

# On a Sequence of Fast Decreasing Polynomial Operators

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**Abstract.** Let  $f$  be a piecewise analytic function on the unit interval (respectively, the unit circle of the complex plane). Starting from the Chebyshev (respectively, Fourier) coefficients of  $f$ , we construct a sequence of fast decreasing polynomials (respectively, trigonometric polynomials) which “detect” the points where  $f$  fails to be analytic, provided  $f$  is not infinitely differentiable at these points.

## 1. Introduction

A fast decreasing sequence of polynomials is a sequence of polynomials which are “large” near some points, and decrease exponentially fast at other points. Such polynomials have been constructed by Gaier, Ivanov, Saff, Totik, and others ([1], [6] and references therein) in connection with the approximation of piecewise analytic functions on a real interval. The error in approximating such functions decays exponentially at the points of analyticity, and (typically) polynomially at the singularities; i.e., at the points where the function fails to be analytic.

In [4], we used segment approximation to **detect** the singularities of a piecewise analytic function on a compact, real interval, provided the number of such points is known in advance. A central observation in our approach is the fact that a piecewise analytic function on an interval (in the sense of [4]) necessarily has jump discontinuities in some derivative at the singularities. Motivated by the work [1] of Gaier, we define in this paper a sequence of fast decreasing polynomials which enables us to detect the location of those singularities of a function where it is not infinitely many times differentiable. Our approach also does not require the prior knowledge of the number of such singularities.

Our construction uses the Chebyshev coefficients of the function. We would like to observe here that the singularity or analyticity of a function is a local property, whereas each of the Chebyshev coefficients represents global information about the function. In [5], we have constructed polynomial operators, based on the Jacobi coefficients, that detect the points where the derivatives of a function

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have jump discontinuities. However, these operators do not define fast decreasing polynomials. In the case of Chebyshev polynomials, we are able to use function-theoretic methods to construct operators which yield fast decreasing polynomials.

Our techniques enable us to detect the singularities of a piecewise analytic function defined on the unit circle as well. In fact, the starting point of our research is to construct fast decreasing trigonometric polynomials for this purpose.

We discuss the construction of fast decreasing trigonometric polynomials in the next section. In Section 3, we point out the connection with Chebyshev polynomials. In Section 4, we present some numerical examples.

## 2. Functions on the Unit Circle

Let  $U$  denote the complex unit circle,  $f : U \rightarrow \mathbb{C}$  be integrable with respect to the arclength of  $U$ . The Fourier coefficients of  $f$  are then defined by

$$\hat{f}_k := \frac{1}{2\pi i} \int_U \frac{f(\omega)}{\omega^{k+1}} d\omega, \quad k = 0, \pm 1, \pm 2, \dots \quad (1)$$

Suppose that both the power series  $\sum_{k=0}^{\infty} \hat{f}_k z^k$  and  $\sum_{k=1}^{\infty} \hat{f}_{-k} z^k$  have radius of convergence equal to 1. Then they define analytic functions on the open unit disc, and have singularities on  $U$  in the sense defined in Hille's book [2, §10.2, p.4 and §10.4, p.15]. We say that  $f$  is piecewise analytic if the total number of singularities on  $U$  of both of these power series is finite. The singularities of  $f$  are then defined to be the singularities of these power series. If  $r \geq 0$  is an integer, we will say that  $f$  has a singularity of order  $r$  at a point  $w_0$  if  $f$  is  $r - 1$  times continuously differentiable at  $w_0$  with respect to the arclength, and the  $r$ th angular derivative (i.e., derivative with respect to the arclength) has a jump discontinuity at  $w_0$ .

Next, we define certain operators related to those defined by Gaier in [1]. For  $m = 0, 1, \dots$ , and  $z \in \mathbb{C}$ , we write

$$s_m(f, z) := \sum_{k=-m}^m \hat{f}_k z^k.$$

Our operators are defined by applying a summability method with the  $s_m$ 's. Towards this end, we write for integers  $\ell \geq 1$ ,  $m \geq 0$ ,

$$h_{\ell, m} := \begin{cases} 0, & \text{if } 0 \leq m \leq \ell - 1, \\ -\frac{1}{\ell}, & \text{if } \ell \leq m \leq 2\ell - 1, \\ \frac{1}{2\ell}, & \text{if } 2\ell \leq m \leq 4\ell - 1, \\ 0, & \text{if } m \geq 4\ell, \end{cases}$$

and  $h_{0,m} := 0$ . For integer  $n \geq 0$ , the summability factors are defined by

$$\begin{aligned} g_{n,m} &:= 2^{-n} \sum_{\ell=0}^n \binom{n}{\ell} \{h_{\ell,4m} + h_{\ell,4m+1} + h_{\ell,4m+2} + h_{\ell,4m+3}\} \\ &= 2^{-n} \sum_{\ell=m+1}^n \binom{n}{\ell} \{h_{\ell,4m} + h_{\ell,4m+1} + h_{\ell,4m+2} + h_{\ell,4m+3}\}. \end{aligned} \quad (2)$$

Our operators are now defined for integer  $n \geq 1$  by

$$\tau_n(f) := \sum_{m=0}^{n-1} g_{n,m} s_m(f) =: \sum_{k=-n+1}^{n-1} \tilde{g}_{n,|k|} \hat{f}_k z^k.$$

We observe that for each integer  $n \geq 1$ ,  $\tau_n$  is a linear operator defined on the class of all functions on  $U$  integrable with respect to the arclength, and for each such function  $f$ ,  $\tau_n(f)$  is a Laurent polynomial of degree at most  $n-1$  (cf. Fig. 2).

For  $f : U \rightarrow \mathbb{C}$ , and  $x \geq 0$ ,  $\|f\|$  denotes  $\sup_{z \in U} |f(z)|$ , and we write

$$E_x(f) := \inf_{T \in \mathbb{H}_x} \|f - T\|,$$

where  $\mathbb{H}_x$  denotes the class of all functions of the form  $\sum_{|k| \leq x} a_k(\cdot)^k$ . Restricted to  $U$ , elements of  $\mathbb{H}_x$  are just trigonometric polynomials of order at most  $x$  in the argument variable.

For integer  $r \geq 0$ , and  $z \in U$ , we define

$$\Gamma_r(z) := \sum_{k=-\infty, k \neq 0}^{\infty} \frac{z^k}{(ik)^{r+1}}.$$

The function  $\Gamma_r$  is analytic on  $U$ , except at  $z = 1$ , where its  $r$ th angular derivative (i.e., derivative with respect to the arclength) has a jump discontinuity. Hence, if  $f : U \rightarrow \mathbb{C}$  has a singularity of order  $r$  at  $w_0 \in U$ , then for a judiciously chosen complex constant  $a(f)$ , the  $r$ th angular derivative of  $f - a(f)\Gamma_r(\cdot/w_0)$  is continuous at  $w_0$ .

In the sequel, we adopt the following notations and conventions. Letters  $c, c_1, c_2, \dots$ , will denote positive constants depending on explicitly indicated quantities only. Their values may be different at different occurrences, even within the same formula. The main theorem of this section is the following.

**Theorem 2.1.** *Let  $f : U \rightarrow \mathbb{C}$  be continuous. For each integer  $n \geq 1$ ,*

$$\|\tau_n(f)\| \leq c_1 e^{-cn} \|f\| + 6E_{n/12}(f). \quad (3)$$

*If  $w_0 \in U$ , and  $f$  is analytic in the disc  $\{z : |z - w_0| < d\}$  for some  $0 < d := d(f, w_0) < 1$ , then*

$$|\tau_n(f, w_0)| \leq \frac{c(f)}{d^4} \left(1 - \frac{d^2}{900}\right)^{n/2}. \quad (4)$$

Suppose that  $f$  has only finitely many singularities, and that the only discontinuities in all angular derivatives of  $f$  at these singularities are jump discontinuities. If  $r \geq 0$  is an integer, there is an arc  $J := J_r(f)$  of the circle, such that if  $w_0 \in U$  is a singularity of  $f$  of order  $r$ , then

$$n^r |\tau_n(f, w)| \geq c(f, r), \quad (ww_0^{-1})^n \in J. \quad (5)$$

In order to prove Theorem 2.1, we need some preparation. First, we define the de la Vallée Poussin operators by  $v_0(f, z) := \hat{f}_0$  and

$$v_\ell(f, z) := \frac{1}{\ell} \sum_{m=\ell}^{2\ell-1} s_m(f, z), \quad \ell = 1, 2, \dots$$

We also need the Euler means of the difference of these operators. Thus, we define for  $n = 0, 1, \dots$

$$\tau_n^*(f, z) := 2^{-n} \sum_{\ell=0}^n \binom{n}{\ell} \left( v_{2\ell}(f, z) - v_\ell(f, z) \right).$$

Next, let

$$F(z) := f(z^4).$$

For  $z = \rho e^{i\theta}$ , we will write  $z^{1/4} := \rho^{1/4} \exp(i\theta/4)$ ,  $\theta \in (-\pi, \pi]$ . If  $z_0 \in U$ ,  $|z - z_0^{1/4}| < \delta < 1$ , then

$$|z^4 - z_0| < |z - z_0^{1/4}|(1 + |z|)(1 + |z|^2) < 15\delta.$$

Hence, if  $f$  is analytic in the disc of radius  $d < 1$ , centered at some  $z_0 \in U$ , then  $F$  is analytic in the disc of radius  $d/15$  around  $z_0^{1/4}$  (and hence, also in the discs of radius  $d/15$  around  $-z_0^{1/4}$  and  $\pm iz_0^{1/4}$ ). Similarly,  $f$  has a singularity of order  $r$  at  $z_0 \in U$  if and only if  $F$  has singularities of order  $r$  at  $\pm z_0^{1/4}$  and  $\pm iz_0^{1/4}$ .

Further, we observe that for integer  $m \geq 0$ ,

$$s_{4m}(F, z) = s_{4m+1}(F, z) = s_{4m+2}(F, z) = s_{4m+3}(F, z) = s_m(f, z^4).$$

Hence, for integer  $\ell \geq 1$ ,

$$\begin{aligned} v_{2\ell}(F, z) - v_\ell(F, z) &= \sum_{m=0}^{4\ell-1} h_{\ell, m} s_m(F, z) \\ &= \sum_{m=0}^{\ell-1} \{h_{\ell, 4m} + h_{\ell, 4m+1} + h_{\ell, 4m+2} + h_{\ell, 4m+3}\} s_m(f, z^4) \\ &=: \sum_{k=-\ell+1}^{\ell-1} \tilde{h}_{\ell, |k|} \hat{f}_k z^{4k}. \end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}
\tau_n^*(F, z) &= 2^{-n} \sum_{\ell=0}^n \binom{n}{\ell} \left( v_{2\ell}(F, z) - v_\ell(F, z) \right) \\
&= 2^{-n} \sum_{m=0}^{n-1} \left\{ \sum_{\ell=m+1}^n \binom{n}{\ell} \left[ h_{\ell, 4m} + h_{\ell, 4m+1} + h_{\ell, 4m+2} + h_{\ell, 4m+3} \right] \right\} s_m(f, z^4) \\
&= \tau_n(f, z^4).
\end{aligned}$$

Thus, Theorem 2.1 will be proved if we prove the following.

**Theorem 2.2.** *Let  $F : U \rightarrow \mathbb{C}$  be continuous, and*

$$F(z) = F(-z) = F(iz) = F(-iz), \quad z \in \mathbb{C}.$$

For each integer  $n \geq 0$ ,

$$\|\tau_n^*(F)\| \leq 0.15e^{-n/18} \|F\| + 6E_{n/3}(F). \quad (6)$$

If  $w_0 \in U$ , and  $F$  is analytic in the disc  $\{z : |z - w_0| < d\}$  for some  $0 < d := d(F, w_0) < 1$ , then

$$|\tau_n^*(F, w_0)| \leq \frac{c(F)}{d^4} \left(1 - \frac{d^2}{4}\right)^{n/2}. \quad (7)$$

Suppose that  $F$  has only finitely many singularities, and that the only discontinuities in all angular derivatives of  $F$  at these singularities are jump discontinuities. If  $r \geq 0$  is an integer, there is an arc  $J := J_r(F)$  of the circle, such that if  $w_0 \in U$  is a singularity of  $F$  of order  $r$ , then

$$n^r |\tau_n^*(F, w)| \geq c(F, r), \quad (ww_0^{-1})^n \in J. \quad (8)$$

*Proof.* It is well known that

$$\|v_\ell(F)\| \leq 3\|F\|, \quad \|v_\ell(F) - v_{2\ell}(F)\| \leq 6E_\ell(F), \quad \ell = 0, 1, \dots.$$

Hence, (6) follows from an estimate for binomial sums (cf. [3]) and Stirling's inequality

$$2^{-n} \sum_{\ell=0}^{\lceil n/3 \rceil - 1} \binom{n}{\ell} \leq 2^{-n} \binom{n}{\lceil n/3 \rceil} \leq \left(\frac{3}{2^{5/3}}\right)^n \frac{3e^{1/(12n)}}{2\sqrt{\pi n}} \leq 0.15e^{(\ln 3 - 5 \ln 2)n},$$

where the last inequality is true for  $n \geq 32$ . For  $n < 32$  the first sum can be estimated directly by computation.

To prove the next assertion of the theorem, we assume without loss of generality that  $w_0 = 1$ , and that  $F$  is analytic in the disc  $\{z : |z - 1| < d\}$  for some  $d$ ,  $0 < d < 0.7$ . Following Gaier [1], we consider the arc  $\gamma$  of the circle  $|z + 1| = \sqrt{4 - d^2}$  intercepted by  $U$  and inside  $U$ . Let  $w$  and  $W$  be the end points of  $\gamma$  in the upper and lower half plane respectively (cf. Figure 1). Both  $w$  and  $W$

are on the arc from  $e^{-i\pi/4}$  to  $e^{i\pi/4}$ . Let  $C_0$  be the Jordan arc from  $e^{-i\pi/4}$  to  $e^{i\pi/4}$ , following  $\gamma$  from  $W$  to  $w$ , and  $U$  otherwise. Let

$$C_1 := C_0 \cup (iC_0) \cup (-C_0) \cup (-iC_0),$$

and  $C_2$  be the reflection of  $C_1$  in the unit circle. We observe that the lens-shaped region between  $\gamma$  and its reflection in  $U$  lies in  $\{z : |z - 1| < d\}$ , and hence,  $F$  is analytic in this region.

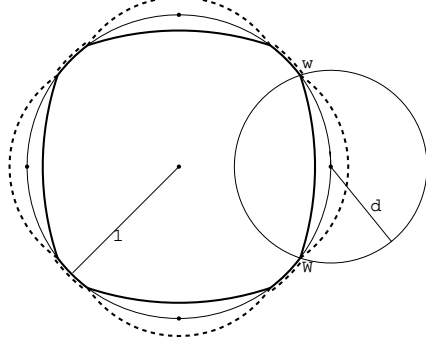


Fig. 1. The unit circle with curves  $C_1$  (thick line) and  $C_2$  (dotted line).

Following Gaier [1], we obtain from (1) and Cauchy's formula that for integer  $m \geq 0$ ,

$$s_m(F, 1) = F(1) + \frac{1}{2\pi i} \oint_{C_1} \frac{F(\zeta)}{(\zeta - 1)} \zeta^m d\zeta - \frac{1}{2\pi i} \oint_{C_2} \frac{F(\zeta)}{(\zeta - 1)} \zeta^{-m-1} d\zeta,$$

and hence, after a little computation, that for integer  $\ell \geq 1$ ,

$$\begin{aligned} v_{2\ell}(F, 1) - v_\ell(F, 1) &= \frac{1}{4\pi i} \oint_{C_1} \frac{F(\zeta)}{(\zeta - 1)^2} \frac{\zeta^{4\ell} - 3\zeta^{2\ell} + 2\zeta^\ell}{\ell} d\zeta \\ &\quad - \frac{1}{4\pi i} \oint_{C_2} \frac{F(\zeta)}{(\zeta - 1)^2} \frac{\zeta^{-4\ell} - 3\zeta^{-2\ell} + 2\zeta^{-\ell}}{\ell} d\zeta, \end{aligned}$$

and for integer  $n \geq 1$ ,

$$\tau_n^*(F, 1) = \frac{1}{4\pi i} \oint_{C_1} \frac{F(\zeta)}{(\zeta - 1)^2} H_n(\zeta) d\zeta - \frac{1}{4\pi i} \oint_{C_2} \frac{F(\zeta)}{(\zeta - 1)^2} H_n(\zeta^{-1}) d\zeta, \quad (9)$$

where

$$H_n(z) := 2^{-n} \sum_{\ell=1}^n \binom{n}{\ell} \frac{z^{4\ell} - 3z^{2\ell} + 2z^\ell}{\ell}.$$

Writing  $h_n(t) = t^{-1}((1+t)^n - 1)$ , we observe that

$$H_n(z) = 2^{-n} \left\{ \int_{z^2}^{z^4} h_n(t) dt - 2 \int_z^{z^2} h_n(t) dt \right\},$$

where the integrations are over any paths joining the indicated end points. For  $\zeta \in C_1$ , we may choose these paths to lie entirely inside  $C_1$  and outside the disc  $\{z : |z| \leq (\sqrt{4-d^2}-1)^{1/4}\}$ . With this choice,

$$|h_n(t)| \leq \frac{(4-d^2)^{n/2} + 1}{(\sqrt{3}-1)^{1/4}}$$

for all  $t$  on all these paths. Therefore, for  $\zeta \in C_1$ ,

$$|H_n(\zeta)| \leq \frac{2\pi}{(\sqrt{3}-1)^{1/4}} \left\{ \left(1 - \frac{d^2}{4}\right)^{n/2} + 2^{-n} \right\} \leq 10.72 \left(1 - \frac{d^2}{4}\right)^{n/2}.$$

Along with (9), this implies

$$|\tau_n^*(F, 1)| \leq \frac{10.72}{(2 - \sqrt{4-d^2})^2} \left(1 - \frac{d^2}{4}\right)^{n/2} c(F),$$

which leads to (7).

Next, let  $w_0$  be a singularity of  $F$ . Without loss of generality, we may again assume that  $w_0 = 1$ . Let  $R$  be the maximum of the orders of all the singularities of  $F$ . Our assumption on  $F$  implies that

$$F(z) = \omega \sum_{\nu=0}^3 \Gamma_r(z i^\nu) + \Phi_1(z) + \Phi_2(z), \quad z \in U,$$

where  $\omega \in \mathbb{C}$ ,  $\Phi_1$  is a (finite) linear combination of terms of the form  $\Gamma_k(z w_j^{-1})$ ,  $w_j \neq 1$ ,  $0 \leq k \leq R$ , and  $\Phi_2$  has  $R$  continuous angular derivatives on  $U$ . In particular,  $\Phi_1$  is analytic at 1, and both  $\Phi_1$  and  $\Phi_2$  satisfy

$$\Phi_j(z) = \Phi_j(iz) = \Phi_j(-z) = \Phi_j(-iz), \quad j = 1, 2.$$

From (6) and Favard's theorem,  $\tau_n^*(\Phi_2, w) = o(n^{-R})$  for  $w \in U$ . From (7),  $\tau_n^*(\Phi_1, w) = o(e^{-c(F)n})$  for  $w$  on some arc  $J_1$  around 1. We observe that

$$\tau_n^*(\Gamma_r(i^\nu \cdot), w) = \tau_n^*(\Gamma_r, i^\nu w)$$

for  $\nu = 0, 1, 2, 3$  and  $w \in U$ . Hence, the estimate (8) will be proved if we prove that

$$n^r |\tau_n^*(\Gamma_r, i^\nu w)| = \mathcal{O}(n^{-1}), \quad \nu = 1, 2, 3, \quad (10)$$

and

$$n^r |\tau_n^*(\Gamma_r, w)| \geq c_r \quad (11)$$

for sufficiently large integer values of  $n$ , and  $w^n$  on some arc of  $U$  which intersects  $J_1$ .

We write  $w =: e^{i\theta}$ ,  $\tau_{n,r} := \tau_n^*(\Gamma_r)$ ,  $v_{\ell,r} := v_\ell(\Gamma_r)$ , and

$$h(t) := \frac{1}{\pi} \begin{cases} 0, & \text{if } t \in [0, \pi/2), \\ t^{-r-1}(2t - \pi), & \text{if } t \in [\pi/2, \pi), \\ t^{-r-1}(2\pi - t), & \text{if } t \in [\pi, 2\pi). \end{cases}$$

A simple summation by parts argument leads to the formula (for integer  $\ell \geq 1$ )

$$v_{2\ell,r}(w) - v_{\ell,r}(w) = 4\pi \left(\frac{2\pi}{4\ell}\right)^r \frac{1}{4\ell} \sum_{k=0}^{4\ell-1} h\left(\frac{2\pi k}{4\ell}\right) \cos(k\theta - (r+1)\pi/2).$$

Next, we recall that for any integer  $p \geq 0$  and  $\ell \geq 1$ ,

$$\ell^{-p} = \frac{\ell!}{(\ell+p)!} \left(1 + \mathcal{O}(\ell^{-1})\right) \leq c(p) \frac{\ell!}{(\ell+p)!}. \quad (12)$$

Hence, an application of Lemma 3.2 of [5] gives

$$\begin{aligned} v_{2\ell,r}(w) - v_{\ell,r}(w) &= 2 \left(\frac{2\pi}{4\ell}\right)^r \left\{ \int_0^{2\pi} h(t) \cos(\ell\alpha t - \beta) dt + \mathcal{O}(\ell^{-1}) \right\} \\ &= 2 \left(\frac{\pi}{2}\right)^r \frac{\ell!}{(\ell+r)!} \left\{ \int_0^{2\pi} h(t) \cos(\ell\alpha t - \beta) dt + \mathcal{O}(\ell^{-1}) \right\}, \end{aligned} \quad (13)$$

where  $\alpha := 2\theta/\pi$ ,  $\beta := (r+1)\pi/2$ .

Let  $\nu \in \{1, 2, 3\}$ . Using this formula with  $i^\nu w$  in place of  $w$ , we get

$$v_{2\ell,r}(i^\nu w) - v_{\ell,r}(i^\nu w) = 2 \left(\frac{\pi}{2}\right)^r \frac{\ell!}{(\ell+r)!} \left\{ \int_0^{2\pi} h(t) \cos(\ell\alpha_\nu t - \beta) dt + \mathcal{O}(\ell^{-1}) \right\},$$

where  $\alpha_\nu = \alpha + \nu$ . For  $|\theta| < \pi/8$ , we have  $|\alpha_\nu| \geq c$ . Hence, using integration by parts, we see that

$$\int_0^{2\pi} h(t) \cos(\ell\alpha_\nu t - \beta) dt = \mathcal{O}(\ell^{-1}). \quad (14)$$

Now, using (12), we obtain

$$\sum_{\ell=1}^n \binom{n}{\ell} \ell^{-p} \leq c(p) \frac{n!}{(n+p)!} \sum_{\ell=1}^n \binom{n+p}{\ell+p} \leq c_1(p) n^{-p} 2^n. \quad (15)$$

Together with (13) and (14), this estimate (with  $p = 1$ ) leads to (10).

Next, we deduce from (13) and (15) that

$$\begin{aligned} \tau_{n,r}(w) &= 2 \left(\frac{\pi}{2}\right)^r \frac{n!}{(n+r)!} \operatorname{Re} \left\{ e^{-i\beta} \int_0^{2\pi} h(t) e^{-ir\alpha t} 2^{-n} \sum_{l=r+1}^{n+r} \binom{n+r}{l} e^{i\alpha t l} dt \right. \\ &\quad \left. + \mathcal{O}(n^{-1}) \right\} \\ &= 2\pi^r \frac{n!}{(n+r)!} \operatorname{Re} \left\{ e^{-i\beta} \int_0^{2\pi} h(t) e^{-ir\alpha t} ((1 + e^{i\alpha t})/2)^{n+r} dt \right. \\ &\quad \left. + \mathcal{O}(n^{-1}) \right\}. \end{aligned}$$

For any  $A > 0$  and  $|\alpha| \leq A/n$ , this yields

$$\begin{aligned} \tau_{n,r}(w) &= 2\pi^r \frac{n!}{(n+r)!} \operatorname{Re} \left\{ e^{-i\beta} \int_0^{2\pi} h(t) e^{in\alpha t} dt + \mathcal{O}(n^{-1}) \right\} \\ &= 2\pi^r \frac{n!}{(n+r)!} \left\{ \int_0^{2\pi} h(t) \cos(n\alpha t - \beta) dt + \mathcal{O}(n^{-1}) \right\}, \end{aligned} \quad (16)$$

where the unspecified constants may depend also on  $A$ . Now, the entire function

$$z \mapsto \int_0^{2\pi} h(t) \cos(zt - \beta) dt$$

is not identically equal to zero, and hence, every neighborhood of 0 contains a real interval  $I$  such that

$$\left| \int_0^{2\pi} h(t) \cos(zt - \beta) dt \right| \geq c_r$$

for all  $z \in I$ . Along with (16), this leads to (11).  $\square$

### 3. Functions on $[-1, 1]$

Let  $f : [-1, 1] \rightarrow \mathbb{R}$ . We will say that  $f$  is piecewise analytic if there exist points  $-1 =: y_0 < y_1 < \dots < y_s := 1$  such that  $f$  is analytic in each of the open intervals  $(y_{j-1}, y_j)$  for  $j = 1, \dots, s$ . The points  $y_j$  will be called the singularities of  $f$ . This notion is more general than that used in [4], where the restriction of  $f$  to each of the *closed* intervals  $[y_{j-1}, y_j]$  was assumed to be analytic. For the analysis in this section, we do have to assume the existence of all the one-sided derivatives of  $f$  at the points  $y_j$ , including at  $\pm 1$ . Our objective is to detect the location of the points  $y_j$  that are in the open interval  $(-1, 1)$ , given the sequence of Chebyshev coefficients of  $f$  (see (17) below for the definition). Unlike [4], we do not require that the number of these ‘‘interior singularities’’ be known in advance. We say that a point  $y \in (-1, 1)$  is a singularity of  $f$  of order  $r$  if  $f$  has  $r$  continuous derivatives in a deleted neighborhood of  $y$ , and the  $r$ th derivative of  $f$  has a jump discontinuity at  $y$ .

We recall that the Chebyshev polynomials are defined by the formula

$$T_n(\cos \theta) := \cos(n\theta), \quad n = 0, 1, 2, \dots, \theta \in [0, \pi].$$

They satisfy the following orthogonality relations

$$\int_{-1}^1 T_n(t) T_m(t) \frac{dt}{\sqrt{1-t^2}} = \begin{cases} \pi, & \text{if } n = m = 0, \\ \pi/2, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

If  $f$  is a continuous function on  $[-1, 1]$ , we may define the Chebyshev coefficients for integer  $k \geq 0$  by

$$f_k^T := \frac{1}{\sqrt{\pi}} \begin{cases} \int_{-1}^1 f(t)(1-t^2)^{-1/2} dt, & \text{if } k = 0, \\ \sqrt{2} \int_{-1}^1 f(t)T_k(t)(1-t^2)^{-1/2} dt, & \text{if } k \neq 0. \end{cases} \quad (17)$$

The partial sums of the Chebyshev expansion of  $f$  are then defined for integer  $m \geq 0$  and  $x \in [-1, 1]$  by

$$S_m(f, x) := \frac{1}{\sqrt{\pi}} f_0^T + \sqrt{\frac{2}{\pi}} \sum_{k=1}^m f_k^T T_k(x).$$

With the summability factors  $g_{n,m}$  as in (2), we may now define the operators  $\mathcal{T}_n$  by

$$\mathcal{T}_n(f, x) := \sum_{m=0}^{n-1} g_{n,m} S_m(f, x), \quad n = 1, 2, \dots, \quad x \in [-1, 1].$$

Finally, we write

$$\|f\|_I := \max_{x \in [-1, 1]} |f(x)|,$$

and for  $y \geq 0$ ,

$$\epsilon_y(f) := \min_{P \in \Pi_y} \|f - P\|_I,$$

where  $\Pi_y$  denotes the class of all algebraic polynomials of degree at most  $y$ .

Theorem 2.1 now leads to the following analogous result.

**Theorem 3.1.** *Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be continuous. For each integer  $n \geq 1$ ,*

$$\|\mathcal{T}_n(f)\|_I \leq c_1 e^{-cn} \|f\|_I + 6\epsilon_{n/12}(f). \quad (18)$$

*If  $x_0 \in [-1, 1]$ , and  $f$  is analytic in the disc  $\{z \in \mathbb{C} : |z - x_0| < d\}$  for some  $0 < d := d(f, x_0) < 1$ , then*

$$|\mathcal{T}_n(f, w_0)| \leq \frac{c(f)}{d^4} (1 - c_1 d^2)^{n/2}. \quad (19)$$

*Suppose that  $f$  fails to be analytic only at finitely many points, and that the only discontinuities in all derivatives of  $f$  at these points are jump discontinuities. We also assume that all the one-sided derivatives of  $f$  at  $\pm 1$  are defined. If  $r \geq 0$  is an integer, there is a subinterval  $J \subseteq [0, \pi]$  such that if  $y =: \cos \phi \in (-1, 1)$  is a singularity of  $f$  of order  $r$ , then*

$$n^r |\mathcal{T}_n(f, \cos \theta)| \geq c(f, r), \quad n|\theta - \phi| \in J. \quad (20)$$

*Proof.* We define  $f^\circ : U \rightarrow \mathbb{R}$  by

$$f^\circ(e^{i\theta}) := f(\cos \theta), \quad \theta \in [-\pi, \pi],$$

and consider the Joukowski transformation

$$z = \frac{1}{2} \left( w + \frac{1}{w} \right), \quad |w| > 1,$$

that maps the exterior of  $[-1, 1]$  conformally onto the exterior of the unit disc in the  $w$ -plane. It is easy to verify from the definitions that

$$S_m(f, \cos \theta) = s_m(f^\circ, e^{i\theta}), \quad \theta \in [-\pi, \pi],$$

and hence, that

$$\mathcal{T}_n(f, \cos \theta) = \tau_n(f^\circ, e^{i\theta}), \quad \theta \in [-\pi, \pi].$$

Together with (3), this implies (18).

We observe that

$$z - \cos \theta = \frac{1}{2w}(w - e^{i\theta})(w - e^{-i\theta}),$$

and that the right-hand side is a rational function of  $w$  with pole at the origin. Therefore, if  $f$  is analytic in the disc  $\{z \in \mathbb{C} : |z - \cos \theta| < d\}$  for some  $\theta \in (0, \pi)$  and  $0 < d < 1$ , then  $f^\circ$  is analytic in discs of radius  $d/2$  around  $e^{i\theta}$  and  $e^{-i\theta}$ . Thus, (4) implies (19).

Our assumption about the one-sided derivatives of  $f$  implies that the only singularities of  $f^\circ$  are at those points  $e^{\pm i\theta}$ , where  $f$  has a singularity at  $\cos \theta$ ,  $\theta \in (0, \pi)$ . Moreover, all the angular derivatives of  $f^\circ$  at these points will have at most jump discontinuities of the same order as the corresponding singularity of  $f$ . We have also assumed the existence of all the one-sided derivatives of  $f$  at  $\pm 1$ . The function  $f^\circ$  is then infinitely differentiable at the points  $\pm 1$  on  $U$ . The estimate (5) then leads to (20).  $\square$

## 4. Numerical Results

We consider the function

$$f(z) := \frac{1}{2} \{ \Gamma_1(z e^{0.8i}) + \Gamma_3(z e^{-i}) \},$$

which has a singularity of order 1 at  $z = z_1 = e^{-0.8i}$  and of order 3 at  $z = z_3 = e^i$ . In the following figures, the quantity  $\arg z$  is on the  $x$ -axis, and the  $y$ -axis denotes the values of the operators in question. The results of our work [5] indicate that the operators

$$V_{1024}(f, z) := v_{512}(f, z) - v_{256}(f, z)$$

are also able to detect the singularity at  $z_1$  and perhaps also at  $z_3$ . We observe that both  $\tau_{1024}(f)$  and  $V_{1024}(f)$  are in  $\mathbb{H}_{1023}$ , and the operator  $V_{1024}$  is obviously much easier to compute. On the other hand, Figure 2 shows the summability factors  $\tilde{g}_{1024,k}$  (respectively  $\tilde{h}_{256,k}$ ) for the operators  $\tau_{1024}$  (respectively  $V_{1024}$ ). It is clear that while both of them are formally of the same order, the real contribution to  $\tau_{1024}$  actually comes from a polynomial of order less than 700.

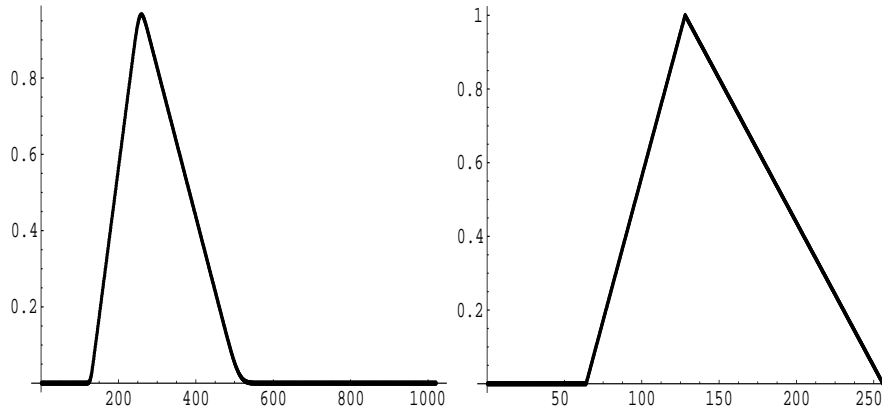


Fig. 2. Left:  $\tilde{g}_{1024,k}$ , Right:  $\tilde{h}_{256,k}$ .

Figure 3 shows that  $\tau_{1024}$  and  $V_{1024}$  are both sufficiently well localized for the detection of the singularity at  $z_1$ .

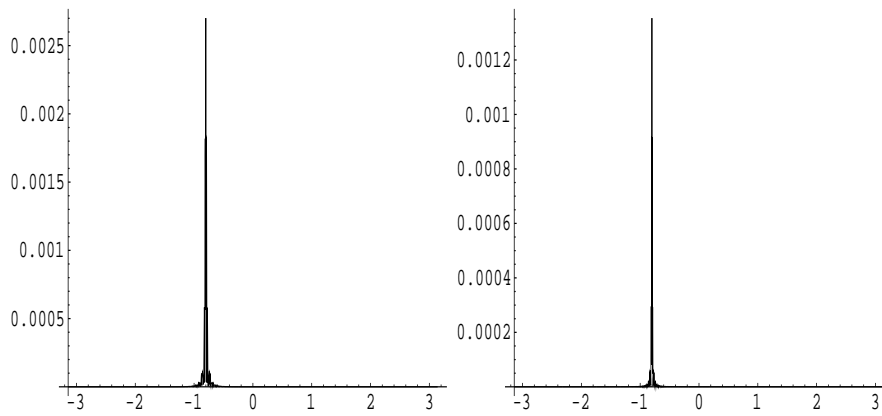


Fig. 3. Left:  $|\tau_{1024}(f)|$ , Right:  $|V_{1024}(f)|$  on  $[-\pi, \pi]$ .

However, as Figure 4 shows, in the presence of this singularity  $z_1$ ,  $V_{1024}$  is not at all able to detect the higher-order singularity at  $z_3$ . The operator  $\tau_{1024}$  is able to do so.

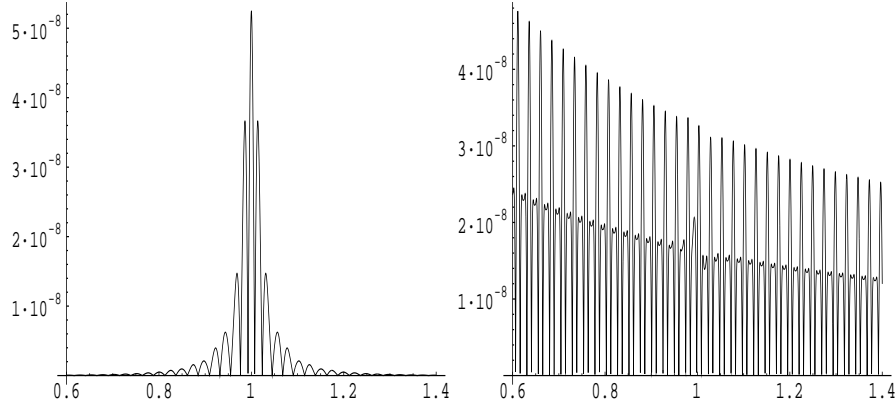


Fig. 4. Left:  $|\tau_{1024}(f)|$ , Right:  $|V_{1024}(f)|$  on  $[0.6, 1.4]$ .

In Figure 5, we examine the behavior of both near  $z_1$ . As we go away from this point, the operator  $\tau_{1024}$  seems to decay slower in the beginning, but then decreases much more rapidly as the theory predicts.

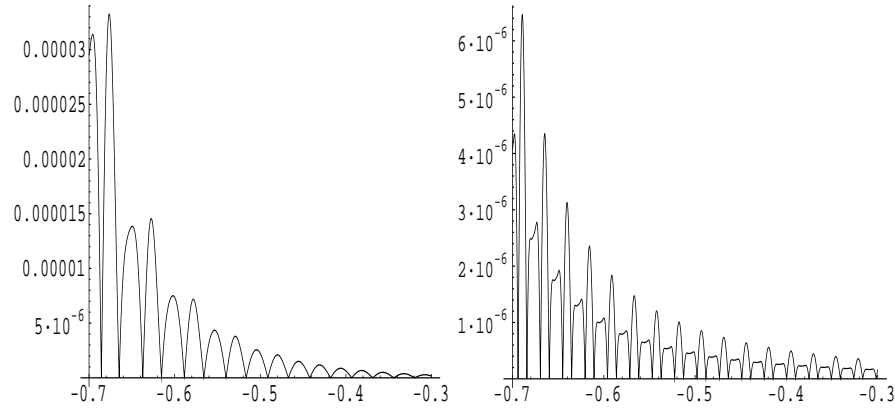


Fig. 5. Left:  $|\tau_{1024}(f)|$ , Right:  $|V_{1024}(f)|$  on  $[-0.7, -0.3]$ .

In Figure 6, it is clear that the oscillations in  $\tau_{1024}$  have already “died” long before we come to  $z_3$ , while those of  $v_{1024}$  are still not sufficiently small to affect the behavior significantly at  $z_3$ .

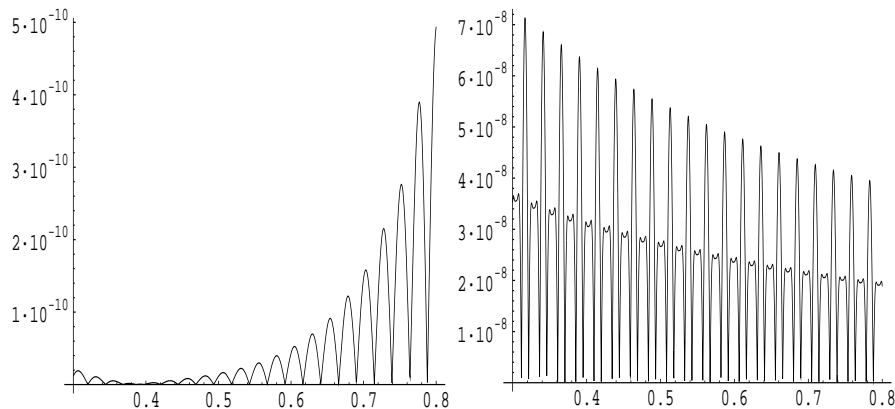


Fig. 6. Left:  $|\tau_{1024}(f)|$ , Right:  $|V_{1024}(f)|$  on  $[0.3, 0.8]$ .

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