \frac{1}{2}\{D_m(\phi) - D_m(\phi + \pi)\}. \end{aligned}$$

We note that

$$D_N^*(\widetilde{\theta}_j - \widetilde{\theta}_k) = N\delta_{j,k},\ D_N(\widetilde{\phi}_m - \widetilde{\phi}_\ell) = (2N+1)\delta_{\ell,m}. \tag{4.17}$$

Lemma 4.4 *For $f \in C(\mathbb{S}^2)$, we have*

$$\begin{aligned} &\mathcal{I}_N f(\boldsymbol{p}(\theta, \phi)) \\ &= \frac{1}{N(2N+1)} \sum_{j=0}^{N-2} \sum_{|m| \le N} f(\boldsymbol{p}(\widetilde{\theta}_j, \widetilde{\phi}_m)) \Big\{ \Big(D_N^*(\theta - \widetilde{\theta}_j) + D_N^*(\theta + \widetilde{\theta}_j) \Big) D_{N,e}(\phi - \widetilde{\phi}_m) \\ &\quad + \Big(D_N^*(\theta - \widetilde{\theta}_j) - D_N^*(\theta + \widetilde{\theta}_j) \Big) D_{N,o}(\phi - \widetilde{\phi}_m) \Big\} \\ &\quad + \frac{1}{N} \{ f(\hat{\mathbf{n}}) D_N^*(\theta) + f(\hat{\mathbf{s}}) D_N^*(\theta - \pi) \} \end{aligned} \tag{4.18}$$

PROOF. In this proof only, we denote the right hand side of (4.18) by $T(\theta,\phi)$. Clearly, each of the summands on the right hand side of (4.18), and hence T, is in $\mathbb{H}_N$. We observe that for all θ, ϕ, $j=0,\cdots,N-2$, $|m|\le N$,

$$\begin{aligned}&\left(D_N^*(-\theta-\widetilde{\theta}_j)+D_N^*(-\theta+\widetilde{\theta}_j)\right)D_{N,e}((\phi+\pi)-\widetilde{\phi}_m)\\ &\quad=\quad\left(D_N^*(\theta-\widetilde{\theta}_j)+D_N^*(\theta+\widetilde{\theta}_j)\right)D_{N,e}(\phi-\widetilde{\phi}_m),\end{aligned}$$

and

$$D_N^*(-\widetilde{\theta}_j)+D_N^*(\widetilde{\theta}_j)=D_N^*(\pi-\widetilde{\theta}_j)+D_N^*(\pi+\widetilde{\theta}_j)=0.$$

Hence, for $j=0,\cdots,N-2$, $|m|\le N$,

$$\left(D_N^*(\theta-\widetilde{\theta}_j)+D_N^*(\theta+\widetilde{\theta}_j)\right)D_{N,e}(\phi-\widetilde{\phi}_m)\in\mathcal{X}_N^*.$$

Similarly, for all θ, ϕ, $j=0,\cdots,N-2$, $|m|\le N$,

$$\begin{aligned}&\left(D_N^*(-\theta-\widetilde{\theta}_j)-D_N^*(-\theta+\widetilde{\theta}_j)\right)D_{N,o}((\phi+\pi)-\widetilde{\phi}_m)\\ &\quad=\quad\left(D_N^*(\theta-\widetilde{\theta}_j)-D_N^*(\theta+\widetilde{\theta}_j)\right)D_{N,o}(\phi-\widetilde{\phi}_m),\end{aligned}$$

and

$$D_N^*(-\widetilde{\theta}_j)-D_N^*(\widetilde{\theta}_j)=D_N^*(\pi-\widetilde{\theta}_j)-D_N^*(\pi+\widetilde{\theta}_j)=0.$$

Moreover,

$$\cos N(\theta-\widetilde{\theta}_j)-\cos N(\theta+\widetilde{\theta}_j)=\cos(N\theta-j\pi-\pi)-\cos(N\theta+j\pi+\pi)=0.$$

Therefore, none of the terms $D_N^*(\theta-\widetilde{\theta}_j)-D_N^*(\theta+\widetilde{\theta}_j)$ can contain a term involving $\sin N\theta$. Thus,

$$\left(D_N^*(\theta-\widetilde{\theta}_j)-D_N^*(\theta+\widetilde{\theta}_j)\right)D_{N,o}(\phi-\widetilde{\phi}_m)\in\mathcal{X}_N^*.$$

It is clear that $D_N^*(\theta)$ and $D_N^*(\theta-\pi)$ are also in $\mathcal{X}_N^*$. Thus, each of the summands on the right hand side in (4.18) may be viewed as functions on $\mathbb{S}^2$, and as such, are in $\mathcal{X}_N$. Thus, $T^\circ\in\mathcal{X}_N$.

Now, for $\ell=0,\cdots,N-2$, $|\nu|\le N$, we may use (4.17) and (4.1) to conclude that

$$\begin{aligned}&T(\widetilde{\theta}_\ell,\widetilde{\phi}_\nu)\\ &\quad=\quad\frac{1}{2N+1}\sum_{|m|\le N}f(\boldsymbol{p}(\widetilde{\theta}_\ell,\widetilde{\phi}_m))\left\{D_{N,e}(\widetilde{\phi}_\nu-\widetilde{\phi}_m)+D_{N,o}(\widetilde{\phi}_\nu-\widetilde{\phi}_m)\right\}\\ &\quad=\quad\frac{1}{2N+1}\sum_{|m|\le N}f(\boldsymbol{p}(\widetilde{\theta}_\ell,\widetilde{\phi}_m))D_N(\widetilde{\phi}_\nu-\widetilde{\phi}_m)=f(\boldsymbol{p}(\widetilde{\theta}_\ell,\widetilde{\phi}_\nu)).\end{aligned}$$

The equation (4.17) also leads to $T^\circ(\hat{\mathbf{n}})=f(\hat{\mathbf{n}})$ and $T^\circ(\hat{\mathbf{s}})=f(\hat{\mathbf{s}})$. $\square$

Our next lemma relates the degrees of approximation of a function $f\in C(\mathbb{S}^2)$ from $\mathcal{X}_N$ with that of $f^*\in C^\circ$ from $\mathbb{H}_N$.

Lemma 4.5 *Let $N \ge 2$ be an integer, $f \in C(\mathbb{S}^2)$. Then*

$$\text{dist }(f^*, \mathbb{H}_N) \le \text{ dist }(f, \mathcal{X}_N) \le 5 \text{ dist }(f^*, \mathbb{H}_{N-1}). \tag{4.19}$$

PROOF. The first inequality in (4.19) is obvious since $\mathcal{X}_N \subset \mathbb{H}_N$. Let $T \in \mathbb{H}_{N-1}$ be chosen so that $\|f^* - T\|_\infty^* = \text{ dist }(f^*, \mathbb{H}_{N-1})$. Then $U(\theta, \phi) := (1/2)[T(\theta, \phi) + T(-\theta, \phi + \pi)]$ satisfies (2.1), and

$$\|f^* - U\|_\infty^* = \text{ dist }(f^*, \mathbb{H}_{N-1}). \tag{4.20}$$

Since U satisfies (2.1), there exist $S_\ell \in \Pi_{N-1}$, $R_\ell \in \Pi_{N-2}$, $|\ell| \le N-1$, such that

$$U(\theta, \phi) = \sum_{\substack{|\ell| \le N-1 \\ \ell \text{ even}}} S_\ell(\cos\theta) e^{i\ell\phi} + \sin\theta \sum_{\substack{|\ell| \le N-1 \\ \ell \text{ odd}}} R_\ell(\cos\theta) e^{i\ell\phi}.$$

For any $\phi \in \mathbb{R}$, we have

$$\max\left\{\left|f(\hat{\mathbf{n}}) - \sum_{\substack{|\ell| \le N-1 \\ \ell \text{ even}}} S_\ell(1) e^{i\ell\phi}\right|, \left|f(\hat{\mathbf{s}}) - \sum_{\substack{|\ell| \le N-1 \\ \ell \text{ even}}} S_\ell(-1) e^{i\ell\phi}\right|\right\} \le \text{ dist }(f^*, \mathbb{H}_{N-1}). \tag{4.21}$$

Therefore,

$$|f(\hat{\mathbf{n}}) - S_0(1)| = \left|\frac{1}{2\pi}\int_0^{2\pi} \{f(\hat{\mathbf{n}}) - \sum_{\substack{|\ell| \le N-1 \\ \ell \text{ even}}} S_\ell(1) e^{i\ell\phi}\} d\phi\right| \le \text{ dist }(f^*, \mathbb{H}_{N-1}). \tag{4.22}$$

Similarly,

$$|f(\hat{\mathbf{s}}) - S_0(-1)| \le \text{ dist }(f^*, \mathbb{H}_{N-1}). \tag{4.23}$$

The estimates (4.21), (4.22), (4.23) lead to

$$\left|\sum_{\substack{1 \le |\ell| \le N-1 \\ \ell \text{ even}}} S_\ell(\pm 1) e^{i\ell\phi}\right| \le 2 \text{ dist }(f^*, \mathbb{H}_{N-1}). \tag{4.24}$$

Now, let

$$\widetilde{S}_\ell(x) = S_\ell(x) - S_\ell(1)(1+x)/2 - S_\ell(-1)(1-x)/2, \qquad 1 \le |\ell| \le N-1, \ \ell \text{ even },$$

and

$$\widetilde{U}(\theta, \phi) = S_0(\cos\theta) + \sum_{\substack{1 \le |\ell| \le N-1 \\ \ell \text{ even}}} \widetilde{S}_\ell(\cos\theta) e^{i\ell\phi} + \sin\theta \sum_{\substack{|\ell| \le N-1 \\ \ell \text{ odd}}} R_\ell(\cos\theta) e^{i\ell\phi}.$$

In view of Theorem 2.1, $\widetilde{U} \in \mathcal{X}_N^*$. It is easy to verify using (4.20), (4.22), (4.23), and (4.24) that $\|f^* - \widetilde{U}\|_\infty^* \le 5 \text{ dist }(f^*, \mathbb{H}_{N-1})$. □

Finally, we are in a position to prove Theorem 2.4.

Proof of Theorem 2.4. In this proof only, let

$$
\begin{aligned}
L_N \quad := \quad \sup_{\theta,\phi\in\mathbb{R}} \Bigg[& \frac{1}{N(2N+1)} \sum_{j=0}^{N-2} \sum_{|m|\le N} \Big\{ \Big| D_N^*(\theta-\widetilde{\theta}_j) + D_N^*(\theta+\widetilde{\theta}_j) \Big| \, |D_{N,e}(\phi-\widetilde{\phi}_m)| \\
& + \Big| D_N^*(\theta-\widetilde{\theta}_j) - D_N^*(\theta+\widetilde{\theta}_j) \Big| \, |D_{N,o}(\phi-\widetilde{\phi}_m)| \Big\} \\
& + \frac{1}{N} \, \{ |D_N^*(\theta)| + |D_N^*(\theta-\pi)| \} \Bigg] \qquad (4.25)
\end{aligned}
$$

Using [9, (3.4), (3.6)], we estimate the discrete sums above by the integral norms of the Dirichlet kernels. Well known bounds on Dirichlet kernels (see for example [5, 19]) now imply that

$$L_N \le c \int_{-\pi}^{\pi} |D_N^*(t)| dt \int_{-\pi}^{\pi} \{ |D_{N,e}(t)| + |D_{N,o}(t)| \} \, dt + c \le c(\log N)^2.$$

The estimate (2.31) is now clear from (4.18).

If $T \in \mathcal{X}_N$, then $\mathcal{I}_N(T) = T$. Therefore, using (2.31), we obtain that for any $T \in \mathcal{X}_N$,

$$\|f - \mathcal{I}_N f\|_\infty = \|f - T - \mathcal{I}_N(f-T)\|_\infty \le \|f-T\|_\infty + c(\log N)^2 \|f-T\|_\infty \le c(\log N)^2 \|f-T\|_\infty.$$

This implies (2.32). □

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