Polynomial operators for spectral approximation of piecewise analytic functions

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Abstract

While classical wavelet analysis is adequate for a characterization of local Besov spaces, we propose a polynomial frame on the unit interval adequate for a characterization of functions analytic at a point on the interval. Thus, *at each point* on the interval, the behavior of the coefficients in our frame expansion can be used to detect whether the function is analytic at that point or not. The corresponding approximation operators yield an exponentially decreasing rate of approximation in the vicinity of points of analyticity and a near best approximation on the whole interval. In spite of this high localization, the construction of our operators are based on the (globally defined) Fourier coefficients in a general orthogonal polynomial expansion. Previously known results in this direction utilize Chebyshev coefficients, and the techniques to obtain these cannot be used for a similar study of general orthogonal polynomial systems. Another novelty of our paper is that while all the previous estimates for localization of polynomial kernels known to us are deduced using such special function properties of the orthogonal polynomials as asymptotics or explicit formulas for the Christoffel–Darboux kernel, we suggest a very simple idea to obtain exponentially localized kernels based on a general system of orthogonal polynomials, for which the Cesàro means of some order are uniformly bounded. The boundedness of these means is known in a number of cases, where no special function properties are known.

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1 Introduction

For integer $n \geq 0$, let Π_n denote the class of all polynomials of degree at most n, $\{T_k \in \Pi_k\}_{k=0}^{\infty}$ be the system of Chebyshev polynomials, defined on $[-1,1]$ by $T_k(\cos\theta) = \cos k\theta$. If $f : [-1,1] \to \mathbb{R}$ is a continuous function, let

$$\hat{f}(k) = (1/\pi)\int_{-1}^{1} f(t)T_k(t)(1-t^2)^{-1/2}dt,$$

$\|f\|_\infty = \max_{x\in[-1,1]}|f(x)|$, and $E_{n,\infty}(f) := \min\limits_{P\in\Pi_n}\|f-P\|_\infty$. It is well known that that f is analytic on $[-1,1]$ if and only if

$$\limsup_{n\to\infty}|\hat{f}(n)|^{1/n} < 1. \tag{1.1}$$

The condition (1.1) is also equivalent to

$$\limsup_{n\to\infty}\|f - \hat{f}(0)/2 - \sum_{k=1}^{n}\hat{f}(k)T_k\|_\infty^{1/n} < 1. \tag{1.2}$$

Similar characterizations are known in the case of a very general class of orthogonal polynomial systems rather the system of Chebyshev polynomials [14, Chapter VII, Section 3.1].

If f fails to be analytic even at only one point on $[-1,1]$, then the behavior of the Fourier–Chebyshev coefficients $\hat{f}(k)$ and the partial sums of the Chebyshev expansions changes drastically. For example, if $f(x) = |x|$, then $\hat{f}(2m+1) = 0$, and $|\hat{f}(2m)| = 2/(4m^2-1)$ for $m = 1, 2, \cdots$. Thus, the sequence $\{\hat{f}(k)\}$ by itself does not reveal either the location of the singularity at 0 or the analytic nature of the function at every other point. Moreover, if P is the best polynomial approximation from Π_n for the function f, then $|f(x) - P(x)|$ remains bounded from below by c/n at points away from the origin as well [1, Theorem 4.1].

In general, if f is a piecewise analytic function, we may still approximate f at a geometrically decaying rate as in (1.2) on intervals away from the singularities by using Chebyshev expansions adapted to the different intervals. In many applications, one may need to use expansions in orthogonal polynomials other than the Chebyshev polynomials [3, 6, 7]. From a computational point of view, it is also desirable to be able to use samples of f rather than the Fourier information.

Thus, two of the important questions in this theory are: (1) Using either the coefficients of f in a *general orthogonal polynomial expansion* or values of f at non–adaptively selected points, determine the locations of singularities of f, (2) Construct a sequence of globally defined operators, in lieu of the partial sums of the Chebyshev expansions, so that the degree of approximation by these operators globally is commensurate with the degree of best polynomial approximation, and decays geometrically fast on intervals where the target function is analytic.

Many mathematicians, including Gaier, Ivanov, Saff, and Totik ([5, 18], and references therein), have studied the second problem. In particular, Gaier constructed a sequence of linear operators $\mathcal{G}_n$ such that for each continuous $f : [-1,1] \to \mathbb{R}$, and integer $n \ge 1$, $\mathcal{G}_n(f) \in \Pi_n$, and satisfies the following conditions:

$$\max_{x\in[-1,1]} |f(x) - \mathcal{G}_n(f,x)| \le M(f)e^{-\alpha n} + c_1 E_{n/6,\infty}(f), \tag{1.3}$$

and if f is analytic in the complex neighborhood $|z - x_0| \le d$ of a point $x_0 \in [-1,1]$, then

$$|f(x_0) - \mathcal{G}_n(f,x_0)| \le M(f)d^{-4}\exp(-cd^2 n),$$

where $M(f)$ is a positive constant depending only on f, and c_1, c, α are absolute positive constants. Here, and in the sequel, rather than using such cumbersome notation as $\Pi_{\lfloor x\rfloor}$, $E_{\lfloor x\rfloor,\infty}$, etc., we will simplify the notation, and use Π_x, respectively, $E_{x,\infty}$, etc. instead. Gaier's construction is based on the Fourier-Chebyshev coefficients of f and depends heavily on a resulting contour integral. Another recent result in this direction is given by Tanner [21]. The results in [21] are formulated for trigonometric polynomial approximation, but can be adapted in a standard way for approximating functions on the interval. Tanner's approximants may be computed based on either the Fourier–Chebyshev coefficients or the values of the function f at zeros of Chebyshev polynomials. These approximants are not polynomials themselves, and it is not clear whether an estimate analogous to (1.3) holds. Also, their construction requires an a priori knowledge of the location of the singularities.

In this paper, we will solve the two problems above using either the coefficients in an expansion with respect to a very general class of orthogonal polynomials, or values of the function at certain points on $[-1,1]$. In Propositions 2.1 and 5.2, we will construct a Littlewood Paley decomposition of the form $f = \sum_{n=0}^{\infty}\sum_{k=1}^{c2^n} b_{k,n}(f)\Psi_{k,n}$ where the convergence is uniform on $[-1,1]$, each $\Psi_{k,n} \in \Pi_{2^{n+3}-1}$, and the coefficients $b_{k,n}(f)$ are obtained as a linear combination of either the coefficients of the orthgonal polynomial expansion of f, or values of f at certain points. The coefficients are computed in the form $b_{k,n}(f) = \tau_n(f, y_{k,n})$ for a linear operator τ_n and suitably chosen points $y_{k,n}$. They satisfy the Riesz condition:

$$\sum_{k,n} |b_{k,n}|^2 \sim \int_{-1}^{1} |f(t)|^2 d\mu(t),$$

where μ is the measure used to define the orthogonal polynomial system. We will demonstrate in Theorems 2.2 and 5.2 the localization of the coefficients by showing that f is analytic at a point $x \in [-1,1]$ if and only if there is a nondegenerate interval $I \subseteq [-1,1]$ containing x such that

$$\limsup_{n\to\infty} \left\{\max_{y_{k,n}\in I} |b_{k,n}(f)|\right\}^{1/2^n} < 1.$$

The partial sums $\sum_{n=0}^{N}\sum_k b_{k,n}(f)\Psi_{k,n}$ are our analogues of the operators $\mathcal{G}_{2^{N+3}}$. We will show in Theorem 2.1, 5.1 that they satisfy an inequality analogous to (1.3), but without the extra term $M(f)e^{-\alpha n}$. The construction of these operators do not require an a priori knowledge of the location of the singularities of the target function, and clearly, these operators are based on a general class of orthogonal polynomials. As shown in [10], the behavior of the coefficients $b_{k,n}(f)$ also characterises the membership of f in different local Besov spaces. We will not elaborate on this aspect in this paper.

As expected, our operators are defined using certain localized kernels of the form $\Phi_n^*(x,y)$ with the property that $|\Phi_n^*(x,y)| \le c_1 \exp(-c_2(x,y)n)$ if $x \ne y$. There are a number of kernels defined in the literature (see [15] for a survey) where the kernels satisfy a bound of the form $c(Q,x,y)/n^Q$ for every integer Q. This rate is not sufficient for our purpose. Moreover, the constructions of such kernels depend heavily on the special function properties of the orthogonal polynomial system in question. In contrast, we require only the existence of a bounded reproducing summability kernel (see Section 5 for precise definition). Freud [4] has proved the existence of such kernels for a very general class of orthogonal polynomials for which no asymptotic expansions are known.

In Section 2, we introduce our ideas in the context of Jacobi polynomials. The theory is illustrated with a few numerical examples in Section 3. In Section 4, we apply this theory to the case of local approximation of functions on the Euclidean unit sphere. In Section 5, the theory in Section 2 is generalized further to the case of arbitrary systems of orthogonal polynomials, subject to certain technical conditions. The proofs of the new results are given in Section 6.

2 Jacobi polynomials

In this paper, let $\alpha, \beta \ge -1/2$. The Jacobi measure is defined by

$$d\mu^{(\alpha,\beta)}(x) = \begin{cases} (1-x)^\alpha (1+x)^\beta dx, & \text{if } x \in (-1,1), \\ 0, & \text{otherwise.} \end{cases}$$

There exists a unique system of polynomials $\{p_k^{(\alpha,\beta)}\}_{k=0}^\infty$, called orthonormalized Jacobi polynomials, with each $p_k \in \Pi_k$, and having a positive leading coefficient such that

$$\int_{-1}^1 p_k^{(\alpha,\beta)} p_j^{(\alpha,\beta)} d\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

If $1 \le p \le \infty$, and $A \subseteq [-1,1]$, the space $L^p(\mu^{(\alpha,\beta)}; A)$ consists of measurable functions f for which

$$\|f\|_{\mu^{(\alpha,\beta)};p,A} := \begin{cases} \left(\int_A |f|^p d\mu^{(\alpha,\beta)} \right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{t \in A} |f(t)|, & \text{if } p = \infty \end{cases}$$

is finite, with the usual convention that two functions are considered equal if they are equal almost everywhere. The space $X^p(\mu^{(\alpha,\beta)}; A)$ denotes $L^p(\mu^{(\alpha,\beta)}; A)$ if $1 \le p < \infty$ and the space of bounded uniformly continuous functions on A (equipped with the supremum norm) if $p = \infty$. The mention of the set A will be omitted if $A = [-1,1]$. In particular, $X^\infty(\mu^{(\alpha,\beta)}) = C[-1,1]$, and we will write $\|f\|_\infty$ rather than the more cumbersome notation $\|f\|_{\mu^{(\alpha,\beta)};\infty}$.

We recall that for real $x \ge 0$, Π_x denotes the class of all algebraic polynomials of degree at most x. For $1 \le p \le \infty$ and $f \in L^p(\mu^{(\alpha,\beta)})$, we define the degree of approximation of f from Π_x by

$$E_{x,p}(\alpha,\beta; f) := \min_{P \in \Pi_x} \|f - P\|_{\mu^{(\alpha,\beta)};p}.$$

Of course, $E_{n,\infty}(\alpha,\beta;f) = E_{n,\infty}(f)$ as defined in the introduction. We adopt the following convention regarding constants: the symbols $c, c_1, \cdots$ will denote generic positive constants, dependent only on such fixed parameters in the discussion as p, α, β, etc. Their value may be different at different occurrences, even within the same formula.

It is readily seen that the partial sum of order 2^n of the Jacobi polynomial expansion of a function $f \in L^1(\mu^{(\alpha,\beta)})$ is given by $\int_{-1}^1 f(y)K_n^{(\alpha,\beta)}(\circ, y)d\mu^{(\alpha,\beta)}(y)$, where the Dirichlet–Darboux kernel K_n is defined by

$$K_n^{(\alpha,\beta)}(x,y) := \sum_{k=0}^{2^n} p_k^{(\alpha,\beta)}(x)p_k^{(\alpha,\beta)}(y).$$

However, the sequence of these partial sums need not converge to f for every $f \in L^1(\mu^{(\alpha,\beta)})$. To get convergent sums, we need to use a summability method. It is known [20, Theorem 9.1.4] that for every continuous f, the Cesáro means of order $K > \alpha+\beta+1$ of the Jacobi polynomial expansion of f converge uniformly to f. In order to get a near best approximation and exponential localization, we need to introduce an operator based on another related kernel.

Let $K \geq \alpha+\beta+2$ be an integer, $h : [0,\infty) \to \mathbb{R}$ be a function which is a K times iterated integral of a function of bounded variation, $h(x) = 1$ for $0 \leq x \leq 1/2$, and $h(x) = 0$ for $x > 1$. Then for $x, y \in \mathbb{C}$, $n = 0, 1, \cdots$, we define the kernel

$$\Phi_n(\mu^{(\alpha,\beta)}; h, x, y) := \sum_{k=0}^{2n} h(k/(2n))p_k^{(\alpha,\beta)}(x)p_k^{(\alpha,\beta)}(y). \tag{2.1}$$

Using a summation by parts argument or directly as in [10], one can prove that

$$\sup_{n\geq 0, x\in[-1,1]} \int_{-1}^1 |\Phi_n(\mu^{(\alpha,\beta)}; h, x, y)|d\mu^{(\alpha,\beta)}(y) < \infty.$$

In addition, it is easy to verify that $\int_{-1}^1 \Phi_n(\mu^{(\alpha,\beta)}; h, x, y)P(y)d\mu^{(\alpha,\beta)}(y) = P(x)$ for every $P \in \Pi_n$. Therefore, the polynomials $\int_{-1}^1 f(y)\Phi_n(\mu^{(\alpha,\beta)}; h, \circ, y)d\mu^{(\alpha,\beta)}(y)$ converge uniformly to f for every continuous f, at a rate comparable to $E_{n,\infty}(\alpha,\beta;f)$. In [10], we have shown that the smoother the h, the better localized the kernels Φ_n are; in particular, if h is infinitely often differentiable, then for every integer Q, $|\Phi_n(\mu^{(\alpha,\beta)}; h, x, y)| \leq c(Q, x, y)/n^Q$. However, this rate is not enough to detect the possibility of analytic continuation of a function near a point. In order to obtain an exponential rate of decay, we use the following kernel instead.

$$\Phi_n^*(\mu^{(\alpha,\beta)}; h, x, y) := \left(\frac{4-(x-y)^2}{4}\right)^n \Phi_{3n}(\mu^{(\alpha,\beta)}; h, x, y). \tag{2.2}$$

The summability operators σ_n^C are defined for $f \in L^1(\mu^{(\alpha,\beta)})$, $x \in \mathbb{C}$, $n = 0, 1, \cdots$, by

$$\sigma_n^C(\alpha,\beta; h, f, x) := \int_{-1}^1 f(y)\Phi_n^*(\mu^{(\alpha,\beta)}; h, x, y)d\mu^{(\alpha,\beta)}(y). \tag{2.3}$$

We note that $\sigma_n^C(\alpha,\beta;h,f)\in\Pi_{8n}$. Since $\Phi_n^*(\mu^{(\alpha,\beta)};h,x,y)$ is a symmetric polynomial of degree $8n$ in each of its variables, one has the representation

$$\Phi_n^*(\mu^{(\alpha,\beta)};h,x,y)=\sum_{k=0}^{8n}\sum_{j=0}^{8n}a_{n;k,j}^{(\alpha,\beta)}(h)p_k^{(\alpha,\beta)}(x)p_j^{(\alpha,\beta)}(y),$$

where, for each integer $n\ge 0$, $(a_{n;k,j}^{(\alpha,\beta)}(h))$ is a symmetric matrix. Defining the Jacobi coefficients of $f\in L^1(\mu^{(\alpha,\beta)})$ by

$$\hat{f}(\alpha,\beta;j)=\int_{-1}^{1}fp_j^{(\alpha,\beta)}d\mu^{(\alpha,\beta)},\qquad j=0,1,\cdots,$$

it follows that

$$\sigma_n^C(\alpha,\beta;h,f)=\sum_{k=0}^{8n}\left(\sum_{j=0}^{8n}a_{n;k,j}^{(\alpha,\beta)}(h)\hat{f}(\alpha,\beta;j)\right)p_k^{(\alpha,\beta)}.$$

Thus, the operators σ_n^C can be computed using finitely many Jacobi coefficients of f.

From a computational point of view, we would like to define discrete versions of these operators, which are obtained using Gauss quadrature formulas. For $m\ge 1$, let $x_{k,m}$, $k=1,\cdots,m$ be the zeros of $p_m^{(\alpha,\beta)}$, and

$$\lambda_{k,m}:=\left(\sum_{j=1}^{m}\left(p_j^{(\alpha,\beta)}(x_{k,m})\right)^2\right)^{-1}$$

be the corresponding Cotes numbers. We define the discretized versions of the operators by

$$\sigma_n^D(\alpha,\beta;h,f,x):=\sum_{k=1}^{8n+1}\lambda_{k,8n+1}f(x_{k,8n+1})\Phi_n^*(\mu^{(\alpha,\beta)};h,x,x_{k,8n+1}).\tag{2.4}$$

The following theorem is our generalization of the result of Gaier in the context of Jacobi polynomials.

Theorem 2.1 *Let* $1\le p\le\infty$, $\alpha,\beta\ge -1/2$, $f\in L^p(\mu^{(\alpha,\beta)})$. *For integer* $n\ge 0$, *let* $\sigma_n(f)$ *denote either* $\sigma_n^C(\alpha,\beta;h,f)$ *or* $\sigma_n^D(\alpha,\beta;h,f)$.
(a) *We have* $\sigma_n(P)=P$ *for* $P\in\Pi_n$, $\|\sigma_n(f)\|_{\mu^{(\alpha,\beta)};p}\le c\|f\|_{\mu^{(\alpha,\beta)};p}$, *and*

$$E_{8n,p}(\alpha,\beta;f)\le\|f-\sigma_n(f)\|_{\mu^{(\alpha,\beta)};p}\le c_1E_{n,p}(\alpha,\beta;f).\tag{2.5}$$

(b) *Let* $f\in C[-1,1]$, $x_0\in[-1,1]$, *and* f *have an analytic continuation to a complex neighborhood of* x_0, *given by* $\{z\in\mathbb{C}:|z-x_0|\le d\}$ *for some* d *with* $0<d\le 2$. *Then*

$$|f(x)-\sigma_n(f,x)|\le c(f,x_0)\exp\left(-n\frac{d^2\log(e/2)}{e^2\log(e^2/d)}\right),\qquad x\in[x_0-d/e,x_0+d/e]\cap[-1,1].\tag{2.6}$$

We note a few interesting features of this theorem. First, we are able to drop the extra term $M(f)e^{-\alpha n}$ in (1.3) at the expense of a higher estimate $c_1E_{n/8,\infty}(f)$ in place of $c_1E_{n/6,\infty}(f)$. Second, our construction can be based either on the coefficients in general Jacobi polynomial expansions, or based on values of the function. Finally, we think that our proofs are simpler than those given by Gaier in [5].

Next, we describe a Littlewood–Paley expansion of functions in $X^p(\mu^{(\alpha,\beta)})$, where the analyticity of the target function at a point can be completely characterised using certain coefficients of this expansion. Towards this end, we define the continuous and discrete frame operators by

$$\begin{aligned}
\tau_n^C(\alpha,\beta;h,f,x) &:= \begin{cases} \sigma_1^C(\alpha,\beta;h,f,x), & \text{if } n=0,\\ \sigma_{2^n}^C(\alpha,\beta;h,f,x)-\sigma_{2^{n-1}}^C(\alpha,\beta;h,f,x), & \text{if } n=1,2,\cdots, \end{cases}\\
\tau_n^D(\alpha,\beta;h,f,x) &:= \begin{cases} \sigma_1^D(\alpha,\beta;h,f,x), & \text{if } n=0,\\ \sigma_{2^n}^D(\alpha,\beta;h,f,x)-\sigma_{2^{n-1}}^D(\alpha,\beta;h,f,x), & \text{if } n=1,2,\cdots. \end{cases}
\end{aligned} \tag{2.7}$$

We note that $\tau_n^C(\alpha,\beta;h,f), \tau_n^D(\alpha,\beta;h,f) \in \Pi_{2^{n+3}}$. The next proposition demonstrates the use of these operators in obtaining a Littlewood–Paley decomposition of functions in $X^p(\mu^{(\alpha,\beta)})$, $1 \le p \le \infty$.

Proposition 2.1 *Let $1 \le p \le \infty$, $\alpha,\beta \ge -1/2$, $f \in X^p(\mu^{(\alpha,\beta)})$. If $N_n \ge 2^{n+3}+1$ are integers, one has the Littlewood–Paley decomposition*

$$\begin{aligned}
f &= \sum_{n=0}^{\infty} \tau_n^C(\alpha,\beta;h,f)\\
&= \sum_{n=0}^{\infty}\sum_{k=1}^{N_n} \lambda_{k,N_n}\tau_n^C(\alpha,\beta;h,f,x_{k,N_n})\left\{K_{n+3}^{(\alpha,\beta)}(\circ,x_{k,N_n})-K_{n-1}^{(\alpha,\beta)}(\circ,x_{k,N_n})\right\},
\end{aligned} \tag{2.8}$$

with convergence in the sense of $X^p(\mu^{(\alpha,\beta)})$. Moreover, we have for every $f \in L^2(\mu^{(\alpha,\beta)})$,

$$\begin{aligned}
c_1\int_{-1}^{1} f(t)^2(1-t)^\alpha(1+t)^\beta dt &\le \sum_{n=0}^{\infty}\sum_{k=1}^{N_n}\lambda_{k,N_n}\tau_n^C(\alpha,\beta;h,f,x_{k,N_n})^2\\
&\le c_2\int_{-1}^{1} f(t)^2(1-t)^\alpha(1+t)^\beta dt.
\end{aligned}$$

If $f \in C[-1,1]$, then $f=\sum_{n=0}^\infty \tau_n^D(\alpha,\beta;h,f)$, with convergence being uniform.

The following theorem demonstrates that, unlike the Fourier–Jacobi coefficients $\hat{f}(\alpha,\beta;j)$, the behavior of the coefficients $\tau_n^C(\alpha,\beta;h,f,x_{k,N_n})$ in the Littlewood–Paley expansion (2.8) for points x_{k,N_n} in a neighborhood of a point x_0 reflects the (analytic) regularity of the function f at x_0.

Theorem 2.2 *Let $\alpha,\beta \ge -1/2$, $x_0 \in [-1,1]$ and $f \in C[-1,1]$. For integer $n \ge 0$, let $\tau_n(f)$ denote either $\tau_n^C(\alpha,\beta;h,f)$ or $\tau_n^D(\alpha,\beta;h,f)$.*
(a) *The function f has an extension as an analytic function in a complex neighborhood of x_0 if and only if there exists a nondegenerate interval $I \subseteq [-1,1]$ containing x_0 such that*

$$\limsup_{n\to\infty}\max_{x\in I}|\tau_n(f,x)|^{1/2^n} < 1. \tag{2.9}$$

(b) *The function f has an extension as an analytic function in a complex neighborhood of x_0 if and only if there exists a nondegenerate interval $I \subseteq [-1,1]$ containing x_0 such that*

$$\limsup_{n\to\infty} \left\{ \max_{x_{k,2^{n+6}+1}\in I} |\tau_n(f, x_{k,2^{n+6}+1})| \right\}^{1/2^n} < 1. \tag{2.10}$$

We note that choosing $N_n = 2^{n+6}+1$ in Proposition 2.1, and using τ_n^C in place of τ_n, (2.10) gives a characterization of regular points of f in terms of the coefficients in the Littlewood–Paley expansion (2.8).

3 Numerical examples

In this section, we illustrate the construction of our localized kernels and their approximation properties using some numerical examples. In our examples below, we use the Chebyshev polynomials; i.e., the polynomials T_n defined by $T_n(\cos\theta) = \cos n\theta$, $\theta \in [0,\pi]$, $n = 0, 1, \cdots$. The polynomials

$$p_n^T = \begin{cases} 1, & \text{if } n = 0, \\ \sqrt{2}T_n, & \text{if } n = 1, 2, \cdots, \end{cases}$$

are orthonormalized with respect to the measure

$$d\mu^T(x) = \frac{d\mu^{(-1/2,-1/2)}(x)}{\pi} = \frac{dx}{\pi(1-x^2)^{1/2}}, \qquad x \in (-1,1).$$

In our numerical computations, we will approximate integrals with respect to $d\mu^T$ by suitable quadrature formulas.

We define the de la Vallée Poussin kernel by

$$\Phi_n^T(x,y) := \sum_{k=0}^{n} p_k^T(x)p_k^T(y) + \sum_{k=n+1}^{2n} (2 - k/n) p_k^T(x)p_k^T(y).$$

We define the discrete de la Vallée Poussin operator by

$$V_n(f,x) = \frac{1}{8n+1} \sum_{k=1}^{8n+1} f\left(\cos\frac{(2k-1)\pi}{16n+2}\right) \Phi_{4n}^T\left(x, \cos\frac{(2k-1)\pi}{16n+2}\right),$$

and note that $V_n(f) \in \Pi_{8n}$. With

$$\Phi_n^{*T}(x,y) = \left(\frac{4-(x-y)^2}{4}\right)^n \Phi_{3n}^T(x,y),$$

we define the exponentially localized operator, denoted here for brevity by σ_n, by

$$\sigma_n(f,x) = \frac{1}{8n+1} \sum_{k=1}^{8n+1} f\left(\cos\frac{(2k-1)\pi}{16n+2}\right) \Phi_n^{*T}\left(x, \cos\frac{(2k-1)\pi}{16n+2}\right).$$

In particular, $\sigma_n(f) \in \Pi_{8n}$.

To illustrate the approximation properties and localization of the operators, we consider two functions, the first of which is

$$f_a(x) := |x - 1/4|, \qquad x \in [-1, 1].$$

To define the second function, we recall that the cardinal B-spline of order 4 is the function defined by (cf. [2, Formula (4.1.12), p. 84])

$$M_4(x) = \frac{1}{6}\left\{x_+^3 - 4(x-1)_+^3 + 6(x-2)_+^3 - 4(x-3)_+^3 + (x-4)_+^3\right\}$$

where $a_+ = \max(a, 0)$. We define $f_b(x) = M_4(2x + 2)$. Thus, f_b is analytic on $(-1, 1)$, except at $\pm 1/2, 0$, where it is twice continuously differentiable.

In this section only, let $\mathcal{C}$ denote the set of 10,000 equidistant points on $[-1, 1]$,

$$\epsilon_n(f, V) := \max_{x\in\mathcal{C}} |f(x) - V_n(f, x)|, \quad \epsilon_n(f, \sigma) := \max_{x\in\mathcal{C}} |f(x) - \sigma_n(f, x)|,$$

and

$$\delta_n(f, V) := \log_2 \frac{\epsilon_n(f, V)}{\epsilon_{2n}(f, V)}, \quad \delta_n(f, \sigma) := \log_2 \frac{\epsilon_n(f, V)}{\epsilon_{2n}(f, V)}.$$

Table 1 shows the decay of errors $\epsilon_n(f, V)$ and $\epsilon_n(f, \sigma)$ for different values of n.

n	$\epsilon_n(f_a, \sigma)$	$\epsilon_n(f_a, V)$	$\epsilon_n(f_b, \sigma)$	$\epsilon_n(f_b, V)$
8	$1.8065 * 10^{-2}$	$1.3609 * 10^{-2}$	$1.3838 * 10^{-4}$	$4.7691 * 10^{-5}$
16	$9.8889 * 10^{-3}$	$8.0887 * 10^{-3}$	$1.684 * 10^{-5}$	$6.0351 * 10^{-6}$
32	$4.8372 * 10^{-3}$	$3.8924 * 10^{-3}$	$2.0823 * 10^{-6}$	$7.5158 * 10^{-7}$
64	$2.3075 * 10^{-3}$	$1.7856 * 10^{-3}$	$2.5918 * 10^{-7}$	$9.3814 * 10^{-8}$

Table 1: Maximum absolute errors.

In light of the direct theorems of approximation theory, the quantities $\delta_n(f_a, V)$ and $\delta_n(f_a, \sigma)$ should be close to 1, and the corresponding quantities for f_b should be close to 3. Table 2 confirms this fact (cf. Theorem 2.2(a).)

n	$\delta_n(f_a, \sigma)$	$\delta_n(f_a, V)$	$\delta_n(f_b, \sigma)$	$\delta_n(f_b, V)$
8	0.8693	0.7506	3.0387	2.9823
16	1.0316	1.0552	3.0156	3.0054
32	1.0678	1.1243	3.0062	3.0021

Table 2: The smoothness index as predicted by δ_n's.

It is clear from Table 1 that the maximum error is less with the de la Vallée Poussin operators than the exponentially localized operators. However, Figure 1 shows with example of $n = 64$ that the later are more localized in the sense that on parts of the interval where the functions are analytic, the error with the exponentially localized operator is substantially smaller than that with the de la Vallée Poussin operators. Figure 1 shows the logplot of the errors in approximating f_a and f_b by the discrete de la Vallée Poussin operators and the exponentially localized operators. Thus, on the y axis in all the subfigures below, the value $-k$ corresponds to the value 10^{-k} for the errors plotted in the figures. We note that only $8n+1$ values of the function are used in the computation of the transforms V_n and σ_n. In particular, in the top left figure in Figure 1, an absolute error of less than 10^{-20} is obtained away from the singularity, using only 513 samples of the function f_a.

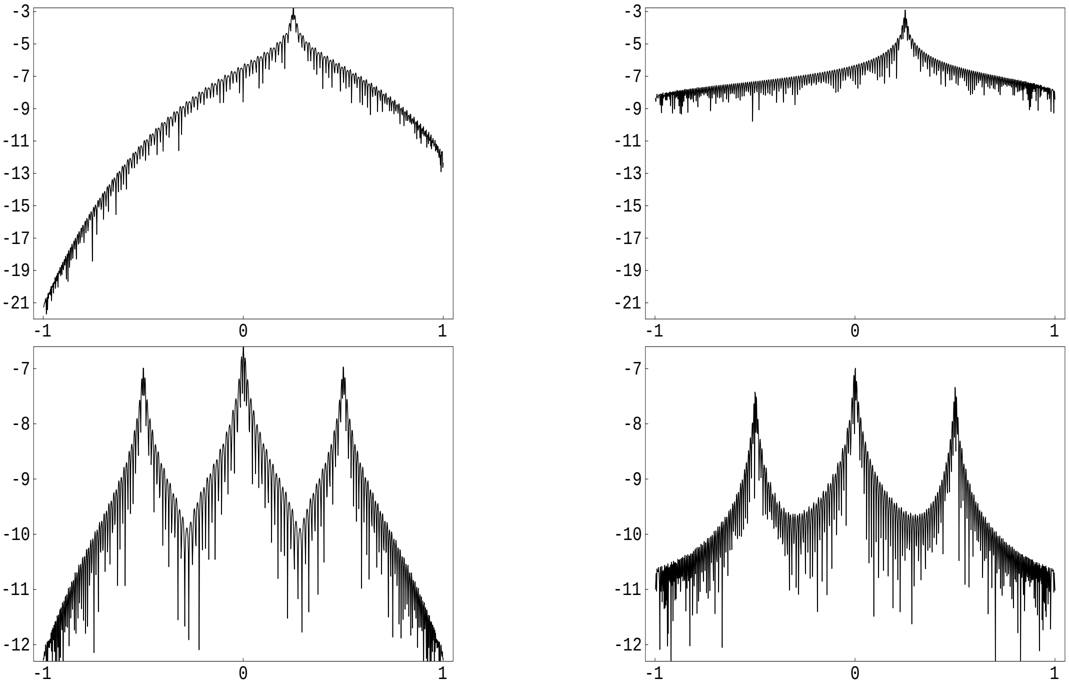

Figure 1: Clockwise, the graphs represent $\log_{10}|f_a-\sigma_{64}(f_a)|$, $\log_{10}|f_a-V_{64}(f_a)|$, $\log_{10}|f_b-V_{64}(f_b)|$, and $\log_{10}|f_b-\sigma_{64}(f_b)|$.

To illustrate this phenomenon further, we evaluated the errors at 2048 points chosen randomly according to the probability measure μ^T (again with polynomials of degree 512), and arranged them in an increasing order. Thus, Figure 2 shows the probability distribution of the errors, where the lowest 15 terms have been discarded as outliers. In the case of f_a, the probability that the error does not exceed 10^{-5} times the minimum of the two maximum errors is 25.88% with the de la Vallée Poussin operators and 79.39% with the exponentially localized operators. In the case of the spline function f_b, the corresponding probabilities are 3.22% and 18.46% respectively. When we compared polynomials of degree 1024 in the case of f_b, the difference became more dramatic, with the probabilities being 5.96% for the de la Vallée Poussin operator, and 40.28% for the exponentially localized operator.

Finally, we note that the plots in Figure 1 are produced with symbolic computations in Mathematica to illustrate an error much less than the usual double precision floating point accuracy would allow. On the other hand, we used Matlab to produce Figure 2, where we obtain the same results as with Mathematica, but up to error of 10^{-16}.

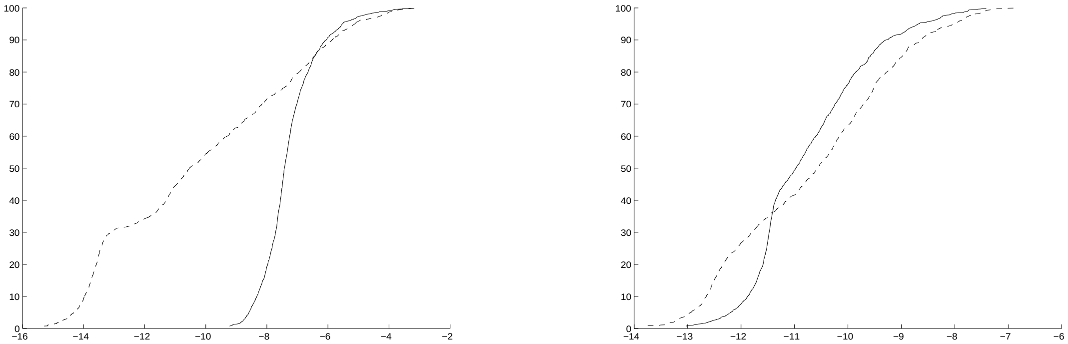

Figure 2: On the left, the continuous line represents on the x axis, the increasing rearrangement of the array $\log_{10}|f_a - V_{64}(f_a)|$ at 2048 randomly chosen points, and on the y axis, the percentage of samples where the error is below the corresponding value. The dashed line represents the corresponding graph for $\log_{10}|f_a - \sigma_{64}(f_a)|$. The graphs on the right represent the same quantities for the function f_b.

4 Approximation on the sphere

The constructions described for the unit interval can be adapted easily for the unit sphere of a Euclidean space. We now sketch these adaptations, and indicate the differences. Let $q \ge 1$ be an integer,

$$\mathbb{S}^q := \{(x_1, \cdots, x_{q+1}) \in \mathbb{R}^{q+1} \ : \ \sum_{j=1}^{q+1} x_j^2 = 1\}.$$

A spherical cap, centered at $\mathbf{x}_0 \in \mathbb{S}^q$ and radius α is defined by

$$\mathbb{S}^q_\alpha(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{S}^q \ : \ \mathbf{x}\cdot\mathbf{x}_0 \ge \cos\alpha\}.$$

We note that for any $\mathbf{x}_0 \in \mathbb{S}^q$, $\mathbb{S}^q_\pi(\mathbf{x}_0) = \mathbb{S}^q$. The surface area (volume element) measure on $\mathbb{S}^q$ will be denoted by μ_q^*, and we write $\mu_q^*(\mathbb{S}^q) =: \omega_q$. The spaces $X^p(\mathbb{S}^q)$ and $C(\mathbb{S}^q)$ on the sphere are defined analogously to the case of the interval.

A spherical polynomial of degree m is the restriction to $\mathbb{S}^q$ of a polynomial in $q+1$ real variables with total degree m. For $x \ge 0$, the class of all spherical polynomials of degree at most x will be denoted by Π_x^q. For integer $\ell \ge 0$, the class of all homogeneous, harmonic, spherical polynomials of degree ℓ will be denoted by $\mathbb{H}_\ell^q$, and its dimension by d_ℓ^q. For each integer $\ell \ge 0$, let $\{Y_{\ell,k} \ : \ k = 1, \cdots, d_\ell^q\}$ be a μ_q^*-orthonormalized basis for $\mathbb{H}_\ell^q$. It is known (cf. [19, 16]) that for any integer $n \ge 0$, $\{Y_{\ell,k} \ : \ \ell = 0, \cdots, n, \ k = 1, \cdots, d_\ell^q\}$ is an orthonormal basis for Π_n^q. The connection with the theory of orthogonal polynomials on $[-1,1]$ is the following addition formula (cf. [16], where the notation is different):

$$\sum_{k=1}^{d_\ell^q} Y_{\ell,k}(\mathbf{x})Y_{\ell,k}(\mathbf{y}) = \omega_{q-1}^{-1} p_\ell^{(q/2-1,q/2-1)}(1) p_\ell^{(q/2-1,q/2-1)}(\mathbf{x}\cdot\mathbf{y}), \qquad \ell = 0, 1, \cdots.$$

It is proved in [13] that if $n \ge 1$ is an integer, $\mathcal{C}$ is a finite set of points on $\mathbb{S}^q$ such that

$$\max_{\mathbf{x}\in\mathbb{S}^q}\min_{\mathbf{y}\in\mathcal{C}} \text{ dist }(\mathbf{x},\mathbf{y}) \le c/n,$$

for a judiciously chosen constant $c > 0$, then there exist nonnegative weights $w_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{C}$ such that

$$\sum_{\mathbf{x}\in\mathcal{C}} w_{\mathbf{x}} P_1(\mathbf{x})P_2(\mathbf{x}) = \int_{\mathbb{S}^q} P_1P_2 d\mu_q^*, \qquad P_1, P_2 \in \Pi_{8n}^q. \tag{4.1}$$

We will say that $\mathcal{C}$ admits a positive quadrature formula of order $16n$.

Since $\mathbb{S}^q$ is not a set of uniqueness of functions analytic on $\mathbb{C}^{q+1}$, we do not see how to define the analogue of analytic continuation of a function on $\mathbb{S}^q$. Nevertheless, one can define an analogue from the approximation theory point of view as follows. Let $\mathbf{x}_0 \in \mathbb{S}^q$, $\mathcal{K}$ be a spherical cap centered at $\mathbf{x}_0$ and $f : \mathcal{K} \to \mathbb{C}$. We write

$$E_n(\mathcal{K}, f) := \inf_{P\in\Pi_n^q} \|f - P\|_{\mu_q^*;\infty,\mathcal{K}}.$$

The class $\mathcal{A}_q(\mathbf{x}_0)$ is defined to be the class of all functions $f \in C(\mathbb{S}^q)$ such that for some spherical cap $\mathcal{K}$ centered at $\mathbf{x}_0$,

$$\limsup_{n\to\infty} E_n(\mathcal{K}, f)^{1/n} < 1.$$

Let $h : [0,\infty) \to \mathbb{R}$ be a q times iterated integral of a function of bounded variation, $h(x) = 1$ if $x \in [0, 1/2]$, and $h(x) = 0$ if $x \in [1,\infty)$. The role of the kernels $K_n^{(\alpha,\beta)}$ and $\Phi_n(\mu^{(\alpha,\beta)}; h, x, y)$ is played respectively by

$$K_n^S(\mathbf{x}\cdot\mathbf{y}) := \omega_{q-1}^{-1}\sum_{\ell=0}^{2^n} p_\ell^{(q/2-1,q/2-1)}(1)p_\ell^{(q/2-1,q/2-1)}(\mathbf{x}\cdot\mathbf{y}) = \omega_{q-1}^{-1}K_n^{(q/2-1,q/2-1)}(1,\mathbf{x}\cdot\mathbf{y})$$

and

$$\begin{aligned}\Phi_n^S(q; h, \mathbf{x}, \mathbf{y}) &:= \omega_{q-1}^{-1}\sum_{\ell=0}^{2n} h(\ell/(2n))p_\ell^{(q/2-1,q/2-1)}(1)p_\ell^{(q/2-1,q/2-1)}(\mathbf{x}\cdot\mathbf{y})\\ &= \Phi_n(\mu^{(q/2-1,q/2-1)}; h, x, y).\end{aligned}$$

For $n = 0, 1, \cdots$, $\mathbf{x}, \mathbf{y} \in \mathbb{S}^q$, let

$$\Phi_n^*(h, \mathbf{x}, \mathbf{y}) = \left(\frac{1 + \mathbf{x}\cdot\mathbf{y}}{2}\right)^n \Phi_{3n}^S(q; h, \mathbf{x}, \mathbf{y}).$$

Let $\{\mathcal{C}_n\}$ be a sequence of finite subsets of $\mathbb{S}^q$, with each $\mathcal{C}_n$ admitting a positive quadrature of order $16n$. We define the analogues of the operators σ_n^C, σ_n^D, τ_n^C, and τ_n^D by

$$\begin{aligned}\sigma_n^{C,S}(h, f, \mathbf{x}) &:= \int_{\mathbb{S}^q} \Phi_n^*(h, \mathbf{x}\cdot\mathbf{y})f(\mathbf{y})d\mu_q^*(\mathbf{y}),\\ \sigma_n^{D,S}(h, f, \mathbf{x}) &:= \sum_{\mathbf{y}\in\mathcal{C}_n} w_{\mathbf{y}}\Phi_n^*(h, \mathbf{x}\cdot\mathbf{y})f(\mathbf{y}),\\ \tau_n^{C,S}(h, f, x) &:= \begin{cases} \sigma_1^{C,S}(h, f, x), & \text{if } n = 0,\\ \sigma_{2^n}^{C,S}(h, f, x) - \sigma_{2^{n-1}}^{C,S}(h, f, x), & \text{if } n = 1, 2, \cdots, \end{cases}\\ \tau_n^{D,S}(h, f, x) &:= \begin{cases} \sigma_1^{D,S}(h, f, x), & \text{if } n = 0,\\ \sigma_{2^n}^{D,S}(h, f, x) - \sigma_{2^{n-1}}^{D,S}(h, f, x), & \text{if } n = 1, 2, \cdots. \end{cases}\end{aligned}$$

The following proposition is the analogue of Proposition 2.1, and can be proved in exactly the same way.

Proposition 4.1 *Let* $1 \le p \le \infty$, $\{\mathcal{C}_n\}$ *be a sequence of finite subsets of* $\mathbb{S}^q$, *with each* $\mathcal{C}_n$ *admitting a positive quadrature of order at least* $16n$. *Then for* $f \in X^p(\mathbb{S}^q)$,

$$f = \sum_{n=0}^{\infty} \tau_n^{C,S}(h, f) = \sum_{n=0}^{\infty} \sum_{\mathbf{y}\in\mathcal{C}_n} w_{\mathbf{y}} \tau_n^{C,S}(h, f, \mathbf{y}) \left\{K_{n+3}^{S}(\circ\cdot\mathbf{y}) - K_{n-1}^{S}(\circ\cdot\mathbf{y})\right\}, \tag{4.2}$$

with the series converging in the sense of $X^p(\mathbb{S}^q)$. *If* $f \in C(\mathbb{S}^q)$, *then also* $f = \sum_{n=0}^{\infty} \tau_n^{D,S}(h, f)$ *with the series converging uniformly. Further, if* $f \in L^2(\mathbb{S}^q)$,

$$c_1 \|f\|_{\mu_q^*;2}^2 \le \sum_{n=0}^{\infty} \|\tau_n^{C,S}(h, f)\|_{\mu_q^*;2}^2 = \sum_{n=0}^{\infty} \sum_{\mathbf{y}\in\mathcal{C}_n} w_{\mathbf{y}} |\tau_n^{C,S}(h, f, \mathbf{y})|^2 \le c_2 \|f\|_{\mu_q^*;2}^2.$$

The analogue of Theorem 2.2 and 2.1 is the following, slightly weaker statement.

Theorem 4.1 *Let* $\{\mathcal{C}_n\}$ *be a sequence of finite subsets of* $\mathbb{S}^q$, *with each* $\mathcal{C}_n$ *admitting a positive quadrature of order at least* $16n$, $f \in C(\mathbb{S}^q)$, *and* $\mathbf{x}_0 \in \mathbb{S}^q$. *Let* σ_n *denote either* $\sigma_n^{C,S}$ *or* $\sigma_n^{D,S}$ *and similarly for* τ_n.
(a) *For integer* $n \ge 1$, *we have*

$$\|f - \sigma_n(h, f)\|_{\mu_q^*;\infty,\mathbb{S}^q} \le cE_n(\mathbb{S}^q, f). \tag{4.3}$$

If $f \in \mathcal{A}_q(\mathbf{x}_0)$, *then there exists a nondegenerate spherical cap* $\mathcal{K} \subseteq \mathbb{S}^q$ *with center at* $\mathbf{x}_0$ *and* $\rho \in (0, 1)$ *(depending upon* $\mathbf{x}_0$ *and* f*) such that*

$$\|f - \sigma_n(h, f)\|_{\mu_q^*;\infty,\mathcal{K}} \le c_1(f, \mathbf{x}_0)\rho^n. \tag{4.4}$$

(b) *The function* $f \in \mathcal{A}_q(\mathbf{x}_0)$ *if and only if there exists a non-degenerate spherical cap* $\mathcal{K} \subseteq \mathbb{S}^q$ *with center at* $\mathbf{x}_0$ *such that*

$$\limsup_{n\to\infty} \|\tau_n(h, f)\|_{\mu_q^*;\infty,\mathcal{K}}^{1/2^n} < 1. \tag{4.5}$$

5 The general case

In this section, we state our main results in a very general form. Thus, instead of the Jacobi measure, we will consider an arbitrary measure, supported on an arbitrary compact subset of $[-1, 1]$. Instead of achieving the discretization of the summability and frame operators using Gauss quadrature formula, we will formulate our "discretization" using general functionals. In order to state our results, we need certain notions from measure theory and potential theory. For the convenience of the reader, we review the measure theoretic notions in Section 5.1; the ideas from potential theory are reviewed in Section 5.2. The generalizations of the operators and new results in Section 2 are given in Section 5.3.

5.1 Measures

We observe that if $N \geq 1$ is an integer, $\{x_k\}_{k=1}^N$, $\{w_k\}_{k=1}^N$ are real numbers, a sum of the form $\sum_{k=1}^N w_k f(x_k)$ can be expressed as a Stieltjes integral $\int f d\nu$, where ν is the measure that associates the mass w_k with each point x_k. The total variation measure in this case is given by $|\nu|(B) = \sum_{x_k \in B} |w_k|$, $B \subset \mathbb{R}$. We prefer to use the integral notation rather than the more explicit sum notation for a number of reasons. First, the precise locations of the points x_k, the values of w_k, and sometimes, even the value of N do not play a significant role in our theory. The use of the integral notation avoids the need to prescribe these quantities explicitly, and develop additional notation for these. Second, we wish our theory to be applicable to all L^p spaces. If $p < \infty$, point evaluations are not well defined for every f in the space, and we have to use some other local measurements, for example, averages over small subintervals around certain points. Again, the details of exactly what these points and the corresponding subintervals are, and even the nature of the local measurements do not play any significant role in our theory. The integral notation allows us to treat both the case of continuous functions and elements of L^p in a unified manner.

Let ν be a (possibly signed) measure on $\mathbb{R}$ that is either positive and finite, or has a bounded variation on $\mathbb{R}$, $|\nu|$ denote ν if ν is a positive measure, and its total variation measure if it is a signed measure. We recall that the support of ν, denoted by supp (ν) is the set of all $x \in \mathbb{R}$ such that $|\nu|(I) > 0$ for every interval I containing x. If $A \subseteq \mathbb{R}$ is $|\nu|$-measurable, $|\nu|(A) > 0$, and $f : A \to \mathbb{R}$ is $|\nu|$-measurable, we write

$$\|f\|_{\nu;p,A} := \begin{cases} \left\{\displaystyle\int_A |f|^p d|\nu|\right\}^{1/p}, & \text{if } 1 \leq p < \infty, \\ |\nu| - \text{ess sup}_{t \in A} |f(t)|, & \text{if } p = \infty. \end{cases}$$

The class of measurable functions f for which $\|f\|_{\nu;p,A} < \infty$ is denoted by $L^p(\nu; A)$, with the standard convention that two functions are considered equal if they are equal $|\nu|$-almost everywhere on A. The class of all uniformly continuous, bounded functions on A (equipped with the norm of $L^\infty(\nu)$) will be denoted by $C(\nu; A)$. The class $X^p(\nu; A)$ will denote $L^p(\nu; A)$ if $1 \leq p < \infty$ and $C(\nu; A)$ if $p = \infty$.

In the sequel, we will assume that μ is a fixed, finite, positive, Borel measure with supp (μ) being an infinite subset of $[-1, 1]$. The mention of the set A will be omitted if $A =$ supp (μ). Thus, for example, we will write $X^p(\mu) = X^p(\mu; \text{supp }(\mu))$ and $C(\mu) = C(\mu; \text{supp }(\mu))$.

We now formulate certain assumptions on our measures.

Definition 5.1 *A sequence $\{\nu_n\}$ will be called an **M–Z (Marcinkiewicz–Zygmund) sequence** if each of the following conditions is satisfied.*

1. *Each ν_n is a Borel, finite, positive or signed measure having bounded variation on $[-1, 1]$.*

2. $$\|T\|_{\nu_n;p} \leq c\|T\|_{\mu;p}, \qquad T \in \Pi_{16n},\ p = 1, \infty. \tag{5.1}$$

3.

$$\int T_1 T_2 d\nu_n = \int T_1 T_2 d\mu, \qquad T_1, T_2 \in \Pi_{8n}. \tag{5.2}$$

If $1 \le p \le \infty$*, a sequence* $\{\nu_n\}$ *will be called* ***p-compatible*** *if each* μ*–measurable function is also* ν_n*–measurable for each* n*, and* $\|f\|_{\nu_n;p} \le c\|f\|_{\mu;p}$ *for every* $f \in X^p(\mu)$*.*

In the case of Jacobi polynomials, it is proved by Lubinsky, Máté, and Nevai [8, Theorem 5] that the measures ν_n that associate the mass $\lambda_{k,8n+1}$ with each $x_{k,8n+1}$, $k = 1, \cdots, 8n+1$, form an ∞–compatible M–Z sequence. In general, it is natural to construct measures to satisfy (5.2) using Gauss quadrature formulas based on the zeros of a sufficiently high degree orthogonal polynomial p_N. However, if supp (μ) is not an interval, then the zeros of the corresponding orthogonal polynomials might not be all in supp (μ), in which case, such a measure would not be ∞–compatible. We will prove the following proposition to demonstrate the existence of ∞–compatible, M–Z sequences of measures supported on finite subsets of supp (μ).

Proposition 5.1 *Let* $\mu(\{x\}) = 0$ *for every* $x \in [-1, 1]$*. Then there exists an* ∞*–compatible M–Z sequence* $\{\nu_n\}$ *of measures such that each of the sets* supp (ν_n) *is a finite subset of* supp (μ)*.*

5.2 Potential theory ideas

In this section, we review briefly certain ideas from potential theory, based on the discussion in [18, Chapter 2.4]. The *logarithmic energy* of a positive measure ν on $\mathbb{C}$ is defined by

$$\mathcal{E}(\nu) := \int\int \log\left(\frac{1}{|x-y|}\right) d\nu(x)d\nu(y),$$

whenever the integral is well defined. For example, we recall (cf. [18, Chapter I, Example 3.5]) that

$$\int\int \log\left(\frac{2}{|x-t|}\right) d\mu^{(-1/2,-1/2)}(x)d\mu^{(-1/2,-1/2)}(t) = \pi^2 \log 4.$$

It follows that if $\alpha, \beta \ge -1/2$ and $\alpha + \beta + 1 > 0$, then

$$\begin{aligned}
&\int\int \log\left(\frac{2}{|x-t|}\right) d\mu^{(\alpha,\beta)}(x)d\mu^{(\alpha,\beta)}(t) \\
\le\ & \left(\frac{2}{\alpha+\beta+1}\right)^{\alpha+\beta+1} (\alpha+1/2)^{\alpha+1/2}(\beta+1/2)^{\beta+1/2} \\
&\quad \times \int\int \log\left(\frac{2}{|x-t|}\right) d\mu^{(-1/2,-1/2)}(x)d\mu^{(-1/2,-1/2)}(t) \\
=\ & \pi^2(\log 4)\left(\frac{2}{\alpha+\beta+1}\right)^{\alpha+\beta+1} (\alpha+1/2)^{\alpha+1/2}(\beta+1/2)^{\beta+1/2}.
\end{aligned}$$

Thus, $\mu^{(\alpha,\beta)}$ has finite logarithmic energy.

If $A \subseteq \mathbb{C}$ is a compact set, the capacity of A, cap (A) is defined by

$$\log(1/\mathrm{cap}\ (A)) = \inf \mathcal{E}(\nu),$$

where the infimum is taken over all unit, positive, Borel measures ν, with supp $(\nu) \subseteq A$. The set A is called *regular* if there exists a function G_A, called the *Green's function for* $\mathbb{C} \setminus A$ *with pole at* ∞, with the following properties: (i) G_A is continuous and nonnegative on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus A$, (ii)

$$\lim_{|z|\to\infty} (G_A(z) - \log|z|) = \log(1/\mathrm{cap}\ (A)),$$

and (iii)

$$\lim_{z\to x} G_A(z) = 0, \qquad x \in A.$$

For example, if $A = [a, b]$, one has cap $([a, b]) = (b - a)/4$, and

$$G_{[a,b]}(z) = \log \frac{|2z - a - b + 2\sqrt{(z-a)(z-b)}|}{b-a}, \qquad z \in \mathbb{C} \setminus [a, b].$$

It is clear that every interval $[a, b]$ is a regular set.

We end this subsection by recalling the well known Bernstein–Walsh inequality (cf. [18, Estimate (2.4), p. 153]).

Lemma 5.1 *Let $A \subseteq [-1, 1]$ be a regular set, $m \geq 0$ be an integer, $P \in \Pi_m$. Then for any $z \in \mathbb{C}$,*

$$|P(z)| \leq \exp(mG_A(z)) \max_{x\in A} |P(x)|. \tag{5.3}$$

In particular, for any $x_0 \in \mathbb{R}$, $\ell, L > 0$,

$$\max_{x\in[x_0-L,x_0+L]} |P(x)| \leq \left(\frac{2L}{\ell}\right)^m \max_{x\in[x_0-\ell,x_0+\ell]} |P(x)|. \tag{5.4}$$

5.3 Polynomial operators

It is well known [4, Chapter 1] that there exists a unique system of polynomials $p_k(x) = \gamma_k x^k + \cdots$, $\gamma_k > 0$, $k = 0, 1, 2, \cdots$, such that for $k, j = 0, 1, \cdots$,

$$\int p_k p_j d\mu = \begin{cases} 1, & \text{if } k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover (cf. [4, Chapter 2]), any function $f \in L^1(\mu)$ is uniquely determined by the sequence of its coefficients

$$\hat{f}(k) := \int f p_k d\mu, \qquad k = 0, 1, \cdots. \tag{5.5}$$

Next, we define the kernels. For $x, y \in \mathbb{C}$, we will write

$$K_n(x, y) := \sum_{k=0}^{2^n} p_k(x)p_k(y), \qquad n = 0, 1, 2, \cdots, \tag{5.6}$$

and $K_{-1}(x,y)=0$. For an integer $n\geq 0$, a function $\Phi_n:\mathbb{C}\times\mathbb{C}\to\mathbb{R}$ will be called a *reproducing summability kernel (of order n)*, if each of the following four conditions is satisfied. For each $x,y\in\mathbb{C}$, $\Phi_n(x,y)=\Phi_n(y,x)$, $\Phi_n(x,\circ)\in\Pi_{2n}$,

$$\int \Phi_n(x,y)P(y)d\mu(y)=P(x), \qquad P\in\Pi_n, \tag{5.7}$$

and

$$\sup_{x\in \mathrm{supp}\ (\mu)}\int |\Phi_n(x,y)|d\mu(y)\leq c, \tag{5.8}$$

where c is a constant independent of n, depending at most on μ and the whole sequence $\{\Phi_n\}$. We assume in the sequel that there exists a sequence $\{\Phi_n\}$ of reproducing summability kernels.

In [10], we have proved that the kernels $\Phi_n(\mu^{(\alpha,\beta)},h,x,y)$ defined in (2.1) are reproducing summability kernels if $\alpha,\beta\geq -1/2$. In [4, Section IV.3], Freud has shown the strong $(C,1)$ summability of a very general class of orthogonal polynomials. If h is an integral of a function of bounded variation, $h(x)=1$ for $0\leq x\leq 1/2$, and $h(x)=0$ for $x>1$, it can be shown using a summation by parts argument that a kernel similar to the one defined in (2.1) with these orthogonal polynomials satisfies all the properties mentioned above. In the case of the Jacobi measure, we were able to use the special function properties of Jacobi polynomials to obtain localization estimates on the kernels $\Phi_n^{(\alpha,\beta)}$ (cf. [10]). These techniques cannot be used to obtain localization estimates on the kernels in general, for example, for the orthogonal polynomial systems discussed by Freud. Nevertheless, our simple construction below allows one to construct exponentially localized kernels based only on the summability estimates. In turn, the localization allows one to use the ideas in [10] to obtain a characterization of local Besov spaces on the interval also in the case of these more general systems of orthogonal polynomials.

For $x,y\in\mathbb{C}$, and $n=0,1,\cdots$, let

$$\Phi_n^*(x,y)=\left(\frac{4-(x-y)^2}{4}\right)^n\Phi_{3n}(x,y).$$

If ν is a Borel, finite, positive or signed measure (with bounded variation), and $f\in L^1(\nu)$, we define the operators

$$\sigma_n(\nu;f,x):=\int\Phi_n^*(x,y)f(y)d\nu(y), \qquad x\in\mathbb{C},\ n=0,1,\cdots.$$

With $\nu=\mu^{(\alpha,\beta)}$ and $\Phi_n(\mu^{(\alpha,\beta)},h,x,y)$ in place of $\Phi_n(x,y)$, $\sigma_n(\nu;f)$ reduces to $\sigma_n^C(\alpha,\beta;h,f)$. We observe that Φ_n^* being a symmetric polynomial in x and y, has an expansion of the form

$$\Phi_n^*(x,y)=\sum_{j=0}^{8n}\sum_{k=0}^{8n}a_{n;j,k}p_j(x)p_k(y),$$

where $(a_{n;j,k})$ is a symmetric matrix. Therefore, taking ν to be the measure μ, we see that

$$\sigma_n(\mu;f)=\sum_{j=0}^{8n}\left(\sum_{k=0}^{8n}a_{n;j,k}\hat{f}(k)\right)p_j$$

is a polynomial with coefficients given as a finite linear combination of the coefficients $\{\hat{f}(k)\}_{k=0}^{8n}$. The more general definition allows us to compute these operators using, for example, values of f.

Our generalization of Theorem 2.1 is the following.

Theorem 5.1 *Let $1 \le p \le \infty$ and $\{\nu_n\}$ be a p-compatible M–Z sequence of measures.*
(a) *We have $\sigma_n(\nu_n; P) = P$ for all $P \in \Pi_n$, and*

$$\|\sigma_n(\nu_n; f)\|_{\mu;p} \le c\|f\|_{\nu_n;p}, \qquad f \in L^p(\nu_n),\ 1 \le p \le \infty. \tag{5.9}$$

Further, for each $f \in X^p(\mu)$,

$$\|f - \sigma_n(\nu_n; f)\|_{\mu;p} \le cE_{n,p}(f). \tag{5.10}$$

(b) *Let $\mathcal{E}(\mu) < \infty$,* supp (μ) *be a regular set, $f \in C(\mu)$, $\{\nu_n\}$ be an ∞-compatible M–Z sequence of measures, $x_0 \in$* supp (μ)*, $0 < d \le 2$, and f have an analytic continuation to a complex neighborhood of $\{z \in \mathbb{C} : |z - x_0| \le d\}$ of x_0. Then*

$$|f(x) - \sigma_n(\nu_n; f, x)| \le c(f, x_0) \exp\left(-n\frac{d^2 \log(e/2)}{e^2 \log(e^2/d)}\right), \quad x \in [x_0 - d/e, x_0 + d/e] \cap \text{supp } (\mu). \tag{5.11}$$

If $\{\nu_n\}$ is a sequence of finite positive or signed Borel measures having a bounded variation on $[-1, 1]$, we define for $z \in \mathbb{C}$

$$\tau_n(\nu_{2^n}; f, z) = \begin{cases} \sigma_1(\nu_1; f, z), & \text{if } n = 0, \\ \sigma_{2^n}(\nu_{2^n}; f, z) - \sigma_{2^{n-1}}(\nu_{2^{n-1}}; f, z), & \text{if } n \ge 1. \end{cases} \tag{5.12}$$

Clearly, the operator τ_n depends upon two measures: ν_{2^n} and $\nu_{2^{n-1}}$. Although we have to mention the measure to distinguish between the general case and the continuous case, when each $\nu_{2^n} = \mu$, we prefer to keep the notation simpler rather than using the more cumbersome notation $\tau_n(\nu_{2^n}, \nu_{2^{n-1}}; f, x)$. In the Jacobi case, we choose $\Phi_n(\mu^{(\alpha,\beta)}; h, x, y)$ in place of $\Phi_n(x, y)$. Choosing each ν_n to be $\mu^{(\alpha,\beta)}$ we obtain $\tau_n(\mu^{(\alpha,\beta)}; f, x) = \tau_n^C(\alpha, \beta; h, f, x)$. We obtain $\tau_n^D(\alpha, \beta; h, f, x)$ by choosing ν_{2^n} to be the measure that associates the mass $\lambda_{k,2^{n+3}+1}$ with each $x_{k,2^{n+3}+1}$, $k = 1, \cdots, 2^{n+3} + 1$.

The following proposition, generalizing Proposition 2.1, shows a representation of any function in X^p, $1 \le p \le \infty$, in terms of the operators and kernels introduced so far. The theorem uses two sequences of measures. The sequence $\{\nu_n\}$ is determined by the kind of information we have regarding the target function f. Thus, if one starts with the coefficients $\{\hat{f}(k)\}$, then each of the measures ν_n is equal to μ. On the other hand, if a set of values of the form $\{f(x_{k,N_n})\}$ are available at a system of points, we should choose ν_n to be an M–Z measure supported at the points $\{x_{k,N_n} : k = 1, \cdots, N_n\}$, if such a measure can be found. The choice of the sequence $\{\mu_n\}$ is required only to satisfy (5.2), and may be used judiciously to obtain a parsimoneous representation, or a representation with other desirable properties depending upon the application.

Proposition 5.2 *Let* $1 \le p \le \infty$, $\{\nu_n\}$ *be a* p*-compatible M–Z sequence,* $\{\mu_n\}$ *be a sequence of measures satisfying (5.2), and* $f \in X^p(\mu)$*. We have*

$$f = \sum_{n=0}^{\infty} \tau_n(\nu_{2^n}; f), \tag{5.13}$$

where the convergence of the series is in the norm of $X^p(\mu)$*. In the case when each* $\nu_{2^n} = \mu$*, we have further*

$$f = \sum_{n=0}^{\infty} \int \tau_n(\mu; f, y) \{K_{n+3}(\circ, y) - K_{n-1}(\circ, y)\} \, d\mu_{2^n}(y). \tag{5.14}$$

Moreover, for $f \in L^2(\mu)$*, we have the frame property:*

$$c_1 \|f\|_{\mu;2}^2 \le \sum_{n=0}^{\infty} \|\tau_n(\mu; f)\|_{\mu;2}^2 = \sum_{n=0}^{\infty} \|\tau_n(\mu; f)\|_{\mu_{2^n};2}^2 \le c_2 \|f\|_{\mu;2}^2. \tag{5.15}$$

Next, we describe the generalization of Theorem 2.2.

Theorem 5.2 *Let* $\mathcal{E}(\mu) < \infty$, $\text{supp}\ (\mu)$ *be a regular set,* $x_0 \in \text{supp}\ (\mu)$, $f \in C(\mu)$*, and* $\{\nu_n\}$ *be an* ∞*-compatible M–Z sequence of measures.*
(a) *The function* f *has an analytic continuation to a complex neighborhood of* x_0 *if and only if there exists a non-degenerate interval* I *with* $x_0 \in I$ *such that*

$$\limsup_{n \to \infty} \|\tau_n(\nu_{2^n}; f)\|_{\mu;\infty,I}^{1/2^n} < 1. \tag{5.16}$$

(b) *The function* f *has an analytic continuation to a complex neighborhood of* x_0 *if and only if there exists a non-degenerate interval* I *with* $x_0 \in I$ *such that*

$$\limsup_{n \to \infty} \|\tau_n(\nu_{2^n}; f)\|_{\nu_{2^{n+3}};\infty,I}^{1/2^n} < 1. \tag{5.17}$$

We note that $\mathcal{E}(\mu) \ge \log(1/2)(\mu([-1,1]))^2$. The condition $\mathcal{E}(\mu) < \infty$ implies, in particular, that $\mu(\{x\}) = 0$ for each $x \in [-1,1]$. Thus, in view of Proposition 5.1, the measures $\{\nu_n\}$ as required in Theorem 5.2 always exist. We observe again that the operators $\tau_n(\nu_{2^n}; f)$ are defined using global information about f; the coefficients $\{\hat{f}(k)\}$ in the case when each ν_{2^n} is equal to μ. Nevertheless, the exponential localization of these operators enables us to obtain the characterization of local analyticity of the function. Similar characterizations of local Besov spaces can also be obtained, using the ideas in [10].

6 Proofs

Proof of Proposition 5.1. This proof follows the ideas in [13]. Without loss of generality, we assume that $\mu([-1,1]) = 1$. Let $n \ge 1$ be an integer. Since Π_n is a finite dimensional space, there exists a constant, to be denoted in this proof only by B_n such that

$$\int_{-1}^{1} |P'(t)|dt \le B_n \int |P(t)|d\mu(t), \qquad P \in \Pi_n. \tag{6.1}$$

Our assumption that $\mu(\{x\}) = 0$ for each $x \in [-1,1]$ implies that the function $x \mapsto \mu([-1,x))$ is a continuous, nondecreasing function on $[-1,1]$, with the range of this function being $[0,1]$. Therefore, there exist intervals I_k with mutually disjoint interiors such that $[-1,1] = \cup I_k$, and $\mu(I_k) \le 1/(4B_n)$ for each I_k. In this proof only, let $\mathcal{I}$ be the set of integers k such that $I_k \cap \text{supp}\ (\mu)$ is not empty, and we choose a point $x_k \in I_k \cap \text{supp}\ (\mu)$ for each $k \in \mathcal{I}$. In view of (6.1), we have for any $P \in \Pi_n$,

$$\begin{aligned} \left| \|P\|_{\mu;1} - \sum_{k\in\mathcal{I}} \mu(I_k)|P(x_k)| \right| &= \left| \sum_{k\in\mathcal{I}} \left(\int_{I_k} |P(t)|d\mu(t) - \int_{I_k} |P(x_k)|d\mu(t) \right) \right| \\ &\le \sum_{k\in\mathcal{I}} \int_{I_k} |P(t) - P(x_k)|d\mu(t) \\ &\le \sum_{k\in\mathcal{I}} \int_{I_k} \int_{I_k} |P'(u)|du d\mu(t) \\ &\le \frac{1}{4B_n} \int_{-1}^{1} |P'(u)|du \le (1/4)\|P\|_{\mu;1}. \end{aligned}$$

Therefore,

$$(3/4)\|P\|_{\mu;1} \le \sum_{k\in\mathcal{I}} \mu(I_k)|P(x_k)| \le (5/4)\|P\|_{\mu;1}. \tag{6.2}$$

In this proof only, let N be the number of elements in $\mathcal{I}$, T denote the linear operator defined by $T(P) = (P(x_k))_{k\in\mathcal{I}}$, and V be the range of T. The estimates (6.2) imply that T is invertible on V. In this proof only, let x^* denote the linear functional defined on V by $x^*(T(P)) = \int P d\mu$. We equip $\mathbb{R}^N$ by the norm $|||(r_k)_{k\in\mathcal{I}}||| = \sum_{k\in\mathcal{I}} \mu(I_k)|r_k|$. The estimates (6.2) imply that the dual norm of x^* with respect to this norm is bounded from above by 4/3. The Hahn–Banach theorem, together with the characterization of the dual of $\mathbb{R}^N$, implies the existence of $w_k \in \mathbb{R}$, $k \in \mathcal{I}$, such that the functional $(r_k)_{k\in\mathcal{I}} \mapsto \sum_{k\in\mathcal{I}} w_k r_k$ extends x^*, and has the dual norm bounded from above by 4/3. In this proof only, let μ_n be the measure that associates the mass w_k with x_k, $k \in \mathcal{I}$. It is easy to check that the total variation measure of μ_n associates the mass $|w_k|$ with each x_k, $k \in \mathcal{I}$. The statement about the extension means that

$$\int P d\mu_n = \int P d\mu, \qquad P \in \Pi_n. \tag{6.3}$$

The statement about the dual norm means that $|w_k| \le (4/3)\mu(I_k)$ for $k \in \mathcal{I}$. Therefore, the estimates (6.2) imply that

$$\|P\|_{\mu_n;1} = \sum_{k\in\mathcal{I}} |w_k||P(x_k)| \le (4/3) \sum_{k\in\mathcal{I}} \mu(I_k)|P(x_k)| \le (5/3)\|P\|_{\mu;1}, \qquad P \in \Pi_n.$$

We note that each μ_n is supported on a finite subset of supp (μ). Therefore, each μ_n is trivially ∞–compatible. Setting $\nu_n = \mu_{16n}$, $n = 1, 2, \cdots$, we have thus shown that the sequence $\{\nu_n\}$ is an ∞-compatible M–Z sequence of measures, with each ν_n supported on a finite subset of supp (μ). $\square$

Proof of Theorem 5.1(a). Let $P \in \Pi_n$, $x \in [-1,1]$, and Q be defined by

$$Q(y) = P(y)\left(\frac{4-(x-y)^2}{4}\right)^n, \qquad y \in \mathbb{R}.$$

Then $Q \in \Pi_{3n}$. Consequently, (5.2) and (5.7) imply that

$$\begin{aligned}
\sigma_n(\nu_n; P, x) &= \int P(y)\left(\frac{4-(x-y)^2}{4}\right)^n \Phi_{3n}(x,y)d\nu_n(y) \\
&= \int Q(y)\Phi_{3n}(x,y)d\nu_n(y) = \int Q(y)\Phi_{3n}(x,y)d\mu(y) \\
&= Q(x) = P(x).
\end{aligned}$$

Since $|4-(x-y)^2| \le 4$ for all $x, y \in [-1,1]$, the conditions (5.1) and (5.8) imply that

$$\sup_{n\ge 0}\sup_{x\in \text{supp }(\mu)} \int |\Phi_n^*(x,y)|d|\nu_n|(y) \le \sup_{n\ge 0}\sup_{x\in \text{supp }(\mu)} \int |\Phi_{3n}(x,y)|d\mu(y) \le c. \tag{6.4}$$

Therefore, for any $f \in L^\infty(\nu_n)$ and $x \in$ supp (μ), we have

$$|\sigma_n(\nu_n; f, x)| \le \int |\Phi_n^*(x,y)||f(y)|d|\nu_n|(y) \le c\|f\|_{\nu_n;\infty}. \tag{6.5}$$

Thus, (5.9) is satisfied if $p = \infty$. If $f \in L^1(\nu_n)$ and $g \in L^\infty(\mu)$, we verify using Fubini's theorem that

$$\int \sigma_n(\nu_n; f, x)g(x)d\mu(x) = \int f(y)\sigma_n(\mu; g, y)d\nu_n(y).$$

Since $\sigma_n(\mu; g) \in \Pi_{8n}$, the condition (5.1) implies that $\|\sigma_n(\mu;g)\|_{\nu_n;\infty} \le c\|\sigma_n(\mu;g)\|_{\mu;\infty}$. Therefore, using (6.5) with μ in place of ν_n, we obtain that for every $f \in L^1(\nu_n)$ and $g \in L^\infty(\mu)$,

$$\begin{aligned}
\left|\int \sigma_n(\nu_n; f, x)g(x)d\mu(x)\right| &\le \int |f(y)\sigma_n(\mu; g, y)|d|\nu_n|(y) \\
&\le \|\sigma_n(\mu;g)\|_{\nu_n;\infty}\|f\|_{\nu_n;1} \le c\|\sigma_n(\mu;g)\|_{\mu;\infty}\|f\|_{\nu_n;1} \\
&\le c\|g\|_{\mu;\infty}\|f\|_{\nu_n;1}.
\end{aligned}$$

Therefore, the Hahn-Banach theorem implies that $\|\sigma_n(\nu_n; f)\|_{\mu;1} \le c\|f\|_{\nu_n;1}$ for every $f \in L^1(\nu_n)$. Thus, we have proved (5.9) for $p = 1, \infty$. An application of the Riesz–Thorin interpolation theorem now yields (5.9) for $1 < p < \infty$.

Consequently, for any $P \in \Pi_n$,

$$E_{8n,p}(f) \le \|f - \sigma_n(\nu_n; f)\|_{\mu;p} = \|f - P - \sigma_n(\nu_n; f-P)\|_{\mu;p} \le c\|f-P\|_{\mu;p}.$$

Since P is arbitrary, this implies (5.10). $\square$

In order to prove Theorem 5.1(b), we need two lemmas. First, we recall a well known fact from the theory of approximation of analytic functions [17, Chapter IX, Section 3]. For the sake of completion, we will sketch a proof.

Lemma 6.1 *Let $x_0 \in \mathbb{R}$, $d > \ell > 0$, and f be analytic on the complex neighborhood $\{z \in \mathbb{C} : |z - x_0| \le d\}$. Then for every integer $n \ge 1$, there exists a polynomial $P \in \Pi_{n-1}$ such that*

$$|f(x) - P(x)| \le c(f, x_0, d/\ell)(\ell/d)^n, \qquad x \in [x_0 - \ell, x_0 + \ell]. \tag{6.6}$$

Proof. With an appropriate affine transform, we may assume that $\ell = 1$, $x_0 = 0$, denote in this proof only, $\delta = d/\ell > 1$, and assume that f is analytic on the disc $|z| \le \delta$. Let P be the partial sum of the power series expansion of f around 0 of degree $n - 1$. Then for $x \in [-1, 1]$,

$$f(x) - P(x) = \frac{x^n}{2\pi i} \oint_{|\xi|=\delta} \frac{f(\xi)d\xi}{\xi^n(\xi - x)}.$$

Since $|x| \le 1$ and $|\xi - x| \ge |\xi| - |x| \ge \delta - 1$ for $x \in [-1, 1]$, $|\xi| = \delta$, we deduce that

$$|f(x) - P(x)| \le \frac{c(f, \delta)}{2\pi\delta^n(\delta - 1)} \oint_{|\xi|=\delta} |d\xi| = c(f, \delta)\delta^{-n}.$$

□

The next lemma helps us to estimate the norms of $\sigma_n(\nu_n; f)$ on small intervals in terms of the norms of f on slightly larger intervals.

Lemma 6.2 *Let $f \in C(\mu)$, $x_0 \in \text{supp}\,(\mu)$, $\ell > 0$, $I = [x_0 - \ell, x_0 + \ell] \cap \text{supp}\,(\mu)$, $J = [x_0 - 2\ell, x_0 + 2\ell] \cap [-1, 1]$, and $\{\nu_n\}$ be an M–Z sequence of measures. Then for every integer $n \ge 0$ and $x \in I$,*

$$|\sigma_n(\nu_n; f, x)| \le c\|f\|_{\nu_n;\infty,J} + c_1 \left(\frac{4 - \ell^2}{4}\right)^n \|f\|_{\nu_n;\infty}. \tag{6.7}$$

Proof. Let $x \in I$. If $y \in [-1, 1] \setminus J$, then

$$\frac{4 - (x - y)^2}{4} \le \frac{4 - \ell^2}{4},$$

and hence,

$$|\Phi_n^*(x, y)| \le \left(\frac{4 - \ell^2}{4}\right)^n |\Phi_{3n}(x, y)|.$$

Therefore, (5.1) and (5.8) imply that

$$\begin{aligned} \left|\int_{[-1,1]\setminus J} \Phi_n^*(x, y)f(y)d\nu_n(y)\right| &\le \left(\frac{4 - \ell^2}{4}\right)^n \|f\|_{\nu_n;\infty} \int |\Phi_{3n}(x, y)|d|\nu_n|(y) \\ &\le c_1 \left(\frac{4 - \ell^2}{4}\right)^n \|f\|_{\nu_n;\infty}. \end{aligned}$$

The estimate (6.4) implies that

$$\left|\int_J \Phi_n^*(x, y)f(y)d\nu_n(y)\right| \le \|f\|_{\nu_n;\infty,J} \int |\Phi_n^*(x, y)|d|\nu_n|(y) \le c\|f\|_{\nu_n;\infty,J}.$$

Together with the definition of $\sigma_n(\nu_n; f, x)$, we are thus led to (6.7). $\square$

PROOF OF THEOREM 5.1(b). In this proof, constants denoted by $c, c_1, \cdots$ may depend upon x_0 and f. In this proof only, let

$$m = n\frac{d^2}{e^2\log(e^2/d)}.$$

Since $d \le 2$ and $e^2\exp(-4/e^2) \ge 4.3 \ge d$, it is not difficult to see that $m \le n$. In view of Lemma 6.1, we find a polynomial $P \in \Pi_m = \Pi_{\lfloor m \rfloor}$, such that

$$|f(x) - P(x)| \le c(e/2)^{-m}, \qquad x \in [x_0 - 2d/e, x_0 + 2d/e] =: J. \tag{6.8}$$

Hence, $|P(x)| \le c_1$ for $x \in [x_0 - 2d/e, x_0 + 2d/e]$, and (5.4) implies that $|P(x)| \le c_2(2e/d)^m$ for $x \in [x_0 - 2, x_0 + 2] \supseteq [-1, 1]$. Consequently,

$$|f(x) - P(x)| \le c_3(2e/d)^m, \qquad x \in \text{supp }(\mu). \tag{6.9}$$

Since the measures ν_n are ∞–compatible, the estimates (6.7), (6.8), and (6.9) imply that for $x \in [x_0 - d/e, x_0 + d/e] \cap \text{supp }(\mu) =: I$ and integer $n \ge 1$,

$$\begin{aligned} |\sigma_n(\nu_n; f - P, x)| &\le c_4\left\{\|f - P\|_{\nu_n;\infty,J} + \left(\frac{4}{4 - d^2/e^2}\right)^n \|f - P\|_{\nu_n;\infty}\right\} \\ &\le c_5\left\{(e/2)^{-m} + (2e/d)^m\left(\frac{4}{4 - d^2/e^2}\right)^n\right\} \\ &\le c_6\{(e/2)^{-m} + (2e/d)^m\exp(-nd^2/e^2)\}. \end{aligned} \tag{6.10}$$

Since $m \le n$, $\sigma_n(\nu_n; P) = P$. Using (6.8) and (6.10) we conclude that for $x \in I$,

$$\begin{aligned} |f(x) - \sigma_n(\nu_n; f, x)| &= |f(x) - P(x) - \sigma_n(\nu_n; f - P, x)| \\ &\le c_7\{(e/2)^{-m} + (2e/d)^m\exp(-nd^2/e^2)\}. \end{aligned}$$

In view of our choice of m,

$$|f(x) - \sigma_n(\nu_n; f, x)| \le c_8\exp\left(-n\frac{d^2\log(e/2)}{e^2\log(e^2/d)}\right), \qquad x \in I.$$

$\square$

PROOF OF PROPOSITION 5.2. We note that $f \in X^p$ implies that $E_{n,p}(f) \to 0$ as $n \to \infty$. The equation (5.13) follows from (5.12) and (5.10). Next, let each $\nu_{2^n} = \mu$. Since $\tau_n(\mu; f) \in \Pi_{2^{n+3}}$, we verify easily that

$$\tau_n(\mu; f, x) = \int \tau_n(\mu; f, y)K_{n+3}(x, y)d\mu(y). \tag{6.11}$$

Further, since $K_{n-1}(x, y)$, as a function of y, is in $\Pi_{2^{n-1}}$, $\tau_n(\mu; K_{n-1}(x, \circ)) = 0$. Therefore,

$$\int \tau_n(\mu; f, y)K_{n-1}(x, y)d\mu(y) = \int f(z)\tau_n(\mu; K_{n-1}(x, \circ), z)d\mu(z) = 0,$$

and (6.11) may be rewritten in the form

$$\tau_n(\mu; f, x) = \int \tau_n(\mu; f, y)\,(K_{n+3}(x, y) - K_{n-1}(x, y))\,d\mu(y).$$

Since μ_{2^n} satisfies the quadrature formula (5.2), this implies (5.14).

In the remainder of this proof only, let

$$\mathbb{P}_m(f, x) := \int f(y)\,(K_m(x, y) - K_{m-1}(x, y))\,d\mu(y), \qquad x \in \mathbb{R},\ m = 0, 1, \cdots.$$

We note that the Parseval identity implies that

$$f = \sum_{m=0}^{\infty} \mathbb{P}_m(f), \qquad \|f\|_{\mu;2}^2 = \sum_{m=0}^{\infty} \|\mathbb{P}_m(f)\|_{\mu;2}^2,$$

where the convergence of the first series is in the sense of $L^2(\mu)$.

Next, we observe that $\mathbb{P}_m(P) = 0$ if $P \in \Pi_{2^{m-1}}$. If $n \ge 0$ is an integer, and $n + 4 \le m$, then $\tau_n(\mu; f) \in \Pi_{2^{n+3}} \subseteq \Pi_{2^{m-1}}$, and $\mathbb{P}_m(\tau_n(\mu; f)) = 0$. Similarly, if $n - 1 \ge m$, then for each $x \in \mathbb{R}$, $K_m(x, \circ) - K_{m-1}(x, \circ) \in \Pi_{2^m} \subseteq \Pi_{2^{n-1}}$, and hence, for each $x, t \in \mathbb{R}$,

$$\int (\Phi^*_{2^n}(y, t) - \Phi^*_{2^{n-1}}(y, t))\,(K_m(x, y) - K_{m-1}(x, y))\,d\mu(y) = 0.$$

Therefore, if $n - 1 \ge m$, then

$$\begin{aligned}
&\mathbb{P}_m(\tau_n(\mu; f)) \\
&\quad = \int \left(\int f(t)\,(\Phi^*_{2^n}(y, t) - \Phi^*_{2^{n-1}}(y, t))\,d\mu(t)\right)(K_m(x, y) - K_{m-1}(x, y))\,d\mu(y) \\
&\quad = \int f(t) \int (\Phi^*_{2^n}(y, t) - \Phi^*_{2^{n-1}}(y, t))\,(K_m(x, y) - K_{m-1}(x, y))\,d\mu(y) d\mu(t) = 0.
\end{aligned}$$

Hence, (5.13) implies that for any integer $m \ge 0$,

$$\begin{aligned}
\|\mathbb{P}_m(f)\|_{\mu,2}^2 &= \left\|\sum_{n=0}^{\infty} \mathbb{P}_m(\tau_n(\mu; f))\right\|_{\mu;2}^2 = \left\|\sum_{n=\max(0,m-3)}^{m} \mathbb{P}_m(\tau_n(\mu; f))\right\|_{\mu;2}^2 \\
&\le \left(\sum_{n=\max(0,m-3)}^{m} \|\mathbb{P}_m(\tau_n(\mu; f))\|_{\mu;2}\right)^2 \\
&\le 4 \sum_{n=\max(0,m-3)}^{m} \|\mathbb{P}_m(\tau_n(\mu; f))\|_{\mu;2}^2 \\
&\le 4 \sum_{n=\max(0,m-3)}^{m} \|\tau_n(\mu; f)\|_{\mu;2}^2.
\end{aligned}$$

This implies

$$\|f\|_{\mu;2}^2 = \sum_{m=0}^{\infty} \|\mathbb{P}_m(f)\|_{\mu,2}^2 \le 4\sum_{m=0}^{\infty}\sum_{n=\max(0,m-3)}^{m} \|\tau_n(\mu;f)\|_{\mu;2}^2 \le 16\sum_{n=0}^{\infty}\|\tau_n(\mu;f)\|_{\mu;2}^2.$$

The proof of the second inequality in (5.15) is similar. Thus, arguing as before, we see that $\tau_n(\mu;\mathbb{P}_m(f)) = 0$ except when $n \le m \le n+3$. So, for any integer $n \ge 0$,

$$\begin{aligned}
\|\tau_n(\mu;f)\|_{\mu;2}^2 &= \left\|\sum_{m=0}^{\infty}\tau_n(\mu;\mathbb{P}_m(f))\right\|_{\mu;2}^2 = \left\|\sum_{m=n}^{n+3}\tau_n(\mu;\mathbb{P}_m(f))\right\|_{\mu;2}^2 \\
&\le \left(\sum_{m=n}^{n+3}\|\tau_n(\mu;\mathbb{P}_m(f))\|_{\mu;2}\right)^2 \le 4\sum_{m=n}^{n+3}\|\tau_n(\mu;\mathbb{P}_m(f))\|_{\mu;2}^2 \\
&\le c\sum_{m=n}^{n+3}\|\mathbb{P}_m(f)\|_{\mu;2}^2.
\end{aligned}$$

Consequently,

$$\sum_{n=0}^{\infty}\|\tau_n(\mu;f)\|_{\mu;2}^2 \le c\sum_{m=0}^{\infty}\|\mathbb{P}_m(f)\|_{\mu;2}^2 = c\|f\|_{\mu;2}^2.$$

□

Proof of Theorem 5.2. We will prove part (a). The equivalence of (a) and (b) is a simple consequence of Lemma 6.2 and the fact that $\sigma_{2^{n+3}}(\nu_{2^{n+3}};P) = P$ for all $P \in \Pi_{2^{n+3}}$. Let $2 > \ell > 0$, (5.16) be satisfied for $J = [x_0 - 2\ell, x_0 + 2\ell] \cap [-1,1]$ in place of I, and $0 < \rho_1 < 1$ and integer N be chosen so that for all integer $n \ge N$,

$$\|\tau_n(\nu_{2^n};f)\|_{\mu;\infty,J} < \rho_1^{2^n}.$$

Since $\tau_n(\nu_{2^n};f) \in \Pi_{2^{n+3}}$ and $\|\tau_n(\nu_{2^n};f)\|_{\mu;\infty} \le c\|f\|_{\mu,\infty}$, we see from Lemma 6.2 applied with μ in place of ν_{2^n} that for every $x \in I := [x_0-\ell, x_0+\ell] \cap \text{supp }(\mu)$,

$$|\tau_n(\nu_{2^n};f,x)| = |\sigma_{2^{n+3}}(\mu;\tau_n(\nu_{2^n};f),x)| \le c\rho_1^{2^n} + c_1(1-\ell^2/4)^{2^n}\|f\|_{\mu,\infty}. \tag{6.12}$$

Since $x_0 \in \text{supp }(\mu)$, $\mu(I) > 0$, and hence, it is easy to see that the restriction of μ to I has a finite logarithmic energy. Therefore, cap $(I) > 0$. In view of a result of Wiener (cf. [18, Theorem 1.1, Appendix A]), I is a regular set. Letting $\rho := \max(\rho_1, (1-\ell^2/4))$, we obtain from (5.3) and (6.12) that for every $z \in \mathbb{C}$, and integer $n \ge N$,

$$|\tau_n(\nu_{2^n};f,z)| \le c(f,x_0,\ell)\left(\rho\exp(G_I(z))\right)^{2^n}.$$

We observe that $0 < \rho < 1$. Therefore, the series $\sum_{n=0}^{\infty}\tau_n(\nu_{2^n};f,z)$ converges uniformly and absolutely on compact subsets of the region $\{z \in \mathbb{C} \ :\ |G_I(z)| < \log(1/\rho)\}$. Since I is regular, this is an open neighborhood of I. In view of (5.13), the sum of this series is an analytic function that coincides with f on I.

The converse assertion follows immediately from Theorem 5.1(b) and the relevant definitions. □

It is proved by Lubinsky, Máté, and Nevai [8, Theorem 5] that the measures ν_n that associate the mass $\lambda_{k,2^{n+3}+1}$ with each $x_{k,2^{n+3}+1}$, $k = 1, \cdots, 2^{n+3}+1$, form an ∞–compatible M–Z sequence. Further, it is proved in [10] that the kernel $\Phi_n(\mu^{(\alpha,\beta)}; h, x, y)$ is a reproducing kernel of order n. Theorem 2.1 (respectively, Theorem 2.2) follows from Theorem 5.1 (respectively, Theorem 5.2) since $\mu^{(\alpha,\beta)}$ has finite logarithmic energy and $[-1, 1]$ is a regular set. Proposition 2.1 follows similarly from Proposition 5.2.

We now turn our attention to the proofs of the results in Section 4. The ideas are the same; we only sketch the proofs when the technical details are essentially different. First, we recall from [12, Theorem 3.3] that if each $\mathcal{C}_n$ admits a positive quadrature formula of order $16n$, and $w_{\mathbf{x}}$, $\mathbf{x} \in \mathcal{C}_n$, are nonnegative weights satisfying (4.1), then

$$\sum_{\mathbf{x}\in\mathcal{C}_n} w_{\mathbf{x}}|P(\mathbf{x})| \le c\int_{\mathbb{S}^q} |P(\xi)|d\mu_q^*(\xi), \qquad P \in \Pi_{16n}^q.$$

Thus, the sequence of measures associating the weight $w_{\mathbf{x}}$ with $\mathbf{x} \in \mathcal{C}_n$ is the spherical analogue of a sequence of M–Z measures.

We further recall an analogue of the Bernstein–Walsh inequality from [9, Estimate (22)]: For $m = 0, 1, \cdots$, $0 < \alpha < \beta \le \pi$,

$$\max_{\mathbf{x}\in\mathbb{S}^q_\beta} |P(\mathbf{x})| \le \left(\frac{\pi(2\beta-\alpha)}{\alpha}\right)^{2m} \max_{\mathbf{x}\in\mathbb{S}^q_\alpha} |P(\mathbf{x})|, \qquad P \in \Pi_m^q. \tag{6.13}$$

In [9], it was assumed that $\beta < \pi$. However, the same estimate holds clearly for $\beta = \pi$ because of continuity.

Proposition 4.1 is proved exactly as Proposition 5.2. There are no new ideas involved, and we omit the proof.

Proof of Theorem 4.1. The proof of (4.3) is similar to that of (5.10). The estimate (4.4) is proved analogously to Theorem 5.1(b) with the following differences. The role of Lemma 5.1 is played by (6.13). The analogue of Lemma 6.2 can be proved in exactly the same way, using the fact that $(1+\mathbf{x}\cdot\mathbf{y})^n < 2^n$ if $\mathbf{x} \ne \mathbf{y}$. The definition of the class $\mathcal{A}_q(\mathbf{x}_0)$ is the substitute for Lemma 6.1. This is the reason why the estimate (4.4) is evidently weaker than the estimate (5.11). Except for these technical differences, the proof of (4.4) follows that of (5.11) in exactly the same way. The proof of part (b) is similar to that of Theorem 5.2, with no new ideas. □

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